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On Some Integral Inequalities Analogs to Hilbert's Inequality

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Abstract

In this paper we give some further extensions of well-known Hilbert's inequality. We give equivalent form in two dimensions as application.

Keywords: *Hilbert's inequality, Hardy-Hilbert's Inequality, equivalent form.*

1 Introduction

The well-known Hilbert's inequality and its equivalent form are presented first:

Theorem A: [4] *If f and g*$\in L^2[0,\infty)$, *then the following inequalities hold and are equivalent*

$$\int_0^\infty \int_0^\infty \frac{f(x)\,g(y)}{x+y}\,dx\,dy \le \pi\left(\int_0^\infty f^2(x)\,dx\right)^{1/2}\left(\int_0^\infty g^2(y)\,dy\right)^{1/2}, \tag{1}$$

and

$$\int_0^\infty \left(\int_0^\infty \frac{f(x)}{x+y}\,dx\right)^2 dy \le \pi^2 \int_0^\infty f^2(x)\,dx \quad , \tag{2}$$

where π and π^2 are the best possible constants.

The classical Hilbert's integral inequality (1) had been generalized by Hardy-Riesz (see [2]) in 1925 as the following result.

If *f*, *g* are nonnegative functions such that $0 < \int_0^\infty f^p(x)dx < \infty$ and

$0 < \int_0^\infty g^q(x)dx < \infty$, where $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\int_0^\infty \int_0^\infty \frac{f(x)\, g(y)}{x+y}\, dx\, dy < \pi \csc\left(\frac{\pi}{p}\right)\left(\int_0^\infty f^p(x)\, dx\right)^{1/p}\left(\int_0^\infty g^q(y)\, dy\right)^{1/q}, \tag{3}$$

where the constant factor $\pi \csc(\pi/p)$ is the best possible. When $p=q=2$, inequality (3) is reduced to (1).

In recent years, a number of mathematicians had given lots of generalizations of these inequalities. We mention here some of these contributions in this direction:

Li et al. [5] have proved the following Hardy- Hilbert's type inequality using the hypotheses of (1):

$$\int_0^\infty \int_0^\infty \frac{f(x)\, g(y)}{x+y+\max\{x,y\}}\, dx\, dy < c\left(\int_0^\infty f^2(x)\, dx\right)^{1/2}\left(\int_0^\infty g^2(y)\, dy\right)^{1/2}, \tag{4}$$

Where the constant factor $c=\sqrt{2}(\pi - 2 - tan^{-1}\sqrt{2}\ ) = 1.7408\ldots$is the best possible.

Y. Li, Y. Qian, and B. He [6] deduced the following result:

Theorem B: If $f, g \geq 0, 0 < \int_0^\infty f^2(x)dx < \infty$ and $0 < \int_0^\infty g^2(x)dx < \infty$, then one has

$$\int_0^\infty \int_0^\infty \frac{|lnx-lny|}{x+y+|x-y|}\, f(x)\, g(y)\, dx\, dy < 4\left(\int_0^\infty f^2(x)\, dx\right)^{1/2}\left(\int_0^\infty g^2(x)\, dx\right)^{1/2}, \tag{5}$$

where the constant factors 4 is the best possible.

More and more results regarding this direction on Hilbert's type inequalities can be found for example in [3, 7, 8].

2 Main Results

In this paper, we give some analogs of Hilbert's type inequality. We will use the following lemma in establishing the main result.

Lemma 2.1: [1] *Let* γ, α, β *be three non-negative real numbers. Then we have the following equations*

$$\int_0^{\infty} \frac{|\ln x - \ln y|^{\gamma}}{\alpha x + \beta y + |x-y|} \left(\frac{x}{y}\right)^{1/2} dy = \int_0^{\infty} \frac{|\ln x - \ln y|^{\gamma}}{\alpha x + \beta y + |x-y|} \left(\frac{y}{x}\right)^{1/2} dx$$

$$= \int_0^1 \frac{2^{\gamma+1}|\ln t|^{\gamma}}{(\alpha+1)+t^2(\beta-1)}\, dt + \int_0^1 \frac{2^{\gamma+1}|\ln t|^{\gamma}}{t^2(\alpha-1)+(\beta+1)}\, dt = A,$$

where $A := A(\gamma, \alpha, \beta) \in [0, \infty]$.

Another result stated in the following theorem [1] is under consideration.

Theorem 2.1: *If* f, g *are real functions such that* $0 < \int_0^{\infty} f^2(x)dx < \infty$, $0 < \int_0^{\infty} g^2(x)dx < \infty$, *then we have*

$$\int_0^{\infty}\int_0^{\infty} \frac{|\ln x - \ln y|^{\gamma}}{\alpha x + \beta y + |x-y|} f(x)\, g(y)\, dx$$

$$\leq A\left(\int_0^{\infty} f^2(x)\, dx\right)^{1/2}\left(\int_0^{\infty} g^2(y)\, dy\right)^{1/2}, \qquad (6)$$

where A is defined in Lemma 2.1 and is the best possible.

In the following theorem, we introduce an equivalent form to inequality (6).

Theorem 2.2: *Suppose* $f \geq 0$ *and* $0 < \int_0^{\infty} f^2(x)\, dx < \infty$, *then*

$$\int_0^{\infty}\left[\int_0^{\infty} \frac{|\ln x - \ln y|^{\gamma}}{\alpha x + \beta y + |x-y|} f(x)\, dx\right]^2 dy \leq A^2 \int_0^{\infty} f^2(x)\, dx\,, \qquad (7)$$

where A is defined in Lemma 2.1. Furthermore, Inequality (7) is equivalent to (6).

Proof: Let

$$I = \int_0^{\infty}\left[\int_0^{\infty} \frac{|\ln x - \ln y|^{\gamma}}{\alpha x + \beta y + |x-y|} f(x)\, dx\right]^2 dy\ . \qquad (8)$$

Setting $x = yz$, $dx = ydz$,then we get

$$I = \int_0^\infty \left[\int_0^\infty \frac{|\ln z|^\gamma}{\alpha z + \beta + |z-1|} f(yz)\, dz\right]^2 dy\,.$$

By Minkowski's inequality for integrals,

$$I \le \left(\int_0^\infty \left[\int_0^\infty \left(\frac{|\ln z|^\gamma}{\alpha z + \beta + |z-1|} f(yz)\right)^2 dy\right]^{1/2} dz\right)^2 .$$

$$I \le \left(\int_0^\infty \frac{|\ln z|^\gamma}{\alpha z + \beta + |z-1|}\left[\int_0^\infty f^2(yz)\, dy\right]^{1/2} dz\right)^2 .$$

Setting $y = u/z$, $dy = (1/z)du$,then by Fubini's Theorem , we obtain

$$I \le \left(\int_0^\infty \frac{|\ln z|^\gamma z^{-1/2}}{\alpha z + \beta + |z-1|}\, dz \left[\int_0^\infty f^2(u)\, du\right]^{1/2}\right)^2 ,$$

$$I \le \left(\int_0^\infty \frac{|ln\, z|^\gamma z^{-1/2}}{\alpha z + \beta + |z-1|}\, dz\right)^2 \int_0^\infty f^2(u)\, du \,,$$

$$= A^2 \int_0^\infty f^2(x)\, dx\,.$$

Thus Inequality (7) holds.

Now, to prove that Inequality (7) is equivalent to (6): Suppose that Inequality (6) holds, and let

$$g(y) = \int_0^\infty \frac{|ln\, x - ln\, y|^\gamma}{\alpha x + \beta y + |x-y|} f(x)\, dx \quad .$$

Hence

$$0 < \int_0^\infty g^2(y)dy = \int_0^\infty \left(\int_0^\infty \frac{|\ln x - \ln y|^\gamma}{\alpha x + \beta y + |x-y|} f(x)dx\right) g(y)dy\,.$$

By Fubini's Theorem and Inequalities (6) ,

$$\int_0^\infty g^2(y)dy = \int_0^\infty \int_0^\infty \frac{|\ln x - \ln y|^\gamma}{\alpha x + \beta y + |x-y|} f(x)\, g(y)\, dx\, dy$$

$$\le A\left(\int_0^\infty f^2(x)\, dx\right)^{1/2} \left(\int_0^\infty g^2(y)\, dy\right)^{1/2} .$$

Notice that by Inequality (7), $g \in L^2$. So the last integral is finite, and hence

$$\left(\int_0^\infty g^2(y)\,dy\right)^{1/2} \le A\left(\int_0^\infty f^2(x)\,dx\right)^{1/2}.$$

Thus

$$\int_0^\infty \left[\int_0^\infty \frac{|\ln x - \ln y|^\gamma}{\alpha x + \beta y + |x-y|}\, f(x)\,dx\right]^2 dy \le A^2 \int_0^\infty f^2(x)\,dx.$$

Conversly, if Inequality (7) holds, then

$$\int_0^\infty \int_0^\infty \frac{|\ln x - \ln y|^\gamma}{\alpha x + \beta y + |x-y|}\, f(x)\,g(y)\,dx\,dy$$

$$= \int_0^\infty \left(\int_0^\infty \frac{|\ln x - \ln y|^\gamma}{\alpha x + \beta y + |x-y|}\, f(x)\,dx\right) g(y)\,dy\,.$$

By Cauchy - Schwarz inequality we get

$$\int_0^\infty \left(\int_0^\infty \frac{|\ln x - \ln y|^\gamma}{\alpha x + \beta y + |x-y|}\, f(x)\,dx\right) g(y)\,dy$$

$$\le \left(\int_0^\infty \left[\int_0^\infty \frac{|\ln x - \ln y|^\gamma}{\alpha x + \beta y + |x-y|}\, f(x)\,dx\right]^2 dy\right)^{1/2} \left(\int_0^\infty g^2(y)\,dy\right)^{1/2}$$

$$\le A\left(\int_0^\infty f^2(x)\,dx\right)^{1/2} \left(\int_0^\infty g^2(y)\,dy\right)^{1/2}.$$

Lemma 2.2: [2] *Let f be a nonnegative integrable function, and $F(x) = \int_0^x f(t)\,dt$, then*

$$\int_0^\infty \left(\frac{F(x)}{x}\right)^p dx < \left(\frac{p}{p-1}\right)^p \int_0^\infty f(x)\,dx\,, \qquad p > 1.$$

Using the above lemma and together with Theorem 2.1, we introduce the following result.

Theorem 2.3: *Let $f, g \ge 0$,*

$$F(x) = \int_0^x f(t)\,dt\,, \qquad G(y) = \int_0^y g(t)\,dt,$$

and assume that $0 < \int_0^\infty f^2(x)dx < \infty$ *and* $0 < \int_0^\infty g^2(y)dy < \infty$, *then we have*

$$\int_0^\infty \int_0^\infty \frac{|\ln x - \ln y|^\gamma}{\alpha x + \beta y + |x-y|} \frac{F(x)}{x} \frac{G(y)}{y}\, dx\, dy$$

$$\leq \mu \left(\int_0^\infty f^2(x)\, dx \right)^{1/2} \left(\int_0^\infty g^2(y)\, dy \right)^{1/2}. \qquad (9)$$

Proof: Let

$$I = \int_0^\infty \int_0^\infty \frac{|\ln x - \ln y|^\gamma}{\alpha x + \beta y + |x-y|} \frac{F(x)}{x} \frac{G(y)}{y}\, dx\, dy \ .$$

By Holder's inequality, we obtain

$$I \leq \left\{ \int_0^\infty \left(\int_0^\infty \frac{|\ln x - \ln y|^\gamma}{\alpha x + \beta y + |x-y|} \left(\frac{x}{y}\right)^{1/2} dy \right) \left(\frac{F(x)}{x} \right)^2 dx \right\}^{1/2}$$

$$\times \left\{ \int_0^\infty \left(\int_0^\infty \frac{|\ln x - \ln y|^\gamma}{\alpha x + \beta y + |x-y|} \left(\frac{y}{x}\right)^{1/2} dx \right) \left(\frac{G(y)}{y} \right)^2 dy \right\}^{1/2}.$$

By using Lemma 2.1,

$$I \leq \left\{ \int_0^\infty A \left(\frac{F(x)}{x} \right)^2 dx \right\}^{1/2} \times \left\{ \int_0^\infty A \left(\frac{G(y)}{y} \right)^2 dy \right\}^{1/2},$$

$$I \leq A \left\{ \int_0^\infty \left(\frac{F(x)}{x} \right)^2 dx \right\}^{1/2} \left\{ \int_0^\infty \left(\frac{G(y)}{y} \right)^2 dy \right\}^{1/2}.$$

Finally, by Lemma 2.2, for *p*=2, we have

$$I \leq 4A \left(\int_0^\infty f^2(x)\, dx \right)^{1/2} \left(\int_0^\infty g^2(y)\, dy \right)^{1/2}.$$

Letting μ= 4*A*, and inequality (9) is proved.

Corollary 2.1: *Let* $\alpha = \beta = 1$ *in* ***Theorem 2.3,*** *then we obtain*

$$\int_0^\infty \int_0^\infty \frac{|\ln x - \ln y|^\gamma}{x + y + |x-y|} \frac{F(x)}{x} \frac{G(y)}{y}\, dx\, dy$$

$$\leq K_\gamma \left(\int_0^\infty f^2(x)\, dx\right)^{1/2} \left(\int_0^\infty g^2(y)\, dy\right)^{1/2}, \tag{10}$$

where the constant

$K_\gamma = \int_0^1 2^{\gamma+1} |\ln h|^\gamma \, dh = 2\gamma K_{\gamma-1}$. Here, $\gamma = 1,2,3, \dots and\ K_0 = 2$.

Proof: The proof of (10) is similar to that of (9), and here we only prove that:

$$K_\gamma = \int_0^1 2^{\gamma+1} |\ln h|^\gamma \, dh = 2\gamma K_{\gamma-1}. \tag{11}$$

We have $K_\gamma = \int_0^\infty \frac{|\ln x - \ln y|^\gamma}{x+y+|x-y|} \left(\frac{x}{y}\right)^{1/2} dy = \int_0^\infty \frac{|\ln t|^\gamma}{1+t+|1-t|} \left(\frac{1}{t}\right)^{1/2} dt$

$$= \int_0^1 \frac{|\ln t|^\gamma}{2} \left(\frac{1}{t}\right)^{1/2} dt + \int_1^\infty \frac{|\ln t|^\gamma}{2t} \left(\frac{1}{t}\right)^{1/2} dt .$$

For the last integral, take $t = s^{-1}$ and rewrite this integral in term of t, We obtain

$$K_\gamma = \int_0^1 \frac{|\ln t|^\gamma}{2} \left(\frac{1}{t}\right)^{1/2} dt + \int_0^1 \frac{|\ln t|^\gamma}{2} \left(\frac{1}{t}\right)^{1/2} dt = \int_0^1 |\ln t|^\gamma \left(\frac{1}{t}\right)^{1/2} dt .$$

Setting $h = t^{1/2}$, we get

$$K_\gamma = \int_0^1 2^{\gamma+1} |\ln h|^\gamma \, dh = 2\gamma K_{\gamma-1} .$$

3 Several Special Cases

We now introduce some special inequalities of (9) by choosing different values for $\gamma,\ \alpha,$ and β.

(1) If $\gamma = \alpha = 0,\ \beta = 1$, then we obtain

$$\int_0^\infty \int_0^\infty \frac{1}{y + |x-y|} \frac{F(x)}{x} \frac{G(y)}{y} \, dx\, dy$$

$$\leq \mu \left(\int_0^\infty f^2(x)\, dx\right)^{1/2} \left(\int_0^\infty g^2(y)\, dy\right)^{1/2}, \tag{12}$$

where μ= 4*A* and from Lemma 2.1,

$$A=\int_0^1 \frac{2}{1}\,dt\ +\int_0^1 \frac{2}{-t^2+2}\,dt = 2+2\int_0^1\left(\frac{1/2\sqrt{2}}{\sqrt{2}-t}+\frac{1/2\sqrt{2}}{\sqrt{2}+t}\right)dt$$

$$=2+\frac{1}{\sqrt{2}}\left(-\ln\left|\sqrt{2}-t\right|\Big|_0^1+\ln\left|\sqrt{2}+t\right|\Big|_0^1\right)\ =\ 3.24646.$$

(2) If $\gamma = 0,\ \alpha = 1,\ \beta = 2$, then

$$\int_0^\infty\int_0^\infty \frac{1}{x+2y+\ |x-y|}\frac{F(x)}{x}\frac{G(y)}{y}\ dx\,dy$$

$$\le \mu\left(\int_0^\infty f^2(x)\,dx\right)^{1/2}\left(\int_0^\infty g^2(y)\,dy\right)^{1/2}, \qquad (13)$$

where μ= 4*A* and from Lemma 2.1,

$$A=\int_0^1\frac{2}{2+t^2}\,dt\ +\int_0^1\frac{2}{3}\,dt = 2\left[\frac{1}{\sqrt{2}}\tan^{-1}\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}\tan^{-1}0\right]+\frac{2}{3}$$
$$=\ 2.2071.$$

(3) If $\gamma = 1,\ \alpha = \beta = 0$, then

$$\int_0^\infty\int_0^\infty \frac{|\ln x\ -\ \ln y|}{|x-y|}\frac{F(x)}{x}\frac{G(y)}{y}\ dx\,dy$$

$$\le \mu\left(\int_0^\infty f^2(x)\,dx\right)^{1/2}\left(\int_0^\infty g^2(y)\,dy\right)^{1/2}, \qquad (14)$$

where μ= 4*A*, and from Lemma 2.1,

$$A=\int_0^1\frac{-4\ \ln t}{1-t^2}\,dt\ +\int_0^1\frac{-4\ \ln t}{-t^2+1}\,dt=-8\int_0^1\frac{\ln t}{1-t^2}\,dt.$$

Since

$$\int_0^1\frac{\ln t}{t-1}t^{-1/2}\,dt\ =\ \pi^2\ .$$

Then we have

$$A=-8\int_0^1\frac{\ln t}{1-t^2}\,dt = 2\pi^2\ .$$

(4) If $\gamma = \alpha = 0,\ \beta = 2$, then

$$\int_0^\infty \int_0^\infty \frac{1}{2y+\ |x-y|} \frac{F(x)}{x} \frac{G(y)}{y}\ dx\, dy$$

$$\leq \mu \left(\int_0^\infty f^2(x)\, dx \right)^{1/2} \left(\int_0^\infty g^2(y)\, dy \right)^{1/2}, \qquad (15)$$

where μ= 4*A* and from Lemma 2.1,

$$A = \int_0^1 \frac{2}{1+t^2}\, dt + \int_0^1 \frac{2}{-t^2+3}\, dt = \frac{\pi}{2} + 2 \int_0^1 \left(\frac{1/2\sqrt{3}}{\sqrt{3}-t} + \frac{1/2\sqrt{3}}{\sqrt{3}+t} \right) dt$$

$$= \frac{\pi}{2} + \frac{1}{\sqrt{3}} \left(-\ln\left|\sqrt{3}-t\right| \Big|_0^1 + \ln\left|\sqrt{3}+t\right| \Big|_0^1 \right) = 1.968\,.$$

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