Gen. Math. Notes, Vol. 7, No. 1, November 2011, pp.59-65
ISSN 2219-7184; Copyright ©ICSRS Publication, 2011
www.i-csrs.org
Available free online at http://www.geman.in

Application of Approximate Best Proximity Pairs

S.A.M. Mohsenalhosseini[1], H. Mazaheri[2], M.A. Dehghan[3] and M. Bagheshahi[4]

[1]Faculty of Mathematics, Valiasr Rafsanjan University, Rafsanjan, Iran
E-mail: amah@vru.ac.ir
[2]Faculty of Mathematics, Yazd University, Yazd, Iran
E-mail: hmazaheri@yazduni.ac.ir
[3]Faculty of Mathematics, Valiasr Rafsanjan University, Rafsanjan, Iran
E-mail: dehghan@vru.ac.ir
[4]Faculty of Mathematics, Yazd University, Yazd, Iran
E-mail: Bagheshahi1385@gmail.com

(Received: 25-5-11/ Accepted: 21-11-11)

Abstract

In this paper we give ky fans approximate best proximity theorem and Picard-Lindeloffs approximate best proximity theorem, then illustrate a few applications where we derive results in fixed point theory, approximation theory, boundary value problems theorem.

Keywords: *Ky Fans approximate best proximity, $\epsilon-$ ky fan, Picard-Lindeloffs approximate best proximity.*

1 Introduction

If f is an operator of a normed space X into itself, $x_0 \in X$ is called a approximate fixed point of f if for some $\epsilon > 0$, $d(f(x_0), x_0) < \epsilon$. Theorems concerning the existence and properties of fixed points and are known as fixed points theorems. Such theorems are the most important tools for proving existence of the solutions to various mathematical models (differential, integral and partial differential equations, etc.) Fixed point theorems of ordered Bananch spaces provide us exact or approximate solutions of boundary-value problems.for details, one can refer to, Amann, [1] Collatz, [2] franklin, [3]. We also Theorems concerning the existence and properties of approximate best proximity are known as approximate best proximity theorems [4]. We show that Approximate best proximity theorems provide us approximate solutions of boundary-value problems, and also such theorems are the most important tools for proving existence of the approximate solutions to various mathematical models (differential, integral and ordinary differential equations, etc.)

Definition 1.1. *[5] Let $f : X \to X$ be a map, where X is a normed space. Then f is said to be contraction if*

$$\exists\alpha \in (0,1) \text{ such that } \|fx - fy\| \leq \alpha\|x - y\|,$$

for all $x, y \in X$.

Definition 1.2. *Let $f(x, y)$ is a continuous and bounded function of two variables in a rectangle $A = \{(x, y) : \ a \leq x \leq b, c \leq y \leq d\}$. Then we define norm$\|f\|_\infty$,*

$$\|f\|_\infty = \sup_{(x,y)\in A} (|f(x, y)|).$$

2 Application

In this section we obtain approximate for Picard theorem.

Theorem 2.1. *Let $f(x, y)$ be a continuous function of two variables in a rectangle $A = \{(x, y) : \ a \leq x \leq b, c \leq y \leq d\}$ and satisfy the contraction condition in the second variable y. Further, let (x_0, y_0) be an interior point of A and $\epsilon > 0$. Then the following boundary value problems for some $\epsilon > 0$.*

$$\begin{cases} |dy/dx - f(x, y)| < \epsilon \\ \qquad\qquad\qquad\qquad (*) \\ \qquad g(x_0) = y_0 \end{cases}$$

has solution $y = g(x)$.

Proof. Since $f(x, y)$ is a continuous function of two variables in a rectangle, then there exists a $M > 0$ such that

$$\|f\|_\infty \leq M.$$

If $y = g(x)$ satisfies (*) for some $0 < \epsilon < M$, then integrating (*) from x_0 to x, we get

$$\int_{x_0}^{x} |\frac{dy}{dx} - f(x, y)|dx = |g(x) - y_0 - \int_{x_0}^{x} f(t, g(t))dt| < \epsilon(x - x_0).$$

Thus we have

$$\begin{aligned} |g(x) - y_0| &\leq |\int_{x_0}^{x} f(t, g(t))dt| + \epsilon(x - x_0) \\ &\leq M(x - x_0) + \epsilon(x - x_0) \\ &\leq M(x - x_0) + M(x - x_0) \end{aligned}$$

If $-\delta + x_0 < x < \delta + x_0$ for $\delta > 0$ then $|g(x) - y_0| \leq 2M\delta$. Now we consider the following set:

$X = \{g \in C[-\delta + x_0, x_0 + \delta] : \ g : [-\delta + x_0, x_0 + \delta] \to \mathbf{R} \ , |g(x) - y_0| < 2M\delta\}$,

Then $(X, \|.\|_\infty)$ is normed space. Let $T : X \to X$ be defined as $Tg = h$, where

$$h(x) = y_0 + \int_{x_0}^{x} f(t, g(t))dt.$$

Since

$$\begin{aligned} |h(x) - y_0| &= |\int_{x_0}^{x} f(t, g(t))dt| \\ &\leq M(x - x_0) \\ &\leq M\delta \\ &\leq 2M\delta. \end{aligned}$$

Thus $h(x) \in X$ and T is well defined. For $g, g_1 \in X$

$$\begin{aligned} \|Tg - Tg_1\|_\infty = \|h - h_1\|_\infty &= Sup|\int_{x_0}^{x}[f(t, g(t)) - f(t, g_1(t)]dt| \\ &\leq Sup\int_{x_0}^{x}|[f(t, g(t)) - f(t, g_1(t)]|dt \\ &\leq qSup\int_{x_0}^{x}|g(t)) - g_1(t)]|dt \\ &\leq q\delta\|g - g_1\|_\infty. \end{aligned}$$

So

$$\|Tg - Tg_1\|_\infty \leq k\|g - g_1\|_\infty$$

where $0 \leq k = q\delta < 1$. Hence T is a contraction mapping of X into itself. By Theorem 2.2, of [4] if $A = B = X$ then $d(A, B) = 0$ and there exists a $\epsilon > 0$ such that

$$\|g^* - Tg^*\|_\infty < \epsilon,$$

that is

$$\|g^*(x) - y_0 - \int_{x_0}^{x} f(t, g^*(t))dt\|_\infty < \epsilon.$$

Thus boundary value problem (*) has has solution $y = g(x)$. ■

Theorem 2.2. *Let the function $k(x, y)$ be defined and measurable in the region*

$$A = \{(x, y) : 0 \leq x, y \leq b\}.$$

Further, let

$$\int_a^b \int_a^b |k(x, y)|^2 dxdy < \infty$$

and $g(x) \in L^2(a, b)$. for scaler μ where $|\mu| \leq \frac{1}{(\int_a^b \int_a^b |k(x,y)|^2 dxdy)^{\frac{1}{2}}}$, then there exists a solution $f(x) \in L^2(a, b)$ and a $\epsilon > 0$ s.t.

$$|f(x) - g(x) - \mu \int_a^b k(x, y)f(y)dy| < \epsilon.$$

Proof. We define $T : L^2(a, b) \to L^2(a, b)$ with $Tf = h$ where

$$h(x) = g(x) + \mu \int_a^b k(x, y)f(y)dy.$$

If $f \in L^2(a, b)$, then $h \in L^2(a, b)$. Since $g \in L^2(a, b)$ and μ is scaler , it is sufficient to show that

$$\int_a^b k(x, y)f(y)dy \in L^2(a, b).$$

By the Cauchy-Schwarz inequality

$$\begin{aligned} |\int_a^b k(x, y)f(y)dy| &\leq \int_a^b |k(x, y)f(y)|dy \\ &\leq (\int_a^b |k(x, y)|^2 dy)^{\frac{1}{2}} (\int_a^b |f(y)|^2 dy)^{\frac{1}{2}} \end{aligned}$$

then

$$|\int_a^b k(x, y)f(y)dy|^2 \leq \int_a^b |k(x, y)|^2 dy \int_a^b |f(y)|^2 dy$$

or

$$\int_a^b |\int_a^b k(x, y)f(y)dy|^2 dx \leq \int_a^b (\int_a^b |k(x, y)|^2 dy)dx \int_a^b (\int_a^b |f(y)|^2 dy)dx,$$

by the hypotheses

$$\int_a^b |\int_a^b k(x, y)f(y)dy|^2 dx < \infty.$$

Thus

$$\int_a^b k(x, y)f(y)dy \in L^2(a, b) \Rightarrow h \in L^2(a, b).$$

Now we show that T is a contraction mapping. We have $\|Tf - Tf_1\|_\infty = \|h - h_1\|_\infty$ where $h(x) = g_1(x) + \mu \int_a^b k(x,y)f(y)dy$

$$\begin{aligned} \|h - h_1\| &= |\mu \int_a^b k(x,y)[f(y) - f_1(y)]dy| \\ &= |\mu|(\int_a^b |[\int_a^b |k(x,y)[f(y) - f_1(y)]dy]|^2 dx)^{\frac{1}{2}} \\ &\leq |\mu|(\int_a^b \int_a^b |k(x,y)|^2 dxdy)^{\frac{1}{2}} (\int_a^b |f(y) - f_1(y)|^2 dy)^{\frac{1}{2}} \end{aligned}$$

by definition of the norm in L_2, we have

$$\|f - f_1\| = (\int_a^b |f(y) - f_1(y)|^2 dy)^{\frac{1}{2}}$$

If

$$|\mu| < \frac{1}{(\int_a^b \int_a^b |k(x,y)|^2 dxdy)^{\frac{1}{2}}}$$

Then

$$\|Tf - Tf_1\|_\infty \leq k\|f - f_1\|_\infty$$

where

$$0 \leq k = |\mu|(\int_a^b \int_a^b |k(x,y)|^2 dxdy)^{\frac{1}{2}} < 1.$$

Thus, T is a contraction. By theorem 2.2 of [4] if $A = B = X$ then $d(A,B) = 0$ and there exists a $\epsilon > 0$ such that:

$$\|f^* - Tf^*\| < \epsilon,$$

that is

$$|f(x) - g(x) - \mu \int_a^b k(x,y)f(y)dy| < \epsilon.$$

Then there exists a solution $f(x) \in L^2(a,b)$ and T has an approximation fixed point. ■

Theorem 2.3. *Let C be a subset of a normed space and $f : C \to C$ be a contraction map. Then there is a $y \in C$ and $\epsilon > 0$ such that*

$$\|y - fy\| < d(fy, C) + \epsilon.$$

Proof. By Theorem 2.2 of [4] the map $f : C \to C$ has an approximate best proximity pair. That is there is a $y \in C$ such that

$$\|y - f(y)\| < \epsilon$$

Therefore $\|y - fy\| < d(fy, C) + \epsilon$. ■

Theorem 2.4. *Let C be a subset of a normed space H such that C is compact. Suppose that the mapping $f : C \to C$ is continuous and*

$$\|fx - fy\| \leq \ \|x - y\|,$$

where $(x, y) \in C \times C$. Then there exists a $y \in C$ and $\epsilon > 0$ such that

$$\|y - fy\| < d(fy, C) + \epsilon.$$

Proof. By Theorem 2.5 of [4] the map $f : C \to C$ has an approximate best proximity pair. That is there is a $y \in C$ such that

$$\|y - f(y)\| < \epsilon$$

Therefore $\|y - fy\| < d(fy, C) + \epsilon$. ■

y satisfying Theorem 2.4, 2.5 is called ky fan approximate.

Theorem 2.5. *consider $X = \boldsymbol{R}$ with $\|x\| = |x|$ and $[a, b] \subset \boldsymbol{R}$; $f : [a, b] \to [a, b]$, a differentiable function such that*

$$|f'(x)| \leq k < 1.$$

Then the equation $|x - f(x)| < \epsilon$, has a solution in $[a, b]$ for some $\epsilon > 0$.

Proof. By Lagrange s mean-value theorem, for any $x, y \in [a, b]$, $f(x) - f(y) = f'(z)(x - y)$, where $y < z < x$. Hence,

$$|f(x) - f(y)| = |f'(z)||x - y| \leq k|x - y|,$$

where $0 \leq k < 1$. Thus, f is a contraction mapping on $[a, b]$ into itself. By Theorem 2.5 of [4] the map $f : [a, b] \to [a, b]$ has an approximate best proximity pair. That is there is a $y \in [a, b]$ such that $|y - f(y)| < \epsilon$. ■

References

[1] H. Amann, Fixed point equations and nonlinear eigenvalue problems in or4'ed Banach spaces, *SIAM Rev.*, 18(2) (1976), 620-709.

[2] L. Collatz, *Functional Analysis and Numerical Mathematics*, Academic Press, New York, NY, (1966).

[3] J. Franklin, *Methods of Mathematical Economics*, Springer Verlag, New York, (1980).

[4] S.A.M. Mohsenalhosseini, H. Mazaheri, and M.A. Dehghan, Approximate best proximity pairs in metric space, *Abstract and Applied Analysis*, 2011(2011), 9 pages.

[5] A.H. Siddiqi, *Functional Analysis with Application*, Tata McGraw-Hill Publishing Company Limited, New Delhi, (1986).

[6] I. Singer, *Best Approximation in Normed Linear Spaces by Elements of Linear Subspaces*, Translated from the Romanian by R. Georgescu, Die Grundlehren der mathematischen Wissenschaften, Band 171 Publishing House of the Academy of the Socialist Republic of Romania, Bucharest, Springer-Verlag, New York-Berlin (1970).

[7] I.A. Rus, *Generalized Contractions and Applications*, Cluj University Press, Cluj-Napoca, (2001).