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An Extension of Fisher's Theorem

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Abstract

A result of Brain Fisher is extended to two pairs of self-maps through the notions of weak compatibility and property EA.

Keywords: *Compatible self-maps, weakly compatible self-maps, property EA and common fixed point.*

1 Introduction

In 1976 Brian Fisher [2] proved the following:

Theorem 1.1: *Let A be a self-map on a complete metric space X satisfying the contractive type inequality*

$$d^2(Ax, Ay) \leq b\, d(x, Ax)d(y, Ay) + c d(x, Ay)d(y, Ax) \text{ for all } x, y \in X, \ \ldots \qquad (1.1)$$

where $0 \leq b, c < 1$. *Then A has a unique fixed point.*

In this paper we extend Theorem 1.1 to two pairs of self-maps using the notion of property EA and weakly compatible maps (*cf.* Section 2 below).

2 Preliminaries

In this paper X denotes a metric space with metric d. Self-maps A and S are commuting if $ASx = SAx$ for all $x \in X$.

Definition 2.1: A and S are *compatible* [3] if

$$\lim_{n\to\infty} d(ASx_n, SAx_n) = 0 \qquad \dots \qquad \text{(2-a)}$$

whenever $\{x_n\}_{n=1}^{\infty}$ is a sequence in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z \qquad \dots \qquad \text{(2-b)}$$

for some $z \in X$.

Note that every commuting pair is compatible. That is compatibility is weaker than the commutativity. However, a compatible pair is commuting (*cf.* [3]).

By altering the asymptotic condition (2-a), later various types of compatibility like A- and S-compatibilities [9], Compatibility of type A (*cf.* [5]), type B (*cf.* [8]), type C (*cf.* [7]), type E (*cf.* [11]) and type P (See [6]) were developed in solving certain functional equations that arise dynamical programming. A nice comparative survey among these types of compatibility was done in [9] and [12].

Definition 2.2: Self maps A and S on X satisfy property EA [1] if there exists a sequence $\{x_n\}_{n=1}^{\infty}$ in X with the choice (2-b)
Obviously compatible and noncompatible pairs satisfy the property EA.

Definition 2.3: Self maps A and S are *weakly compatible* [4] if they commute at their coincidence points.

It was shown that every compatible pair is weakly compatible but the converse is not true [4], and the notions of weakly compatibility and property EA are independent [10].

3 Main Result and Remarks

Theorem 3.1: *Let A, B, S and T be self-maps on X satisfying the inclusions*

$$A(X) \subset T(X) \; and \; B(X) \subset S(X) \qquad \dots \qquad (3)$$

and the inequality

$$d^2(Ax, By) \le b\, d(Ax, Sx)\, d(By, Ty) + c d(Sx, By) d(Ty, Ax)$$

$$\textit{for all } x, y \in X, \quad \ldots \quad (4)$$

with the same choice of the constants b and c as in Theorem 1.

If one of S(*X*) *and T*(*X*) *is complete and*

(a) *Either* (*A*, *S*) *or* (*B*, *T*) *satisfies property EA*
(b) *The pairs* (*A*, *S*) *and* (*B*, *T*) *are weakly compatible.*

Then A, B, S and T have a unique common fixed point.

Proof. Suppose that A and S satisfy the property EA. By the inclusion $A(X) \subset T(X)$, we can find another sequence $\{y_n\}_{n=1}^{\infty}$ in X such that

$Ax_n = Ty_n$ for all n so that from (2-b)

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ty_n = z. \quad \ldots \quad (5)$$

Let $q = \lim_{n\to\infty} By_n$. We prove below that $q = z$.

Writing $x = x_n$ and $y = y_n$ in the inequality (4), we get

$$d^2(Ax_n, By_n) \le b\, d(Ax_n, Sx_n)\, d(By_n, Ty_n) + c d(Sx_n, By_n) d(Ty_n, Ax_n).$$

Applying the limit as $n \to \infty$ in this and using (5) it follows that

$d^2(z, q) \le b.0 + c.0$ so that $d^2(z, q) = 0$ or $d(z, q) = 0$. That is, $q = z$.

Hence $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ty_n = \lim_{n\to\infty} By_n = z$. $\quad \ldots \quad$ (6)

Similarly we can prove (6) if the pair (B, T) satisfies the property EA.

Case *A*: Suppose that $T(X)$ is complete subspace of X.

Note that $\{Ty_n\}_{n=1}^{\infty}$ is Cauchy and convergent sequence in $T(X)$. Therefore $z \in T(X)$. That is $z = Tq$ for some $q \in X$. Now we show that q is a coincidence point of B and T.

Taking $x = x_n$ and $y = q$ in the inequality (4) and using (6) we get

$$d^2(Ax_n,Bq) \leq b.d(Ax_n,Sx_n)\, d(Bq,Tq) + c.d(Sx_n,Bq)d(Tq,Ax_n)$$

or $$d^2(Tq, Bq) \leq b.0 + c.0 = 0.$$

Hence $Tq = Bq$, that is q is a coincidence point of T and B.

Again $B(X) \subset S(X)$ implies that $Bq \in S(X)$ or $Bq = Sr$ for some $r \in X$.

Then from the inequality (4) with $x = r$, $y = q$ we get

$$d^2(Ar,Bq) \leq b.d(Ar,Sr)d(Bq,Tq) + c.d(Sr,Bq)d(Tq,Ar).$$

Using $Bq = Tq = Sr$ in this, we see that $d^2(Ar,Sr) \leq 0$ or $Ar = Sr$. Hence

$$Ar = Sr = Bq = Tq. \qquad \ldots \qquad (7)$$

In other words, r is a coincidence point of A and S and q is a coincidence point of B and T.

Case *B*: Suppose that $S(X)$ is complete subspace of X.

Since $\{Sx_n\}_{n=1}^{\infty}$ is a Cauchy sequence and convergent sequence in $S(X)$ we see that $z \in S(X)$ or $z = Tp$ for some $p \in X$.

Now we write $x = x_n$ and $y = p$ in the inequality (4). Then

$$d^2(Ax_n,Bp) \leq b.d(Ax_n,Sx_n)d(Bp,Tp) + c.d(Sx_n,Bp)d(Tp,Ax_n)$$

or $d^2(Tp, Bp) \leq b.0 + c.\,0 = 0$ so that $Tp = Bp$ or that p is a coincidence point of T and B.

Again $B(X) \subset S(X)$ implies that $Bp \in S(X)$ or $Bp = Sv$ for some $v \in X$.

Then from the inequality (4) with $x = v$ and $y = p$, we get

$$d^2(Av, Bp) \leq b.d\,(Av, Sv)\, \mathrm{d}(Bp, Tp) + c.d\,(Sv, Bp)\, \mathrm{d}(Tp, Av).$$

Using $Tp = Bp = Sv$, this gives

$$d^2(Av,Sv) \leq b.d(Av,Sv)d(Tp,Tp) + c.d\,(Bp,Bp)\, d(Tp,Av) = 0 \text{ or } Av = Sv.$$

Thus v is a coincidence point of A and S and p is a coincidence point of B and T.

Since the pairs (A, S) and (B, T) are weakly compatible, we find that

$ASr = SAr$ and $BTq = TBq$. This implies $Az = Sz$ and $Bz = Tz$.

Now from the inequality (4) with $x = y = z$, it follows that

$$d^2(Az, Bz) \leq b.d(Az, Sz)d(Bz, Tz) + c.d(Sz, Bz)d(Tz, Az)$$

$$\leq b.d\,(Sz, Sz)d(Tz, Tz) + c.d\,(Az, Bz)\,\mathrm{d}(Bz, Az)$$

$\Rightarrow \quad (1 - c)\, d^2(Az, Bz) \leq 0 \quad \Rightarrow \quad d^2(Az, Bz) = 0$ or $Az = Bz$.

Thus $\quad Az = Sz = Bz = Tz \quad \ldots \quad (8)$

Now we prove that $Az = z$.

From the inequality (4) with $x = z$ and $y = q$, we have

$$d^2(Az, Bq) \leq b.d(Az, Sz)d(Bz, Tq) + c.\ d(Sz, Bq)d(Tq, Az) \leq b\cdot 0 + c.d^2(Az, z)$$

$\Rightarrow \quad (1 - c)\, d^2(Az, z) \leq 0$ or $Az = z$.

Hence $Az = Sz = Bz = Tz = z$. Thus z is a common fixed point of A, S, B and T.

Uniqueness: Let z, z' be two common fixed points of A, S, B and T.

From the inequality (4) with $x = z$ and $y = z'$, we get

$$d^2(Az, Sz') \leq b.d(Az, Sz)d(Bz', Tz') + c.d(Sz, Bz')d(Tz', Az) \leq 0 + c.d(z, z')d(z',z)$$

or $\quad d^2(z, z') \leq c.d^2(z, z')$ so that $z = z'$.

Hence the fixed point is unique.

Remark 3.1: Writing $B = A$ and $S = T = I$, the identity map on X in Theorem 3.1, we get (1) from (4) as a special case. It is also known that the identity map commutes and hence is weakly compatible with every map. Further from the proof of Theorem 1.1, the sequence $\{A^n x\}_{n=1}^{\infty}$ is Cauchy for each $x \in X$. Therefore if X is complete, this converges to some $z \in X$ and its convergence is equivalent to the property EA of the pair (A, I), that is the condition (a) of Theorem 3.1.

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