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New Class of Univalent Functions with Negative Coefficients Defined by Ruscheweyh Derivative

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Abstract

In this paper, we have discussed a subclass $S(\gamma,\alpha,\mu,\lambda)$ *of univalent functions with negative coefficients defined by Ruscheweyh derivative in the unit disk* $U = \{z \in C : |z| < 1\}$. *We obtain basic properties like coefficient inequality, distortion and covering theorem, radii of starlikeness, convexity and close-to-convexity, extreme points, Hadamard product, closure theorems and convolution operator for functions belonging to our class.*

Keywords: *Univalent function, Ruscheweyh derivative, Distortion theorem, Radius of starlikeness, Extreme points, Hadamard product.*

1 Introduction

Let K denote the class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are analytic and univalent in the open unit disk $U = \{z \in C : |z| < 1\}$.

If a function f is given by (1) and g is defined by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n, \tag{2}$$

is in the class K, then the convolution (or Hadamard product) of f and g is defined by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \quad z \in U. \tag{3}$$

Let S denote the subclass of K consisting of functions of the form:

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \qquad a_n \geq 0. \tag{4}$$

We aim to study the subclass $S(\gamma, \alpha, \mu, \lambda)$ consisting of function $f \in S$ and satisfying:

$$\left| \frac{\gamma((D^{\lambda} f(z))' - \frac{D^{\lambda}(f(z)}{z})}{\alpha(D^{\lambda}(f(z))' + (1-\gamma)\frac{D^{\lambda} f(z)}{z}} \right| < \mu, \quad z \in U, \tag{5}$$

for $0 \leq \gamma < 1, 0 \leq \alpha < 1,\ 0 < \mu < 1$ and $D^{\lambda} f(z)$ is defined as follow:

$$D^{\lambda} f(z) = z - \sum_{n=2}^{\infty} a_n B_n(\lambda) z^n,$$

where

$$B_n(\lambda) = \frac{(\lambda+1)(\lambda+2)\cdots(\lambda+n-1)}{(n-1)!},\ \lambda > -1, z \in U. \tag{6}$$

This function is called the Ruscheweyh derivative [5], [6] of f of order λ denoted by $D^{\lambda} f$.

Another classes studied by Atshan and Kulkarni [2] and Darus [3] consisting of functions of the form (4).

2 Coefficient Inequality

In the following theorem, we obtain a necessary and sufficient condition for function to be in the class $S(\gamma,\alpha,\mu,\lambda)$.

Theorem 1: *Let the function f be defined by (4). Then* $f\in S(\gamma,\alpha,\mu,\lambda)$ *if and only if*

$$\sum_{n=2}^{\infty}[\gamma(n-1)+\mu(n\alpha+1-\gamma)]B_n(\lambda)a_n \le \mu(\alpha+(1-\gamma)), \tag{7}$$

where $0<\mu<1$ $0\le\gamma<1, 0\le\alpha<1,$ and $\lambda>-1$. The result (7) is sharp for the function

$$f(z)=z-\frac{\mu(\alpha+(1-\gamma))}{[\gamma(n-1)+\mu(n\alpha+1-\gamma)]B_n(\lambda)}z^n, \qquad n\ge 2.$$

Proof: Suppose that the inequality (7) holds true and $|z|=1$ .Then we obtain

$$\left|\gamma((D^{\lambda}f(z))'-\frac{D^{\lambda}f(z)}{z})\right|-\mu\left|\alpha(D^{\lambda}f(z))'+(1-\gamma)\frac{D^{\lambda}f(z)}{z}\right|$$

$$=\left|-\gamma\sum_{n=2}^{\infty}(n-1)B_n(\lambda)a_nz^{n-1}\right|-\mu\left|\alpha+(1-\gamma)-\sum_{n=2}^{\infty}(n\alpha+1-\gamma)B_n(\lambda)a_nz^{n-1}\right|$$

$$\le\sum_{n=2}^{\infty}[\gamma(n-1)+\mu(n\alpha+1-\gamma)]B_n(\lambda)a_n-\mu(\alpha+(1-\gamma))\le 0.$$

Hence, by maximum modulus principle, $f\in S(\gamma,\alpha,\mu,\lambda)$. Now assume that $f\in S(\gamma,\alpha,\mu,\lambda)$ so that

$$\left|\frac{\gamma((D^{\lambda}f(z))'-\frac{D^{\lambda}f(z)}{z})}{\alpha((D^{\lambda}f(z))'+(1-\gamma)\frac{D^{\lambda}f(z)}{z}}\right|<\mu, \qquad z\in U$$

Hence

$$\left|\gamma((D^{\lambda}f(z))'-\frac{D^{\lambda}f(z)}{z})\right|<\mu\left|\alpha(D^{\lambda}f(z))'+(1-\gamma)\frac{D^{\lambda}f(z)}{z}\right|.$$

Therefore, we get

$$\left|-\sum_{n=2}^{\infty}\gamma(n-1)B_n(\lambda)a_n z^{n-1}\right| < \mu\left|\alpha+(1-\gamma)-\sum_{n=2}^{\infty}(n\alpha+1-\gamma)B_n(\lambda)a_n z^{n-1}\right|,$$

Thus

$$\sum_{n=2}^{\infty}[\gamma(n-1)+\mu(n\alpha+1-\gamma)]B_n(\lambda)a_n \le \mu(\alpha+(1-\gamma))$$

and this completes the proof.

Corollary 1: *Let the function* $f \in S(\gamma,\alpha,\mu,\lambda)$ *.Then*

$$a_n \le \frac{\mu(\alpha+(1-\gamma))}{[\gamma(n-1)+\mu(n\alpha+1-\gamma)]B_n(\lambda)}, \quad n \ge 2.$$

3 Distortion and Covering Theorem

We introduce the growth and distortion theorems for the functions in the class $S(\gamma,\alpha,\mu,\lambda)$

Theorem 2: *Let the function* $f \in S(\gamma,\alpha,\mu,\lambda)$. *Then*

$$|z|-\frac{\mu(\alpha+(1-\gamma))}{[\gamma+\mu(2\alpha+1-\gamma)](1+\lambda)}|z|^2 \le |f(z)| \le |z|+\frac{\mu(\alpha+(1-\gamma))}{[\gamma+\mu(2\alpha+1-\gamma)](1+\lambda)}|z|^2.$$

The result is sharp and attained

$$f(z)=z-\frac{\mu(\alpha+(1-\gamma))}{[\gamma+\mu(2\alpha+1-\gamma)](1+\lambda)}z^2.$$

Proof:

$$|f(z)|=\left|z-\sum_{n=2}^{\infty}a_n z^n\right| \le |z|+\sum_{n=2}^{\infty}a_n|z|^n$$

$$\le |z|+|z|^2\sum_{n=2}^{\infty}a_n.$$

By Theorem 1, we get

$$\sum_{n=2}^{\infty}a_n \le \frac{\mu(\alpha+(1-\gamma))}{[\gamma+\mu(2\alpha+1-\gamma)]B_n(\lambda)}. \tag{8}$$

Thus

$$|f(z)| \le |z| + \frac{\mu(\alpha+(1-\gamma))}{[\gamma+\mu(2\alpha+1-\gamma)](1+\lambda)}|z|^2.$$

Also

$$|f(z)| \ge |z| - \sum_{n=2}^{\infty} a_n |z|^n$$

$$\ge |z| - |z|^2 \sum_{n=2}^{\infty} a_n$$

$$\ge |z| - \frac{\mu(\alpha+(1-\gamma))}{[\gamma+\mu(2\alpha+1-\gamma)](1+\lambda)}|z|^2.$$

Theorem 3: *Let* $f \in S(\gamma,\alpha,\mu,\lambda)$. *Then*

$$1 - \frac{\mu(\alpha+(1-\gamma))}{[\gamma+\mu(2\alpha+1-\gamma)](1+\lambda)}|z| \le |f'(z)| \le 1 + \frac{\mu(\alpha+(1-\gamma))}{[\gamma+\mu(2\alpha+1-\gamma)](1+\lambda)}|z|$$

With equality for

$$f(z) = z - \frac{\mu(\alpha+(1-\gamma))}{[\gamma+\mu(2\alpha+1-\gamma)](1+\lambda)} z^2.$$

Proof: Notice that

$$(\lambda+1)[\gamma+\mu(2\alpha+1-\gamma)]\sum_{n=2}^{\infty} na_n \le \sum_{n=2}^{\infty} n[\gamma(n-1)+\mu(n\alpha+1-\gamma)]B_n(\lambda)a_n$$

$$\le \mu(\alpha+(1-\gamma)), \tag{9}$$

from Theorem 1. Thus

$$|f'(z)| = \left|1 - \sum_{n=2}^{\infty} na_n z^{n-1}\right| \le 1 + \sum_{n=2}^{\infty} na_n |z|^{n-1}$$

$$\le 1 + |z|\sum_{n=2}^{\infty} na_n$$

$$\le 1 + |z| \frac{\mu(\alpha+(1-\gamma))}{[\gamma+\mu(2\alpha+1-\gamma)](1+\lambda)}. \tag{10}$$

On the other hand

$$|f'(z)| = \left|1 - \sum_{n=2}^{\infty} na_n z^{n-1}\right| \geq 1 - \sum_{n=2}^{\infty} na_n |z|^{n-1}$$

$$\geq 1 - |z| \sum_{n=2}^{\infty} na_n$$

$$\geq 1 - |z| \frac{\mu(\alpha + (1-\gamma))}{[\gamma + \mu(2\alpha + 1 - \gamma)](1+\gamma)}. \tag{11}$$

Combining (10) and (11), we get the result.

4 Radii of Starlikeness, Convexity and Close-to-Convexity

In the following theorems, we obtain the radii of starlikeness, convexity and close- to-convexity for the class $S(\gamma, \alpha, \mu, \lambda)$.

Theorem 4: *Let* $f \in S(\gamma, \alpha, \mu, \lambda)$ *.Then* f *is p-valently starlike in* $|z| < R_1$ *of order* $\delta, 0 \leq \delta < 1,$ *where*

$$R_1 = \inf_n \left\{ \frac{(1-\delta)(\gamma(n-1) + \mu(n\alpha + 1 - \gamma))B_n(\lambda)}{(n-\delta)\mu(\alpha + (1-\gamma))} \right\}^{\frac{1}{n-1}}, \quad n \geq 2. \tag{12}$$

Proof: f is p-valently starlike of order $\delta, 0 \leq \delta < 1$ if

$$\operatorname{Re}\left\{ \frac{zf'(z)}{f(z)} \right\} > \delta.$$

Thus it is enough to show that

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| = \left| \frac{-\sum_{n=2}^{\infty} (n-1) a_n z^{n-1}}{1 - \sum_{n=2}^{\infty} a_n z^{n-1}} \right| \leq \frac{\sum_{n=2}^{\infty} (n-1) a_n |z|^{n-1}}{1 - \sum_{n=2}^{\infty} a_n |z|^{n-1}}.$$

Thus

$$\left|\frac{zf'(z)}{f(z)}-1\right| \leq 1-\delta \qquad \text{if} \qquad \sum_{n=2}^{\infty}\frac{(n-\delta)}{(1-\delta)}a_n|z|^{n-1} \leq 1. \tag{13}$$

Hence by Theorem 1, (13) will be true if

$$\frac{n-\delta}{1-\delta}|z|^{n-1} \leq \frac{(\gamma(n-1)+\mu(n\alpha+1-\gamma))B_n(\lambda)}{\mu(\alpha+(1-\gamma))}$$

or if

$$|z| \leq \left[\frac{(1-\delta)(\gamma(n-1)+\mu(n\alpha+1-\gamma))B_n(\lambda)}{(n-\delta)\mu(\alpha+(1-\gamma))}\right]^{\frac{1}{n-1}}, \qquad n \geq 2. \tag{14}$$

The theorem follows easily from (14).

Theorem 5: *Let* $f \in S(\gamma,\alpha,\mu,\lambda)$. *Then* f *is p-valently convex in* $|z| < R_2$ *of order* $\delta, 0 \leq \delta < 1$, *where*

$$R_2 = \inf_n \left\{\frac{(1-\delta)(\gamma(n-1)+\mu(n\alpha+1-\gamma))B_n(\lambda)}{n(n-\delta)\mu(\alpha+(1-\gamma))}\right\}^{\frac{1}{n-1}}, \; n \geq 2. \tag{15}$$

Proof: f is p- valently convex of order $\delta, 0 \leq \delta < 1$ if

$$\operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\} > \delta.$$

Thus it is enough to show that

$$\left|\frac{zf''(z)}{f'(z)}\right| = \left|\frac{-\sum_{n=2}^{\infty} n(n-1)a_n z^{n-1}}{1-\sum_{n=2}^{\infty} na_n z^{n-1}}\right| \leq \frac{\sum_{n=2}^{\infty} n(n-1)a_n|z|^{n-1}}{1-\sum_{n=2}^{\infty} na_n|z|^{n-1}}.$$

Thus

$$\left|\frac{zf''(z)}{f'(z)}\right| \leq 1-\delta \qquad \text{if} \qquad \sum_{n=2}^{\infty}\frac{n(n-\delta)a_n|z|^{n-1}}{1-\delta} \leq 1. \tag{16}$$

Hence by Theorem 1, (16) will be true if

$$\frac{n(n-\delta)|z|^{n-1}}{1-\delta} \le \frac{(\gamma(n-1)+\mu(n\alpha+1-\gamma))B_n(\lambda)}{\mu(\alpha+(1-\gamma))}$$

or if

$$|z| \le \left[\frac{(1-\delta)(\gamma(n-1)+\mu(n\alpha+1-\gamma))B_n(\lambda)}{n(n-\delta)\mu(\alpha+(1-\gamma))}\right]^{\frac{1}{n-1}}, \quad n \ge 2. \tag{17}$$

The theorem follows easily from (17).

Theorem 6: *Let* $f \in S(\gamma,\alpha,\mu,\lambda)$*. Then* f *is p-valently close-to-convex in* $|z| < R_3$ of order $\delta, 0 \le \delta < 1$, where

$$R_3 = \inf_n \left\{\frac{(1-\delta)(\gamma(n-1)+\mu(n\alpha+1-\gamma))B_n(\lambda)}{n\mu(\alpha+(1-\gamma))}\right\}^{\frac{1}{n-1}}, \; n \ge 2. \tag{18}$$

Proof: f is p-valently close-to-convex of order $\delta, 0 \le \delta < 1$ if

$$\mathrm{Re}\{f'(z)\} > \delta.$$

Thus it is enough to show that

$$|f'(z)-1| = \left|-\sum_{n=2}^{\infty} na_n z^{n-1}\right| \le \sum_{n=2}^{\infty} na_n |z|^{n-1}.$$

Thus

$$|f'(z)-1| \le 1-\delta \qquad \text{if} \quad \sum_{n=2}^{\infty} \frac{na_n|z|^{n-1}}{1-\delta} \le 1. \tag{19}$$

Hence by Theorem 1, (19) will be true if

$$\frac{n|z|^{n-1}}{1-\delta} \le \frac{(\gamma(n-1)+\mu(n\alpha+1-\gamma))B_n(\lambda)}{\mu(\alpha+(1-\gamma))}$$

or if

$$|z| \leq \left[\frac{(1-\delta)(\gamma(n-1)+\mu(n\alpha+1-\gamma))B_n(\lambda)}{n\mu(\alpha+(1-\gamma))} \right]^{\frac{1}{n-1}}, \quad n \geq 2. \tag{20}$$

The theorem follows easily from (20).

5 Extreme Points

In the following theorem, we obtain extreme points for the class $S(\gamma,\alpha,\mu,\lambda)$.

Theorem 7: *Let* $f_1(z) = z$ *and*

$$f_n(z) = z - \frac{\mu(\alpha+(1-\gamma))}{[\gamma(n-1)+\mu(n\alpha+1-\gamma)]B_n(\lambda)} z^n, \quad \text{for } n = 2,3,\cdots.$$

Then $f \in S(\gamma,\alpha,\mu,\lambda)$ if and only if it can be expressed in the form

$$f(z) = \sum_{n=1}^{\infty} \theta_n f_n(z), \text{ where } \quad \theta_n \geq 0 \text{ and } \sum_{n=1}^{\infty} \theta_n = 1.$$

Proof: Assume that $f(z) = \sum_{n=1}^{\infty} \theta_n f_n(z),$ hence we get

$$f(z) = z - \sum_{n=2}^{\infty} \frac{\mu(\alpha+(1-\gamma))\theta_n}{[\lambda(n-1)+\mu(n\alpha+1-\gamma)]B_n(\lambda)} z^n.$$

Now, $f \in S(\gamma,\alpha,\mu,\lambda),$ since

$$\sum_{n=2}^{\infty} \frac{[\gamma(n-1)+\mu(n\alpha+1-\gamma)]B_n(\lambda)}{\mu(\alpha+(1-\gamma))} . \frac{\mu(\alpha+(1-\gamma))\theta_n}{[\gamma(n-1)+\mu(n\alpha+1-\gamma)]B_n(\lambda)} = \sum_{n=2}^{\infty} \theta_n = 1-\theta_1 \leq 1.$$

Conversely, suppose $f \in S(\gamma,\alpha,\mu,\lambda).$ Then we show that f can be written in the form $\sum_{n=1}^{\infty} \theta_n f_n(z).$

Now $f \in S(\gamma,\alpha,\mu,\lambda)$ implies from Theorem 1

$$a_n \leq \frac{\mu(\alpha+(1-\gamma))}{[\gamma(n-1)+\mu(n\alpha+1-\gamma)]B_n(\lambda)}.$$

Setting $\theta_n = \dfrac{[\gamma(n-1)+\mu(n\alpha+1-\gamma)]B_n(\lambda)}{\mu(\alpha+(1-\gamma))} a_n, n = 2,3,\ldots$ and $\theta_1 = 1-\sum_{n=2}^{\infty}\theta_n,$

we obtain $f(z) = \sum_{n=1}^{\infty}\theta_n f_n(z).$

6 Hadamard Product

In the following theorem, we obtain the convolution result for functions belongs to the class $S(\gamma,\alpha,\mu,\lambda)$.

Theorem 8: *Let* $f, g \in S(\gamma,\alpha,\mu,\lambda)$. *Then* $f * g \in S(\gamma,\alpha,\mu,\lambda)$ *for*

$$f(z) = z-\sum_{n=2}^{\infty}a_n z^n, g(z) = z-\sum_{n=2}^{\infty}b_n z^n \text{ and } (f*g)(z) = z-\sum_{n=2}^{\infty}a_n b_n z^n,$$

where

$$\lambda \geq \frac{\mu^2(\alpha+(1-\gamma))\gamma(n-1)}{[\gamma(n-1)+\mu(n\alpha+1-\gamma)]^2 B_n(\lambda)-\mu^2(\alpha+(1-\gamma))(n\alpha+1-\gamma)}.$$

Proof: $f \in S(\gamma,\alpha,\mu,\lambda)$ and so

$$\sum_{n=2}^{\infty}\frac{[\gamma(n-1)+\mu(n\alpha+1-\gamma)]B_n(\lambda)}{\mu(\alpha+(1-\gamma))}a_n \leq 1, \quad (21)$$

and

$$\sum_{n=2}^{\infty}\frac{[\gamma(n-1)+\mu(n\alpha+1-\gamma)]B_n(\lambda)}{\mu(\alpha+(1-\gamma))}b_n \leq 1. \quad (22)$$

We have to find the smallest number λ such that

$$\sum_{n=2}^{\infty}\frac{[\gamma(n-1)+\ell(n\alpha+1-\gamma)]B_n(\lambda)}{\ell(\alpha+(1-\gamma))}a_n b_n \leq 1. \quad (23)$$

By Cauchy-Schwarz inequality

$$\sum_{n=2}^{\infty}\frac{[\gamma(n-1)+\mu(n\alpha+1-\gamma)]B_n(\lambda)}{\mu(\alpha+(1-\gamma))}\sqrt{a_nb_n}\le 1. \qquad (24)$$

Therefore it is enough to show that

$$\frac{[\gamma(n-1)+\lambda(n\alpha+1-\gamma)]B_n(\lambda)}{\ell(\alpha+(1-\gamma))}a_nb_n \le \frac{[\gamma(n-1)+\mu(n\alpha+1-\gamma)]B_n(\lambda)}{\mu(\alpha+(1-\gamma))}\sqrt{a_nb_n}.$$

That is

$$\sqrt{a_nb_n}\le\frac{[\gamma(n-1)+\mu(n\alpha+1-\gamma)]\lambda}{[\gamma(n-1)+\ell(n\alpha+1-\gamma)]\mu}. \qquad (25)$$

From (24)

$$\sqrt{a_nb_n}\le\frac{\mu(\alpha+(1-\gamma))}{[\gamma(n-1)+\mu(n\alpha+1-\gamma)]B_n(\lambda)}.$$

Thus it is enough to show that

$$\frac{\mu(\alpha+(1-\gamma))}{[\gamma(n-1)+\mu(n\alpha+1-\gamma)]B_n(\lambda)}\le\frac{[\gamma(n-1)+\mu(n\alpha+1-\gamma)]\lambda}{[\gamma(n-1)+\ell(n\alpha+1-\gamma)]\mu},$$

which simplifies to

$$\lambda\le\frac{\mu^2(\alpha+(1-\gamma))\gamma(n-1)}{[\gamma(n-1)+\mu(n\alpha+1-\gamma)]^2B_n(\lambda)-\mu^2(\alpha+(1-\gamma))(n\alpha+1-\gamma)}.$$

7 Closure Theorems

We shall prove the following closure theorems for the class $S(\gamma,\alpha,\mu,\lambda)$.

Theorem 9: *Let* $f_j\in S(\gamma,\alpha,\mu,\lambda),\ j=1,2,\cdots,s.$ *Then*

$$g(z)=\sum_{j=1}^{s}c_j f_j(z)\in S(\gamma,\alpha,\mu,\lambda)$$

For $f_j(z)=z-\sum_{n=2}^{\infty}a_{n,j}z^n$, where $\sum_{j=1}^{s}c_j=1.$

Proof:

$$g(z)=\sum_{j=1}^{s}c_j f_j(z)$$

$$=z-\sum_{n=2}^{\infty}\sum_{j=1}^{s}c_j a_{n,j}z^n=z-\sum_{n=2}^{\infty}e_n z^n,$$

where $e_n=\sum_{j=1}^{s}c_j a_{n,j}$. Thus $g(z)\in S(\gamma,\alpha,\mu,\lambda)$ if

$$\sum_{n=2}^{\infty}\frac{[\gamma(n-1)+\mu(n\alpha+1-\gamma)]B_n(\lambda)}{\mu(\alpha+(1-\gamma))}e_n\le 1,$$

that is, if

$$\sum_{n=2}^{\infty}\sum_{j=1}^{s}\frac{[\gamma(n-1)+\mu(n\alpha+1-\gamma)B_n(\lambda)}{\mu(\alpha+(1-\gamma))}c_j a_{n,j}$$

$$=\sum_{j=1}^{s}c_j\sum_{n=2}^{\infty}\frac{[\gamma(n-1)+\mu(n\alpha+1-\gamma)]B_n(\lambda)}{\mu(\alpha+(1-\gamma))}a_{n,j}\le\sum_{j=1}^{s}c_j=1.$$

Theorem 10: *Let* $f,g\in S(\gamma,\alpha,\mu,\lambda)$. *Then*

$$h(z)=z-\sum_{n=2}^{\infty}(a_n^2+b_n^2)z^n$$

Belongs to $S(\gamma,\alpha,\ell,\lambda)$, where

$$\lambda\ge\frac{2\gamma(n-1)\mu^2(\alpha+(1-\gamma))}{[\gamma(n-1)+\mu(n\alpha+1-\gamma)]^2B_n(\lambda)-2\mu^2(\alpha+(1-\gamma))(n\alpha+1-\gamma)}.$$

Proof: *Since* $f,g\in S(\gamma,\alpha,\mu,\lambda)$, *so Theorem 1 yields*

$$\sum_{n=2}^{\infty}\left[\frac{(\gamma(n-1)+\mu(n\alpha+1-\gamma))B_n(\lambda)}{\mu(\alpha+(1-\gamma))}a_n\right]^2 \leq 1$$

and

$$\sum_{n=2}^{\infty}\left[\frac{(\gamma(n-1)+\mu(n\alpha+1-\gamma))B_n(\lambda)}{\mu(\alpha+(1-\gamma))}b_n\right]^2 \leq 1.$$

We obtain from the last two inequalities

$$\sum_{n=2}^{\infty}\frac{1}{2}\left[\frac{[\gamma(n-1)+\mu(n\alpha+1-\gamma)]B_n(\lambda)}{\mu(\alpha+(1-\gamma))}\right]^2(a_n^2+b_n^2)\leq 1. \tag{26}$$

But $h(z)\in S(\gamma,\alpha,\ell,\lambda)$, if and only if

$$\sum_{n=2}^{\infty}\frac{[\gamma(n-1)+\ell(n\alpha+1-\gamma)]B_n(\lambda)}{\ell(\alpha+(1-\gamma))}(a_n^2+b_n^2)\leq 1, \tag{27}$$

where $0<\ell<1$, however (26) Implies (27) if

$$\frac{[\gamma(n-1)+\ell(n\alpha+1-\gamma)]B_n(\lambda)}{\ell(\alpha+(1-\gamma))}\leq\frac{1}{2}\left[\frac{[\gamma(n-1)+\mu(n\alpha+1-\gamma)]B_n(\lambda)}{\mu(\alpha+(1-\gamma))}\right]^2.$$

Simplifying, we get

$$\ell\geq\frac{2\gamma(n-1)\mu^2(\alpha+(1-\gamma))}{[\gamma(n-1)+\mu(n\alpha+1-\gamma)]^2B_n(\lambda)-2\mu^2(\alpha+(1-\gamma))(n\alpha+1-\gamma)}.$$

8 Convolution Operator

Definition 1 [4]: *The Gaussian hypergeometric function denoted by*

$${}_2F_1(a,b;c;z)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}.\frac{z^n}{n!},\ |z|<1,$$

where $c>b>0, c>a+b$ and

$$(x)_n = \begin{cases} x(x+1)(x+2)\cdots(x+n-1)\ for & n=1,2,3,\cdots \\ 1 & n=0. \end{cases}$$

Definition 2 [1]: *For every* $f \in S$, *we define the convolution operator* $W_{a,b,c}(f)(z)$ *as below:*

$$W_{a,b,c}(f)(z) = {}_2F_1(a,b;c;z) * f(z)$$

$$= z - \sum_{n=2}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} a_n z^n,$$

where ${}_2F_1(a,b;c;z)$ is Gaussian hypergeometric function (see [1] and [4]) introduced in Definition 1.

Theorem 11: *Let* f *is given by (4) be in the class* $S(\gamma,\alpha,\mu,\lambda)$. *Then the convolution operator* $W_{a,b,c}(f)$ *is in the class* $S(\gamma,\alpha,\mu,\lambda)$ *for* $|z| \le r(\mu,\ell)$,

where

$$r(\mu,\ell) = \inf_n \left\{ \frac{\ell[\gamma(n-1)+\mu(n\alpha+1-\gamma)]}{\mu[\gamma(n-1)+\ell(n\alpha+1-\gamma)]\frac{(a)_n (b)_n}{(c)_n n!}} \right\}^{\frac{1}{n-1}}.$$

The result is sharp for the function

$$f_n(z) = z - \frac{\mu(\alpha+(1-\gamma))}{[\gamma(n-1)+\mu(n\alpha+1-\gamma)]B_n(\lambda)} z^n,\ n = 2,3,\cdots.$$

Proof: Since $f \in S(\gamma,\alpha,\mu,\lambda)$, we have

$$\sum_{n=2}^{\infty} \frac{[\gamma(n-1)+\mu(n\alpha+1-\gamma)]B_n(\lambda)}{\mu(\alpha+(1-\gamma))} a_n \le 1.$$

It is sufficient to show that

$$\sum_{n=2}^{\infty} \frac{[\gamma(n-1)+\ell(n\alpha+1-\gamma)]B_n(\lambda)\frac{(a)_n (b)_n}{(c)_n n!} a_n}{\ell(\alpha+(1-\gamma))} \le 1. \tag{28}$$

Note that (28) is satisfied if

$$\frac{[\gamma(n-1)+\ell(n\alpha+1-\gamma)]B_n(\lambda)\dfrac{(a)_n(b)_n}{(c)_n n!}}{\ell(\alpha+(1-\gamma))} a_n |z|^{n-1} \le \frac{[\gamma(n-1)+\mu(n\alpha+1-\gamma)]B_n(\lambda)}{\mu(\alpha+(1-\gamma))} a_n,$$

solving for $|z|$, we get the result.

References

[1] G.E. Andrews, R. Askey and R. Roy, *Special Functions*, Cambridge University Press, Cambridge, U.K., (1999).

[2] W.G. Atshan and S.R. Kulkarni, On a new class of analytic functions with negative coefficients, *Analele Universitătii Oradea, Fasc. Mathematica, Tom*, XVI (2009), 43-51.

[3] M. Darus, Some subclasses of analytic functions, *J. Math. Comp. Sci. (Math. Ser.)*, 16(3) (2003), 121-126.

[4] Y.C. Kim and F. Rønning, Integral transforms of certain subclasses of analytic function, *J. Math. Appl.*, 258(2001), 466-489.

[5] S. Ruscheweyh, New criteria for univalent functions, *Proc. Amer. Math. Soc.*, 49(1975), 109-115.

[6] H.M. Srivastava and S. Owa (Eds), *Current Topics in Analytic Function Theory*, World Scientific, Singapore, (1992).