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On n-Normal Operators

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Abstract

In this paper we introduce n-normal operators on a Hilbert space H. We give some basic properties of these operators. In general an n-normal operators need not be a normal operator, a hyponormal operator.

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1 Introduction

Throughout this paper, $B(H)$ denotes to the algebra of all bounded linear operators acting on a complex Hilbert space H. An operator T is said to be normal if $T^*T = TT^*$, (it is well known that normal operators have translation-invariant property, i.e., if T is a normal operator, then $(T - \lambda)$ is a normal operator for every $\lambda \in \mathbb{C}$); self adjoint if $T^* = T$; positive if $T^* = T$ and $\langle Tx, x\rangle \geq 0$ for all $x \in H$; and projection if $T^2 = T = T^*$. For an operator $T \in H$, if $\|Tx\| = \|x\|$ for all $x \in H$ (or equivalently $T^*T = I$), then T is called an isometry. An onto isometry is called unitary. An operator $T \in B(H)$ is called partial isometry if T^*T is projection. An operator T on H is called subnormal if there exists a Hilbert space K with H is a subspace of K and a normal operator N on K such that $NH \subseteq H$ and $N|H = T$; T is hyponormal if $T^*T \geq TT^*$. Let $T \in B(H)$ and $x \in H$. The sequence $\{T^n x\}_{n=0}^{\infty}$ is called orbit of x under T, and is denoted by $orb(T, x)$. If $orb(T, x)$ is dense in H, then x is called a hypercyclic vector for T. An operator $T \in B(H)$ is called scalar of order m if it possesses a spectral distribution of order m, i.e., if there is a continuous unital morphism $\phi : C_0^m(\mathbb{C}) \longrightarrow B(H)$ such that $\phi(z) = T$ where z stands for the identity function on $\mathcal{C}$ and $C_0^m(\mathbb{C})$ for the space of compactly supported functions on $\mathbb{C}$, continuously differentiable of order m, $0 \leq m \leq \infty$. An operator $T \in B(H)$ is called subscalar if it is similar to the restriction of a scalar operator to an invariant subspace.

2 n-normal operators

Definition 2.1. *$T \in B(H)$ is called an n-normal operator if $T^nT^* = T^*T^n$.*

Proposition 2.2. *Let $T \in B(H)$. Then T is n-normal if and only if T^n is normal where $n \in \mathbb{N}$.*

Proof. Let T is n-normal, $T^nT^* = T^*T^n$. Therefore

$$T^n(T^*)^n = T^*T^n(T^*)^{n-1} = T^*(T^nT^*)(T^*)^{n-2} = (T^*)^2T^n(T^*)^{n-2} = (T^*)^nT^n.$$

Then T^n is normal. Now, let T^n is normal. Since $T^nT = TT^n$, by Fuglede theorem [8], $T^*T^n = T^nT^*$. Therefore T is n-normal. □

It is clear that a bounded normal operator is n-normal for any n. The converse is not true. Indeed if $T = \begin{pmatrix} i & 2 \\ 0 & -i \end{pmatrix}$, then T is 2-normal which is not normal. And all nonzero nilpotent operators are n-normal operators, for $n \geq k$ where k the index of nilpotance, but they are not normal. It is well known that if T is normal, then it is hyponormal. And if T is normal and T^k is compact for some k, then T is compact by [8]. The following example shows that these need not be true in case of n-normal operator.

Example 2.3. *Let $H = \ell^2$ and $e_1, e_2, \ldots$be standard orthogonal basis for ℓ^2. Define T on H by* $Te_i = \begin{cases} e_1, & i=1 \\ e_{i+1}, & i=2j \\ 0, & i=2j+1 \end{cases}$ $, j = 1, 2, \cdots$. *Then $T^2 = P$, where P is the orthogonal projection on the space span by e_1. So T is 2-normal but neither T nor T^* is hyponormal.*
Now, since T^2 is a projection on one-dimensional space , it is compact. However, since range of T contains an infinite orthonormal set $\{e_i, i = 1, 3, 5, \cdots\}$, T is not compact.

The following example shows that there exists an operator which is subnormal but not n-normal for any $n \in N$.

Example 2.4. *Let U be unilateral shift on ℓ^2 (i.e., $U(\alpha_0, \alpha_1, \cdots) = (0, \alpha_0, \alpha_1, \cdots)$. Then U is subnormal but for any $n \in N$, U^n is not normal.*

It is well known that if T is hyponormal and compact, then T is normal. But we note that the nilpotent operator $T = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}$ an n-normal operator, which is compact but not normal. Thus T is not hyponormal.

Theorem 2.5. *The set of all n-normal operators on H is closed subset of $B(H)$ which is closed under scalar multiplication.*

Proof. First if T is n-normal, and α is scalar, then $(\alpha T)^n(\alpha T)^* = \alpha^n\overline{\alpha}(T^nT^*) = \overline{\alpha}\alpha^n(T^*T^n)$ and $(\overline{\alpha}T^*)(\alpha^nT^n) = (\alpha T)^*(\alpha T)^n$. Hence αT is n-normal. Now, suppose that (T_k) is sequence of n-normal operators converging to T in $B(H)$. Now, $\|T^nT^* - T^*T^n\| \leq \|T^nT^* - T_k^nT_k^*\| + \|T_k^*T_k^n - T^*T^n\| \longrightarrow 0$ as $k \longrightarrow \infty$. Hence $T^*T^n = T^nT^*$. Thus T is n-normal. □

Proposition 2.6. *Let $T \in B(H)$ be n-normal. Then*

1. *T^* is n-normal.*
2. *If T^{-1} exists, then (T^{-1}) is n-normal.*
3. *If $S \in B(H)$ is unitary equivalent to T, then S is n-normal.*
4. *If M is a closed subspace of H such that M reduces T, then $S = T/M$ is an n-normal operator.*

Proof. (1) Since T is n-normal, T^n is normal. So $(T^n)^* = (T^*)^n$ is normal, T^* is an n-normal operator.
(2) Since T is n-normal, T^n is normal. Since $(T^n)^{-1} = (T^{-1})^n$ is normal, T^{-1} is an n-normal operator.
(3) Let T be an n-normal operator and S be unitary equivalent of T. Then there exists unitary operator U such that $S = UTU^*$ so $S^n = UT^nU^*$. Since T^n is normal, S^n is normal. Therefore S is n-normal.
(4) Since T is n-normal, T^n is normal. So T^n/M is normal. And since M is invariant under T, $T^n/M = (T/M)^n$. Thus $(T/M)^n$ is normal. So T/M is n-normal.
Now, the following example shows that the class of 2-normal operators may not have the translation-invariant property. □

Example 2.7. *Let $T = \begin{pmatrix} 0 & T_1 \\ 0 & 0 \end{pmatrix}$, where $T_1 : H_1 \longrightarrow H$. Then T is 2-normal operator. But $[(T-\lambda)^{*2}, (T-\lambda)^2] = \begin{pmatrix} -4 \mid \lambda \mid^2 T_1T_1^* & 0 \\ 0 & 4 \mid \lambda \mid^2 T_1^*T_1 \end{pmatrix}$ not necessarily equal to zero unless $\lambda = 0$. Hence $(T-\lambda)^2$ is not normal. So $(T-\lambda)$ is not necessarily 2-normal operator.*

Theorem 2.8. *If S, T are commuting n-normal operators, then ST is an n-normal operator.*

Proof. Since S, T are commuting n-normal operators, S^n, T^n are commuting normal operator. So S^nT^n is a normal operator. Since $S^nT^n = (ST)^n$, $(ST)^n$ is normal. Hence ST is n-normal. □

The following example shows that Theorem 2.8 is not necessarily true if S, T are not commuting.

Example 2.9. *Let* $S = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ *and* $T = \begin{pmatrix} i & 2 \\ 0 & -i \end{pmatrix}$ *be operators on the Hilbert space* $\mathbb{C}^2$. *Then* S *and* T *are 2-normal. We note that* $ST = \begin{pmatrix} i & 2 \\ 0 & i \end{pmatrix} \neq \begin{pmatrix} i & -2 \\ 0 & i \end{pmatrix} = TS$. *But as* $(ST)^2 = \begin{pmatrix} -1 & 4i \\ 0 & -1 \end{pmatrix}$ *is not normal,* ST *is not 2-normal.*

Corollary 2.10. *If* T *is n-normal, Then* T^m *is n-normal for any positive integer* m.

The following example shows that sum of two commuting n-normal operators need not be n-normal.

Example 2.11. *Let* $S = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. *Then* S *and* T *are commuting 2-normal. But* $S+T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $(S+T)^2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ *is not normal. Thus* $S+T$ *is not* 2*-normal. We note here* S *is a selfadjoint operator.*

Proposition 2.12. *Let* T, S *be commuting n-normal operator, such that* $(S+T)^*$ *commutes with* $\sum_{k=1}^{n-1}\binom{n}{k}S^{n-k}T^k$. *Then* $(S+T)$ *is an n-normal operator.*

Proof. Since $(S+T)^n(S+T)^* = (\sum_{k=0}^{n}\binom{n}{k}S^{n-k}T^k)(S^*+T^*)$, $(S+T)^n(S+T)^* = S^nS^* + \sum_{k=1}^{n-1}\binom{n}{k}S^{n-k}T^k(S+T)^* + T^nS^* + S^nT^* + T^nT^*$. And since $(S+T)^*$ is commuting with $\sum_{k=1}^{n-1}\binom{n}{k}S^{n-k}T^k$, $(S+T)^n(S+T)^* = S^*S^n + (S+T)^*\sum_{k=1}^{n-1}\binom{n}{k}S^{n-k}T^k + S^*T^n + T^*S^n + T^*T^n$. So $(S+T)^n(S+T)^* = (S+T)^*(S^n+T^n) + (S+T)^*(\sum_{k=1}^{n-1}\binom{n}{k}S^{n-k}T^k)$. Hence $(S+T)^n(S+T)^* = (S+T)^*(\sum_{k=0}^{n}\binom{n}{k}S^{n-k}T^k) = (S+T)^*(S+T)^n$. □

Lemma 2.13. *If $S, T \in \mathbb{B}(H)$ are 2-normal operators and $ST + TS = 0$, then $T + S$ and ST are 2-normal.*

Proof. Since $ST + TS = 0$, $S^2T^2 = T^2S^2$. So $(S+T)^2 = S^2 + T^2$ is normal. Thus $(S + T)$ is an 2-normal operator.
Now since $ST + TS = 0$, $(ST)^2 = -S^2T^2 = -T^2S^2$. Hence by Theorem 2.8, ST is a 2-normal operator. □

Now we state some well known lemmas which we shall need.

Lemma 2.14. *Let P, Q be the projections on closed subspaces M, N respectively. Then $M \perp N$ if and only if $PQ = 0$.*

Lemma 2.15. *If T is normal, then $Tx = \lambda x$ if and only if $T^*x = \overline{\lambda}x$.*

Lemma 2.16. *If P is the projection on a closed subspace M of H, then M reduces of T if and only if $TP = PT$.*

Theorem 2.17. *Let T be an operator on finite dimensional Hilbert space H, $\lambda_1, ..., \lambda_m$ be eigenvalues of T such that $\lambda_i^n \neq \lambda_j^n$,$i \neq j$, $M_1, ..., M_m$ the corresponding eigenspaces, and $P_1, ..., P_m$ the projections on $M_1, ..., M_m$ respectively. Then M_i's are pairwise orthogonal and they span H if and only if T is n-normal operator.*

Proof. Assume M_i's are pairwise orthogonal and they span H. Then for $x \in H$, $x = x_1 + x_2 + ... + x_m$, $x_i \in M_i$, $T^nx = T^nx_1 + ... + T^nx_m = \lambda_1^n x_1 + ... + \lambda_m^n x_m$.

Since P_i's are projection on eigenspace M_i's which are pairwise orthogonal, by lemma 2.14 $P_ix = x_i$. Hence $Ix = x_1 + ...x_m = P_1x + ... + P_mx = (P_1 + ... + P_m)x$ for every $x \in H$. Thus $I = \Sigma_{i=1}^n P_i$. Since $T^nx = \lambda_1^n x_1 + ... + \lambda_m^n x_m = \lambda_1^n P_1x + ... + \lambda_m^n P_mx = (\lambda_1^n P_1 + ... + \lambda_m^n P_m)x$ for all $x \in H$. So $T^n = \sum_{i=1}^m \lambda_i^n P_i$. Hence $T^{*n} = \overline{\lambda_1}^n P_1 + ... + \overline{\lambda_m}^n P_m$. Since M_i's are pairwise orthogonal, $P_iP_j = \begin{cases} P_i, & \text{if } i = j; \\ 0, & \text{if } i \neq j. \end{cases}$ So $T^nT^{*n} = |\lambda_1|^{2n}P_1 + ... + |\lambda_m|^{2n}P_m$ and $T^{*n}T^n = |\lambda_1|^{2n}P_1 + ... + |\lambda_m|^{2n}P_m$. Thus T^n is normal, i.e., T is an n-normal operator.
Suppose T is an n-normal operator. Then T^n is a normal operator. We claim that M_i's are pairwise orthogonal. Let x_i, x_j be vectors in M_i, M_j, $(i \neq j)$ such that $T^nx_i = \lambda_i^n x_i$ and $T^nx_j = \lambda_j^n x_j$. Then $\lambda_i^n \langle x_i, x_j \rangle = \langle \lambda_i^n x_i, x_j \rangle = \langle T^nx_i, x_j \rangle = \langle x_i, T^{*n}x_j \rangle = \langle x_i, \overline{\lambda_j}^n x_j \rangle = \lambda_j^n \langle x_i, x_j \rangle$. So $(\lambda_i^n - \lambda_j^n)\langle x_i, x_j \rangle = 0$. Since $\lambda_i^n \neq \lambda_j^n$, $\langle x_i, x_j \rangle = 0$. This shows that M_i's are pairwise orthogonal.
Let $M = M_1 + ... + M_m$. Then M is a closed subspace of H. Let P be associated projection onto M. Then $P = P_1 + ... + P_m$. Since T^n is normal, each M_i reduces T^n. It follows that $T^nP = PT^n$. Consequently $M^\perp$ is invariant under T^n. Suppose that $M^\perp \neq \{0\}$. Let $T_1 = T^n/M^\perp$. Then T_1 is an operator on non-trivial finite dimensional complex Hilbert space $M^\perp$ with empty point spectrum which is impossible. Therefore $M^\perp = \{0\}$. i.e., $M = H$. □

Theorem 2.18. *Let $T_1, ..., T_m$ be n-normal operators in $B(H)$. Then $(T_1 \oplus ... \oplus T_m)$ and $(T_1 \otimes ... \otimes T_m)$ are n-normal operators.*

Proof. Since $(T_1 \oplus ... \oplus T_m)^n (T_1 \oplus ... \oplus T_m)^* = (T_1^n \oplus ... \oplus T_m^n)(T_1^* \oplus ... \oplus T_m^*) = T_1^n T_1^* \oplus ... \oplus T_m^n T_m^* = T_1^* T_1^n \oplus ... \oplus T_m^* T_m^n = (T_1^* \oplus ... \oplus T_m^*)(T_1^n \oplus ... \oplus T_m^n) = (T_1 \oplus ... \oplus T_m)^* (T_1 \oplus ... \oplus T_m)^n$. Then $(T_1 \oplus ... \oplus T_m)$ is an n-normal operator. Now, for $x_1, ...x_m \in H$ $(T_1 \otimes .. \otimes T_m)^n (T_1 \otimes .. \otimes T_m)^* (x_1 \otimes .. \otimes x_m)$
$= (T_1^n \otimes .. \otimes T_m^n)(T_1^* \otimes .. \otimes T_m^*)(x_1 \otimes .. \otimes x_m) = T_1^n T_1^* x_1 \otimes .. \otimes T_m^n T_m^* x_m$,
$= T_1^* T_1^n x_1 \otimes .. \otimes T_m^* T_m^n x_m = (T_1^* \otimes .. \otimes T_m^*)(T_1^n \otimes .. \otimes T_m^n)(x_1 \otimes .. \otimes x_m)$,
$= (T_1 \otimes .. \otimes T_m)^* (T_1 \otimes .. \otimes T_m)^n (x_1 \otimes .. \otimes x_m)$. So $(T_1 \otimes ... \otimes T_m)^n (T_1 \otimes ... \otimes T_m)^* = (T_1 \otimes ... \otimes T_m)^* (T_1 \otimes ... \otimes T_m)^n$. Thus $(T_1 \otimes ... \otimes T_m)$ is n-normal. □

Proposition 2.19. *$(T - \lambda)$ is an n-normal operator for every $\lambda \in \mathbb{C}$ if and only if T is a normal operator.*

Proof. Since $(T - \lambda)$ is n-normal for every $\lambda \in \mathbb{C}$, $(T - \lambda)^* (T - \lambda)^n = (T - \lambda)^n (T - \lambda)^*$. Hence $(T^* - \overline{\lambda})(\sum_{k=1}^n (-1)^k \binom{n}{k} T^{n-k} \lambda^k) = (\sum_{k=1}^n (-1)^k \binom{n}{k} T^{n-k} \lambda^k)\ (T^* - \overline{\lambda})$. So $(\sum_{k=1}^n (-1)^k \binom{n}{k} T^* T^{n-k} \lambda^k) - (\sum_{k=1}^n (-1)^k \binom{n}{k} T^{n-k} \lambda^k)\overline{\lambda} = (\sum_{k=1}^n (-1)^k \binom{n}{k} T^{n-k} T^* \lambda^k) - (\sum_{k=1}^n (-1)^k \binom{n}{k} T^{n-k} \lambda^k)\overline{\lambda}$. Therefore
$\sum_{k=1}^n (-1)^k \binom{n}{k} (\lambda)^k (T^* T^{n-k} - T^{n-k} T^*) = 0$. From the left side of the last equation we get the term which $k = n$ is zero. Hence $\sum_{k=1}^{n-1} (-1)^k \binom{n}{k} (\lambda)^k (T^* T^{n-k} - T^{n-k} T^*) = 0$. Thus $(-1)^{n-1} n (\lambda)^{n-1} (T^* T - T T^*) + \sum_{k=1}^{n-2} (-1)^k \binom{n}{r} (\lambda)^k (T^* T^{n-k} - T^{n-k} T^*) = 0$. Put $\lambda = re^{i\theta}$, $0 \leq \theta \leq 2\pi$, $r > 0$, we get
$(-1)^{n-1} n (re^{i\theta})^{n-1} (T^* T - T T^*) + \sum_{k=1}^{n-2} (-1)^k \binom{n}{k} (re^{i\theta})^k (T^* T^{n-k} - T^{n-k} T^*) = 0$.
So $(-1)^{n-1} (T^* T - T T^*) + \frac{1}{n(re^{i\theta})^{n-1}} (\sum_{k=1}^{n-2} (-1)^k \binom{n}{k} (re^{i\theta})^k (T^* T^{n-k} - T^{n-k} T^*)) = 0$.
Let $r \longrightarrow \infty$. Then $T^* T - T T^* = 0$. Hence T is normal. The converse is trivial. □

Proposition 2.20. *Let $T \in B(H)$ with the Cartesian decomposition $T = A + iB$ where A and B are selfadjoint operators. Then T is 2-normal operator if and only if B^2 commutes with A, and A^2 commutes with B.*

Proof. Suppose $B^2 A = A B^2$ and $A^2 B = B A^2$. Then $T^2 T^* = (A + iB)^2 (A - iB) = (A^2 + iAB + iBA - B^2)(A - iB) = A^3 - iA^2 B - B^2 A + iB^3 + iABA + AB^2 +$

iBA^2+BAB and $T^*T^2 = A^3-AB^2+iA^2B+iABA-iBA^2+iB^3+BAB+B^2A$. Since $B^2A = AB^2$ and $A^2B = BA^2$, $T^2T^* = T^*T^2$. Hence T is 2-normal. Now let T be 2-normal. So $T^2T^* = T^*T^2$. Hence $-B^2A + iBA^2 - iA^2B + AB^2 = -AB^2 + iA^2B - iBA^2 + B^2A$, $(AB^2 - B^2A) + i(BA^2 - A^2B) = 0$. Let $T_1 = AB^2 - B^2A$, $T_2 = BA^2 - A^2B$. Then $T_1^* = -T_1$, $T_2^* = -T_2$ (i.e., T_1, T_2 are skew hermition) and $T_1 + iT_2 = 0$. So $-T_1 + iT_2 = 0$. This gives $T_1 = AB^2 - B^2A = 0$. Similarly, $B^2A = AB^2$. □

It is clear that a 2-normal operator is a $2k$-normal operator and a 3-normal operator is a $3k$-normal operator. The following examples show that a 2-normal operator need not be 3-normal operator and vice versa.

Example 2.21. *Let* $T = \begin{pmatrix} 2 & 1 \\ 0 & -2 \end{pmatrix}$. *Then* $T^2 = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$ *is a normal operator. But* $T^3 = \begin{pmatrix} 8 & 4 \\ 0 & -8 \end{pmatrix}$ *is not normal. So* T *is* 2*-normal but it is not* 3*-normal.*

Example 2.22. *Let* $T = \begin{pmatrix} 2 & 2 \\ -2 & 0 \end{pmatrix}$. *Then* $T^3 = \begin{pmatrix} -8 & 0 \\ 0 & -8 \end{pmatrix}$ *is a normal operator. But* $T^2 = \begin{pmatrix} 0 & 4 \\ -4 & -4 \end{pmatrix}$ *is not normal. So* T *is* 3*-normal but it is not* 2*-normal.*

Proposition 2.23. *Suppose* T *is both* k*-normal and* $(k+1)$*-normal for some positive integer* k*. Then* T *is* $(k+2)$*-normal. And hence* T *is* n*-normal for all* $n \geq k$*.*

Proof. Since T is k-normal, $T^kT^* = T^*T^k$. Hence $TT^kT^*T = TT^*T^kT$. So $T^{k+1}T^*T = TT^*T^{k+1}$. Since T is $(k+1)$-normal, $T^*T^{k+2} = T^{k+2}T^*$. Thus T is $(k+2)$-normal. □

Corollary 2.24. *If* T *is* 2*-normal and* 3*-normal, then* T *is an* n*-normal for all* $n \geq 2$*.*

The following example shows a 2-normal and 3-normal operator may not be normal.

Example 2.25. *Let* $T = \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix}$ *be an operator acting in two-dimensional complex Hilbert space. Then* T *is* 2*-normal,* 3*-normal, and hence it is* n*-normal for all* $n \geq 2$ *but it is not normal.*

Proposition 2.26. *Suppose* T *is a* k*-normal operator for a positive integer* k *and it is a partial isometry. Then* T *is a* $(k+1)$*-normal operator. And hence* T *is* n*-normal for all* $n \geq k$*.*

Proof. Since T is partial isometry, $TT^*T = T$ by [5, p.250]. Hence $TT^*T^k = T^k$ and $T^kT^*T = T^k$. Since T is k-normal, $T^{k+1}T^* = T^k$ and $T^*T^{k+1} = T^k$. Thus $T^{k+1}T^* = T^*T^{k+1}$. Therefore T is $(k+1)$-normal. And hence by Proposition 2.23 T is n-normal for all $n \geq k$. □

Corollary 2.27. *If T is 2-normal and partial isometry, then T is n-normal for all integer $n \geq 2$.*

We note that, in Example 2.25 if a equal to 1, then T is a 2-normal operator and a partial isometry but not normal.

Lemma 2.28. *Let T be k-normal and $(k+1)$-normal. If either T or T^* is injective, then T is normal.*

Proof. Since T is $(k+1)$-normal, $T^{k+1}T^* = T^*T^{k+1}$. And since T is k-normal, $T^{k+1}T^* = T^kT^*T$. Hence $T^k(TT^* - T^*T) = 0$. Since T is injective, $TT^* - T^*T = 0$. Thus T is normal. In case T^* is injective, since T^* is k-normal and $(k+1) - normal$, T^* is normal. Hence T is normal. □

Proposition 2.29. *Let $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where $a, b, c, d \in \mathbb{C}$. Then T is 2-normal if and only if $(a+d) = 0$ and $(|b| = |c|$ or $b(\overline{d} - \overline{a}) = \overline{c}(d-a))$.*

Proof. Suppose $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is 2-normal. Then $T^2 = \begin{pmatrix} a^2+bc & ab+bd \\ ac+dc & cb+d^2 \end{pmatrix}$ is normal. Hence $|ab+bc| = |ac+dc|$ and $(ab+bd)(\overline{(cd+d^2)} - \overline{(a^2+bc)}) = \overline{(ac+dc)}((cb+d^2)-(a^2+bc))$. Since $|b(a+d)| = |c(a+d)|$ and $b(a+d)(\overline{cb}+\overline{d^2}-\overline{a^2}-\overline{bc}) = \overline{c}\overline{(a+d)}(cb+d^2-a^2-bc)$, $|b||a+d| = |c||a+d|$ and $b(a+d)(\overline{d^2}-\overline{a^2}) = \overline{c}(\overline{a}+\overline{d})(d^2-a^2)$. Hence $|b||a+d| = |c||a+d|$ and $b(a+d)(\overline{d}-\overline{a})(\overline{d}+\overline{a}) = \overline{c}(\overline{a}-\overline{d})(d-a)(d+a)$. So $|b||a+d| = |c||a+d|$ and $b(\overline{d}-\overline{a})|a+d|^2 = \overline{c}(d-a)|a+d|^2$. Thus $|b| = |c|$ or $|a+d| = 0$ and $b(\overline{d}-\overline{a}) = \overline{c}(d-a)$ or $|a+d|^2 = 0$. □

By giving similar arguments that in the last Proposition one can prove the following.

Proposition 2.30. *Let $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where $a, b, c, d \in \mathbb{C}$. Then T is 3-normal if and only if $(a^2+bc+ad+d^2) = 0$ and $(|b| = |c|$ or $\overline{c}(d-a) = b(\overline{d}-\overline{a})$.*

Next, we characterize when a two-dimensional upper triangular complex matrix is n-normal.

Proposition 2.31. *For $n \geq 2$ we have $T = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$ is n-normal if and only if $b(a^{n-1} + a^{n-2}c + ... + c^{n-1}) = 0$.*

Proof. Let $T = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$. Then T is n-normal if and only if

$$T^n = \begin{pmatrix} a^n & b(a^{n-1} + a^{n-2}c + ... + c^{n-1}) \\ 0 & c^n \end{pmatrix},$$

is normal if and only if $| \ b(a^{n-1} + a^{n-2}c + ... + c^{n-1}) \ |= 0$ if and only if $b(a^{n-1} + a^{n-2}c + ... + c^{n-1}) = 0$. □

Example 2.32. *Consider $n = 3$ in the last Proposition. Then T is a 3-normal operator if and only if $b(a^2 + ac + c^2) = 0$. Take $a = 2$, $b = 1$, and $c = -1 + \sqrt{3}i$. Then $T = \begin{pmatrix} 2 & 1 \\ 0 & -1 + \sqrt{3}i \end{pmatrix}$ is 3-normal. Note that $T^3 = \begin{pmatrix} 8 & 0 \\ 0 & 8 \end{pmatrix}$ is normal. Thus T is 3-normal.*

We note that by use the last Proposition we may get an n-normal operator but not normal.

Proposition 2.33. *Let $T \in B(H)$, $F = T^n + T^*$, and $G = T^n - T^*$. Then T is an n-normal operator if and only if G commutes with F.*

Proof. $FG = GF$ if and only if $(T^n + T^*)(T^n - T^*) = (T^n - T^*)(T^n + T^*)$ if and only if $T^{2n} - T^nT^* + T^*T^n - T^{*2} = T^{2n} + T^nT^* - T^*T^n - T^{*2}$ if and only if $T^nT^* - T^*T^n = 0$ if and only if T is an n-normal. □

Proposition 2.34. *Let $T \in B(H)$, $B = T^nT^*$, $F = T^n + T^*$, and $G = T^n - T^*$. If T is an n-normal, then B commutes with F and G.*

Proof. Since T is an n-normal, $BF = T^nT^*(T^n + T^*) = T^nT^*T^n + T^nT^*T^* = T^nT^nT^* + T^*T^nT^* = (T^n + T^*)T^nT^* = FB$. By similar way we can prove that $BG = GB$. □

Proposition 2.35. *Let T be a weighted shift with nonzero weights $\{\alpha_k\}_{k=0}^{\infty}$. Then T is n-normal if and only if $| \ \alpha_{k-n} \ | ... | \ \alpha_{k-1} \ |=| \ \alpha_k \ | ... | \ \alpha_{k+n-1} \ |$ for $k = n, n+1, ...$*

Proof. Let $\{e_k\}_{k=0}^{\infty}$ be an orthogonal basis of Hilbert space H. Since $T^ne_k = \alpha_k...\alpha_{k+n-1}$
e_{k+n} and $T^{*n}e_k = \overline{\alpha_{k-1}}...\overline{\alpha_{k-n}}e_{k-n}$, $T^nT^{*n}e_k =| \ \alpha_{k-1} \ |^2 \ ... \ | \ \alpha_{k-n} \ |^2 \ e_k$ and $T^{*n}T^ne_k =$
$| \ \alpha_k \ |^2 ... | \ \alpha_{k+n-1} \ |^2 \ e_k$. Thus T^n is normal if and only if $| \ \alpha_k \ |^2 ... | \ \alpha_{k+n-1} \ |^2 =| \ \alpha_{k-1} \ |^2 ... | \ \alpha_{k-n} \ |^2$ for $k = n, n+1, ...$ □

Proposition 2.36. *Let $T \in B(H)$ be an n-normal operator and invertible. Then T and T^{-1} have a common nontrivial closed invariant subspace.*

Proof. . Since T is n-normal and invertible, T^n and $(T^{-1})^n$ are normal. Hence by [1, Corollary 4.5] T^n and $(T^{-1})^n$ both have no hypercyclic vector. Thus by [7], T and T^{-1} both have no hypercyclic vector. Therefore by [2], T and T^{-1} have a common nontrivial closed invariant subspace. □

Let λ be the coordinate in $\mathbb{C}$ and $d_\mu(\lambda)$, denotes planar Lebesgue measure. Let D be a bounded open subset of $\mathbb{C}$. We shall denote by $L^2(D,H)$ the Hilbert space of measurable function $f : D \longrightarrow H$ such that

$$\|f\|_{2,D} = \{\int_D \|f(\lambda)\|^2 d_\mu(\lambda)\}^{\frac{1}{2}} < \infty.$$

The space of functions $f \in L^2(D,H)$ that are analytic in D (i.e., $\overline{\partial} f = 0$) is denoted by

$$A^2(D,H) = L^2(D,H) \cap \hat{\text{O}}(U,H).$$

$A^2(D,H)$ is called the Bergman space for D.
Let D be a bounded open subset of D and m a fixed non-negative integer. The vector valued Sobolev space $W^m(D,H)$ with respect to $\overline{\partial}$ and of order m will be the space of those functions $f \in L^2(D,H)$ whose derivatives $\overline{\partial} f, ..., \overline{\partial^m} f$ in the sense of distributions also belong to $L^2(D,H)$. Endowed with the norm $\|f\|^2_{W^m} = \sum_{i=0}^m \|\overline{\partial^i} f\|^2_{2,D}$. $W^m(D,H)$ becomes a Hilbert space contained continuously in $L^2(D,H)$.

Theorem 2.37. *Let D be an arbitrary bounded disk in $\mathbb{C}$. If $T \in B(H)$ is 2-normal with the property that $\sigma(T) \cap (-\sigma(T)) = \emptyset$, then the operator*

$$\lambda - T : W^2(D,H) \longrightarrow L^2(D,H)$$

is one to one.

Proof. Let $f \in W^2(D,H)$ such that $(\lambda - T)f = 0$ i.e.,

$$\|(\lambda - T)f\|_{W^2} = 0. \tag{1}$$

Then, for $i = 1, 2$, we have

$$\|(\lambda - T)\overline{\partial^i} f\|_{2,D} = 0. \tag{2}$$

Hence for $i = 1, 2$, we get $\|(\lambda^2 - T^2)\overline{\partial^i} f\|_{2,D} = 0$. For $i = 1, 2$. Since T^2 is normal,

$$\|(\overline{\lambda}^2 - T^{*2})\overline{\partial^i} f\|_{2,D} = 0. \tag{3}$$

Since $\lambda - T$ is invertible for $\lambda \in D \backslash \sigma(T)$, the equation 2 implies that $\|\overline{\partial^i} f\|_{2,D\backslash\sigma(T)} = 0$. Therefore

$$\|(\overline{\lambda} - T^*)\overline{\partial^i} f\|_{2,D\backslash\sigma(T)} = 0. \tag{4}$$

Since $\sigma(T)\cap(-\sigma(T)) = \emptyset$ and $\sigma(T^*) = \sigma(T)^*$, $\overline{\lambda}+T^*$ is invertible for $\lambda \in \sigma(T)$. therefore, from equation 3, we have

$$\|(\overline{\lambda} - T^*)\overline{\partial^i} f\|_{2,\sigma(T)} = 0. \tag{5}$$

Hence from 4 and 5, we get

$$\|(\overline{\lambda} - T^*)\overline{\partial^i} f\|_{2,D} = 0. \tag{6}$$

By [6, Proposition 2.1], we obtain

$$\|(I - P)f\|_{2,D} = 0, \tag{7}$$

where P denotes the orthogonal projection of $L^2(D, H)$ onto the Bergman space
$A^2(D, H)$. Hence $(\lambda - T)Pf = (\lambda - T)f = 0$. Since T has SVEP, $f = Pf = 0$. Hence $\lambda - T$ is one to one. □

Lemma 2.38. *Let $T \in B(H)$ be an 2-normal operator with property for $\sigma(T) \cap (-\sigma(T)) = \emptyset$. If V is an isometry, then the operator $\lambda - VTV^* : W^2(D, H) \longrightarrow L^2(D, H)$ is one to one.* ■

Proof. Let $f \in W^2(D, H)$ such that $(\lambda - VTV^*)f = 0$. Then$(\lambda - T)V^*f = 0$. Hence for $i = 0, 1, 2$ $(\lambda - T)V^*\overline{\partial^i} f = 0$. By Theorem 2.37, for $i = 0, 1, 2$, $V^*\overline{\partial^i} f = 0$. Hence for $i = 0, 1, 2$, $VTV^*\overline{\partial^i} f = 0$. Thus $\lambda\overline{\partial^i} f = 0$ for $i = 0, 1, 2$. By [6, Proposition 2.1] with $T = (0)$, we get $\|(I - P)f\|_{2,D} = 0$, where P denotes the orthogonal projection of $L^2(D, H)$ onto the Bergman space $A^2(D, H)$. Hence $\lambda f = \lambda Pf = 0$. By [4, Corollary 10.7], there exists a constant $c > 0$ such that

$c\|Pf\|_{2,D} \leq \|\lambda Pf\|_{2,D} = 0$. So $f = Pf = 0$. Thus $\lambda - VTV^*$ is one to one.

□

Proposition 2.39. *Let $T \in B(H)$ be an n-normal operator. If T is quasinilpotent, then T is nilpotent, and hence T is subscalar.*

Proof. Since T is quasinilpotent, $\sigma(T) = \{0\}$. Hence by the spectral mapping theorem we get $\sigma(T^n) = \sigma(T)^n = \{0\}$. Thus T^n is quasinilpotent and normal. So $T^n = 0$ i.e., T is nilpotent and T is algebraic operator and hence by [3], T is subscalar. □

Proposition 2.40. *Let $T \in B(H)$ be a 2-normal Operator with the property that $\sigma(T) \cap (-\sigma(T)) = \emptyset$. Then T is subscalar of order 2.*

Proof. Consider an arbitrary bounded disk $D \subset \mathbb{C}$ which contains $\sigma(T)$ and the quotient space $H(D) = W^2(D,H)/\overline{(\lambda - T)W^2(D,H)}$ endowed with the Hilbert space norm. The class of a vector or an operator A on $H(D)$ will be denoted respectively by $\tilde{f}$, $\tilde{A}$. Let M be the operator of multiplication by λ on $W^2(D,H)$. Then M is a scalar operator of order 2 and has a spectral distribution ϕ. Let $S = \tilde{M}$. Since $(\lambda - T)W^2(D,H)$ is invariant under every operator M_f, $f \in C_0^2(C)$, we infer that S is a scalar operator of order 2 with spectral distribution $\tilde{\phi}$.
Consider the natural map $V : H \longrightarrow H(D)$ denoted by $Vh = 1 \tilde{\otimes} h$, for $h \in H$, where $1 \otimes h$ denotes the constant function sending $\lambda \in D$ to h. Then $VT = SV$. In particular $R(V)$ is an invariant subspace for S. Now we shall prove that V is one to one and has closed range.
Let $\{h_n\}$, $\{f_n\}$ be sequences respectively in H, $W^2(D,H)$ such that

$$\lim_{n \longrightarrow \infty} \|(\lambda - T)f_n + 1 \otimes h\|_{W^2} = 0. \tag{8}$$

It suffices to show that $\lim_{n \longrightarrow \infty} h_n = 0$.
By the definition of the norm of Sobolev space 8 implies that

$$\lim_{n \longrightarrow \infty} \|(\lambda - T)\overline{\partial^i} f_n\|_{2,D} = 0. \tag{9}$$

$\lim_{n \longrightarrow \infty} \|(\lambda - T)\overline{\partial^i} f_n\|_{2,D} = 0$ Since T^2 is normal, for $i = 1,2$

$$\lim_{n \longrightarrow \infty} \|(\overline{\lambda}^2 - T^{*2})\overline{\partial^i} f_n\|_{2,D}\|\overline{\partial^i} f_n\|_{2,D} = 0. \tag{10}$$

Since $\lambda - T$ invertible for $\lambda \in D\backslash\sigma(T)$, 9 implies that $\lim_{n \longrightarrow \infty} \|\overline{\partial^i} f_n\|_{2,D\backslash\sigma(T)} = 0$. Therefore

$$\lim_{n \longrightarrow \infty} \|(\overline{\lambda} - T^*)\overline{\partial^i} f_n\|_{2,D\backslash\sigma(T)} = 0. \tag{11}$$

Since for $\sigma(T) \cap (-\sigma(T)) = \emptyset$ and $\sigma(T^*) = \sigma(T)^*$, $\lambda + T^*$ is invertible for $\lambda \in \sigma(T)$. Therefor from 10 we have

$$\lim_{n \longrightarrow \infty} \|(\overline{\lambda} - T^*)\overline{\partial^i} f_n\|_{2,\sigma(T)} = 0. \tag{12}$$

Hence by 11 and 12 we get

$$\lim_{n \longrightarrow \infty} \|(\overline{\lambda} - T^*)\overline{\partial^i} f_n\|_{2,D} = 0. \tag{13}$$

By [6, Proposition 2.1], we obtain

$$\lim \|(I - P)f_i\|_{2,D} = 0, \tag{14}$$

where P denotes the orthogonal projection of $L^2(D,H)$ onto the Bergman space
$A^2(D,H)$. Substituting 14 into 8, we get $\lim_{n \longrightarrow \infty} \|(\lambda - T)Pf_n + 1 \otimes h_n\|_{2,D} = 0$.
Let Γ be a curve in D Surrounding $\sigma(T)$. Then for $\lambda \in \Gamma$

$$\lim_{n\longrightarrow\infty} \|Pf_n(\lambda) + (\lambda - T)^{-1}(1 \otimes h)\| = 0$$

uniformly. Hence by Riesz-Dunford functional

$$\lim_{n\longrightarrow\infty} \|\frac{1}{2\pi i}\int_\Gamma Pf_n(\lambda)d\lambda + h_n\| = 0.$$

But since $\frac{1}{2\pi i}\int_\Gamma Pf_n(\lambda)d\lambda = 0$, by Cauchy's theorem calculus, $\lim_{n\longrightarrow\infty} h_n = 0$. Thus V is one to one and has closed range. □

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