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Some Examples on Weak Symmetries

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Abstract

In this paper some examples of weakly symmetric manifold, weakly Ricci symmetric manifold, conformally flat weakly symmetric manifold and weakly projective symmetric manifold are constructed.

Keywords: *Conformally flat manifold, Kähler manifold, projectively flat manifold, weakly symmetric manifold, weakly Ricci symmetric manifold.*

1 Introduction

The idea of weakly symmetric and weakly Ricci symmetric manifolds is introduced by L. Tamassy and T. Q. Binh [1]. After then, these ideas are extended by M. Prvanovic [6], U. C. De and S. Bandyopdhyay [2] and also by the other differential geometers.

A Riemannian manifold is said to be weakly symmetric if the curvature tensor R of the manifold satisfies

$$\begin{aligned}(\nabla_X R)(Y,Z,U,V) =& A(X)R(Y,Z,U,V) + B(Y)R(X,Z,U,V)\\ &+ C(Z)R(Y,X,U,V) + D(U)R(Y,Z,X,V)\\ &+ E(V)R(Y,Z,U,X),\end{aligned} \tag{1}$$

and if the Ricci tensor S of the manifold satisfies

$$(\nabla_X S)(Y,Z) = A(X)S(Y,Z) + B(Y)S(X,Z) + C(Z)S(Y,X), \tag{2}$$

then manifold is called weakly Ricci symmetric manifold. A, B, C, D, E are simultaneously non-vanishing 1-forms and X, Y, Z, U, V are vector fields.

In 1995, Prvanovic [6] proved that if the manifold be weakly symmetric satisfying equation (1) then $B = C = D = E$.

In this paper we have assumed that $B = C = D = E = \omega$, such that $g(X, \rho) = \omega(X)$, and therefore, the equations (1) and (2) can be written as

$$\begin{aligned}(\nabla_X R)(Y, Z, U, V) =& A(X)R(Y, Z, U, V) + \omega(Y)R(X, Z, U, V)\\ &+ \omega(Z)R(Y, X, U, V) + \omega(U)R(Y, Z, X, V)\\ &+ \omega(V)R(Y, Z, U, X),\end{aligned} \tag{3}$$

and

$$(\nabla_X S)(Y, Z) = A(X)S(Y, Z) + \omega(Y)S(X, Z) + \omega(Z)S(Y, X), \tag{4}$$

where $g(X, \alpha) = A(X)$.

2 Example of Weakly Symmetric Manifold

Example 2.1. Let g be a metric in manifold $\mathbb{R}^n$ defined by

$$ds^2 = \varphi(dx^1)^2 + K_{\alpha\beta}dx^\alpha dx^\beta + 2dx^1 dx^n, \tag{5}$$

where $[K_{\alpha\beta}]$ is a non-singular symmetric matrix with entries as constants such that α, β varies from 2 to $(n-1)$ and φ is a function of $x^1, x^2, ..., x^{n-1}$.
Roter [7] found the non-zero components of the Christoffel symbols Γ^i_{jk}, curvature tensor R_{hijk} and the Ricci tensor R_{ij} as follows

$$\Gamma^\beta_{11} = -\frac{1}{2}K^{\alpha\beta}\varphi_{\cdot\alpha}, \quad \Gamma^n_{11} = \frac{1}{2}\varphi_{\cdot 1}, \quad \Gamma^n_{1\alpha} = \frac{1}{2}\varphi_{\cdot\alpha}, \tag{6}$$

and

$$R_{1\alpha\beta 1} = \frac{1}{2}\varphi_{\cdot\alpha\beta}, \qquad R_{11} = \frac{1}{2}K^{\alpha\beta}\varphi_{\cdot\alpha\beta}, \tag{7}$$

where "." denotes the partial differentiation and $[K^{\alpha\beta}]$ is the inverse matrix of $[K_{\alpha\beta}]$. If we take $K_{\alpha\beta}$ as $\delta_{\alpha\beta}$ $(= \{^{1 \quad if \alpha=\beta}_{0, \quad oherwise})$ and $\varphi = K_{\alpha\beta}x^\alpha x^\beta f(x^1)$ then φ becomes

$$\varphi = \sum_{\alpha=2}^{n-1}(x^\alpha)^2 f(x^1), \tag{8}$$

where f is an arbitrary non-zero function of x^1.
From (8), we easily get

$$\varphi_{\cdot\alpha\alpha} = 2f(x^1), \quad \varphi_{\cdot\alpha\beta} = 0, \alpha \neq \beta. \tag{9}$$

Equations (7) and (9) imply that the non-zero components of curvature tensor R_{hijk} and their derivatives $R_{hijk.l}$ respectively are

$$R_{1\alpha\alpha 1} = f(x^1), \tag{10}$$

and

$$R_{1\alpha\alpha 1.1} = \frac{d}{dx^1} f(x^1). \tag{11}$$

Now, from equation (3), the condition of weakly symmetric manifold for the non-zero components of the curvature tensor R_{hijk} becomes

$$R_{1\alpha\alpha 1.1} = A_1 R_{1\alpha\alpha 1} + \omega_1 R_{1\alpha\alpha 1} + \omega_\alpha R_{1\alpha\alpha 1} + \omega_\alpha R_{1\alpha\alpha 1} + \omega_1 R_{1\alpha\alpha 1}. \tag{12}$$

If we take

$$A_i = \{ {}^{\phi(x^1), \quad i=1}_{0, \quad otherwise,} \tag{13}$$

and

$$\omega_i = \{ {}^{\frac{1}{2}[\frac{\frac{d}{dx^1} f(x^1)}{f(x^1)} - \phi(x^1)], \quad i=1}_{0, \quad otherwise,} \tag{14}$$

then by the use of (14), equation (12) reduces to

$$R_{1\alpha\alpha 1.1} = (A_1 + 2\omega_1) R_{1\alpha\alpha 1}. \tag{15}$$

Now, from (10), (13) and (14), we easily get the right hand side of (15)

$$(A_1 + 2\omega_1) R_{1\alpha\alpha 1} = \frac{d}{dx^1} f(x^1). \tag{16}$$

Hence, from (11) and (16), we can say that this is an example of weakly symmetric manifold.

3 Example of Weakly Ricci Symmetric Manifold

Example 3.1. Since equation (7) implies that the non-zero component of Ricci tensor R_{ij} and its derivative $R_{ij.l}$ are

$$R_{11} = (n-2) f(x^1), \tag{17}$$

and

$$R_{11.1} = (n-2) \frac{d}{dx^1} f(x^1). \tag{18}$$

Therefore, from equation (4), the condition of weakly Ricci symmetric manifold for non-zero component of Ricci tensor R_{ij} becomes

$$R_{11.1} = A_1 R_{11} + \omega_1 R_{11} + \omega_1 R_{11}. \tag{19}$$

or

$$R_{11.1} = (A_1 + 2\omega_1)R_{11}. \tag{20}$$

From (13), (14) and (17), the right hand side of the equation (20) becomes

$$(A_1 + 2\omega_1)R_{11} = (n-2)\frac{d}{dx^1}f(x^1). \tag{21}$$

Therefore, from (18) and (21), we can say that this is an example of weakly Ricci symmetric manifold.

4 Example of Conformally Flat Weakly Symmetric Manifold

Example 4.1. We know that the Weyl conformal curvature tensor C on an n-dimensional manifold is defined as

$$\begin{aligned} C(X,Y,Z,T) = R(X,Y,Z,T) &- \frac{1}{(n-2)}[S(Y,Z)g(X,T) \\ &- S(X,Z)g(Y,T) + S(X,T)g(Y,Z) - S(Y,T)g(X,Z)] \\ &+ \frac{r}{(n-1)(n-2)}[g(Y,Z)g(X,T) - g(X,Z)g(Y,T)]. \end{aligned} \tag{22}$$

Now, from (5) we have $g_{ni} = g_{in} = 0$ for $i \neq 1$ but then $g^{11} = 0$ and hence the scalar curvature tensor $r = g^{ij}R_{ij} = g^{11}R_{11} = 0$. Therefore, equation (22) gives the non-zero component of the conformal curvature tensor C_{hijk} as follows

$$C_{1\alpha\alpha 1} = R_{1\alpha\alpha 1} - \frac{1}{(n-2)}g_{\alpha\alpha}R_{11}. \tag{23}$$

Using (10) and (17) in (23) and taking $g_{\alpha\alpha} = 1$, we get

$$C_{1\alpha\alpha 1} = 0. \tag{24}$$

Hence, from equation (24) and example (2.1), we can say that this is an example of conformally flat weakly symmetric manifold

5 Example of Weakly Projective Symmetric Manifold

Example 5.1. It is well known that the projective curvature tensor P on an n-dimensional Riemannian manifold is defined by

$$\begin{aligned} P(X,Y,Z,U) = R(X,Y,Z,U) &- \frac{1}{(n-1)}[S(Y,Z)g(X,U) \\ &- S(X,Z)g(Y,U)]. \end{aligned} \tag{25}$$

Therefore, the non-zero component of the projective curvature tensor P_{hijk} becomes

$$P_{1\alpha\alpha 1} = R_{1\alpha\alpha 1} - \frac{1}{(n-1)} g_{\alpha\alpha} R_{11}. \tag{26}$$

Using (10) and (17) in (26), we get

$$P_{1\alpha\alpha 1} = \frac{1}{(n-1)} f(x^1), \tag{27}$$

and the derivative of projective curvature tensor becomes

$$P_{1\alpha\alpha 1.1} = \frac{1}{(n-1)} \frac{d}{dx^1} f(x^1). \tag{28}$$

Now, the condition of weakly projective symmetric manifold, given by

$$\begin{aligned}(\nabla_X P)(Y,Z,U,V) =& A(X)P(Y,Z,U,V) + B(Y)P(X,Z,U,V)\\ &+ C(Z)P(Y,X,U,V) + D(U)P(Y,Z,X,V)\\ &+ E(V)P(Y,Z,U,X),\end{aligned} \tag{29}$$

reduces to

$$P_{1\alpha\alpha 1.1} = A_1 P_{1\alpha\alpha 1} + \omega_1 P_{1\alpha\alpha 1} + \omega_\alpha P_{1\alpha\alpha 1} + \omega_\alpha P_{1\alpha\alpha 1} + \omega_1 P_{1\alpha\alpha 1}, \tag{30}$$

for non-zero component of projective curvature tensor P_{hijk}.
From (14), above equation can be written as

$$P_{1\alpha\alpha 1.1} = (A_1 + 2\omega_1) P_{1\alpha\alpha 1}. \tag{31}$$

Using (27) in (31), we have

$$P_{1\alpha\alpha 1.1} = \frac{1}{(n-1)} (A_1 + 2\omega_1) f(x^1). \tag{32}$$

From (10) and (32), we can write

$$P_{1\alpha\alpha 1.1} = \frac{1}{(n-1)} (A_1 + 2\omega_1) R_{1\alpha\alpha 1}. \tag{33}$$

Using (16) in (33), we get

$$P_{1\alpha\alpha 1.1} = \frac{1}{(n-1)} \frac{d}{dx^1} f(x^1). \tag{34}$$

Hence, from (28) and (34), we can say that this is an example of weakly projectively symmetric manifold.

6 Conclusion

From examples (2.1), (3.1), (4.1) and (5.1), it is clear that the Riemannian manifold $\mathbb{R}^n$ with metric (5) defined by W. Roter [7] verify the properties of weakly symmetric manifold, weakly Ricci symmetric manifold, conformally flat weakly symmetric manifold and weakly projective symmetric manifold respectively.

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