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A Three Step Implicit Hybrid Linear Multistep Method for the Solution of Third Order Ordinary Differential Equations

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Abstract

A Linear Multistep Hybrid Method (LMHM) with continuous coefficients is considered and directly applied to solve third order Initial Value Problems (IVPs). The continuous method is used to obtain Multiple Finite Difference Methods (MFDMs) each of order 5 which are combined as simultaneous numerical integrators to provide a direct solution to IVPs over sub-intervals which do not overlap. The convergence of the MFDMs is discussed by conveniently representing the MFDMs as a block method and verifying that the block method is zero-stable and consistent. The superiority of the MFDMs over the existing methods is established numerically.

Keywords: *Multiple finite difference methods, third order, boundary value problem, block methods, multistep methods.*

1 Introduction

The mathematical formulation of physical phenomena in science and engineering often leads to initial value problems of the form:

$$y''' = f(x, y), \; y(a) = y_0 \;, \; y'(a) = \eta_0 \;, \; y''(a) = \eta_1 \tag{1}$$

However, only a limited number of numerical methods are available for solving (1) directly without reducing to a first order system of initial value problems. Some authors have proposed solution to second order initial value problems of ordinary differential equations using different approaches (see Awoyemi [1], Awoyemi and Idowu [2], Fatunla [3], Lambert [4] and Adee et al. [5]) ; in particular Awoyemi and Idowu [2]. Awoyemi [1] derived a p-stable linear multistep method for general second order initial value problems of ordinary differential equations which is to be used in form of predictor-corrector forms and like most linear multistep methods, they require starting values from Runge-Kutta methods or any other one-step methods. The predictors are also developed in the same way as correctors. Moreover, the block methods in Fatunla [3] are discrete and are proposed for non-stiff special second order ordinary differential equations in form of a predictor- corrector integration process. Also like other linear multistep methods they are usually applied to the initial value problems as a single formula but they are not self-starting; and they advance the numerical integration of the ordinary differential equations in one-step at a time, which leads to overlapping of the piecewise polynomials solution Model. There is the need to develop a method which is self-starting, eliminating the use of predictors with better accuracy and efficiency. Recently, several researches (Jator [6,7], Jator and Li [8], Mohammed et al.[9] and Mohammed [10]) proposed LMMs for the direct solution of the general second order IVPs, which were showed to be zero stable and were implemented without the need for either predictors or starting values from other methods. Jator [11] used the LMMs developed for IVPs and additional methods obtained from the same continuous k-step LMM to solve second order BVPs with Dirichlet and Neumann boundary conditions and also Olabode and Yusuph [12] developed a linear multistep method for the direct solution of initial value problems of ordinary differential equations for special third order initial value problem. This study, therefore propose a block hybrid multistep method for the direct solution of third order initial value problems of ordinary differential equations.

The paper is organized as follows. In Section 2, we derive a continuous approximation Y (x) for the exact solution y(x). Section 3 is devoted to the specification of the methods and how the MFDMs are obtained. Analysis, stability region and implementation of MFDM are discussed in section 4. Numerical examples are given in Section 5 to show the efficiency of the MFDMs. Finally, the conclusion of the paper is discussed in Section 6.

2 Development of Methods

In this section, our objective is to derive Hybrid Linear Multi-step Method (HLMM) of the form

$$\sum_{j=0}^{r-1}\alpha_j y_{n+j} = h^3\sum_{j=0}^{s-1}\beta_j f_{n+j} + h^3\beta_v f_{n+v} \tag{2}$$

where α_j, β_j and β_v are unknown constants and v_j is not an integer. We note that α_k =1, $\beta_j \neq 0$, α_0 and β_0 do not both vanish. In order to obtain (2), we proceed by seeking to approximate the exact solution y(x) of the form

$$Y(x) = \sum_{j=0}^{r+s-1} l_j x^j , \tag{3}$$

Where $x \in [a,b]$, l_j are unknown coefficients to be determined and $1 \leq r < k$, $S > 0$ are the number of interpolation and collocation points respectively. We then construct our continuous approximation by imposing the following conditions.

$$Y(x_{n+j}) = y_{n+j}\,, \quad j = 0,1.2,......, r-1 \tag{4}$$

$$Y'''(x_{n+\mu}) = f_{n+\mu} \tag{5}$$

Equation (4) and (5) lead to a system of (r+s) equations which is solved by Cramer's rule to obtain l_j . Our continuous approximation is constructed by substituting the values of l_j into equation (3). After some manipulation, the continuous method is expressed as

$$Y(x) = \sum_{j=0}^{r-1}\alpha_j(x) y_{n+j} + h^3\sum_{j=0}^{s-1}\beta_j(x) f_{n+j} + h^3\beta_v(x) f_{n+v} \tag{6}$$

where $\alpha_j(x)$, $\beta_j(x)$ and $\beta_v(x)$ are continuous coefficients. We note that since equation (1) involves first and second derivatives, the first and second derivative formula

$$\begin{aligned} Y'(x) &= \frac{1}{h}\left(\sum_{j=0}^{r-1}\alpha_j'(x) y_{n+j} + h^3\sum_{j=0}^{s-1}\beta_j'(x) f_{n+j} + h^3\beta_v'(x) f_{n+v}\right) \\ Y''(x) &= \frac{1}{h^2}\left(\sum_{j=0}^{r-1}\alpha_j''(x) y_{n+j} + h^3\sum_{j=0}^{s-1}\beta_j''(x) f_{n+j} + h^3\beta_v''(x) f_{n+v}\right) \end{aligned} \tag{7}$$

Equation (7) is easily obtained from (6) and is then used to provide the first and second derivatives for the methods by imposing the condition

$$Y'(x) = \delta(x),\ Y''(x) = \gamma(x) \tag{8}$$

$$Y'(a) = \delta_0,\ Y''(a) = \gamma_0 \tag{9}$$

3 Specification of the Methods

To derive these methods, we use Eq.(6) to obtained a continuous 3-step HLM method with the following specification:
r=3,s=5,k=3, $v = \frac{8}{3}, \gamma_i(x) = x^i, i = 0,1,...,7$. We also express $\alpha_j(x), \beta_j(x)$ and $\beta_v(x)$ as a functions of t, where $t = \frac{x - x_n}{h}$ to obtain the continuous form as follows:

$$\alpha_0(x) = \left(1 - \frac{3}{2}t + \frac{1}{2}t^2\right),\ \alpha_1 = \left(2t - t^2\right),\ \alpha_2 = \left(-\frac{1}{2}t + \frac{1}{2}t^2\right)$$
$$\beta_0(x) = \frac{1}{20160}\left(1170t - 3157t^2 + 3360t^3 - 1855t^4 + 567t^5 - 91t^6 + 6t^7\right)$$
$$\beta_1(x) = \frac{1}{8400}\left(2798t - 3815t^2 + 1680t^4 - 812t^5 + 161t^6 - 12t^7\right)$$
$$\beta_2(x) = \frac{1}{1680}\left(-230t + 427t^2 - 420t^4 + 287t^5 - 70t^6 + 6t^7\right)$$
$$\beta_{\frac{8}{3}}(x) = \frac{1}{11200}\left(1566t - 2835t^2 + 2835t^4 - 2079t^5 + 567t^6 - 54t^7\right)$$
$$\beta_3(x) = \frac{1}{5040}\left(-306t + 553t^2 - 560t^4 + 420t^5 - 119t^6 + 12t^7\right) \tag{10}$$

The MFDMs are obtained by evaluating (10) at $x = \left\{x_{n+3}, x_{n+\frac{8}{3}}\right\}$ to obtain the following

$$y_{n+3} = y_n - 3y_{n+1} + 3y_{n+2} + \frac{h^3}{800}\left[5f_n + 376f_{n+1} + 460f_{n+2} - 81f_{n+\frac{8}{3}} + 40f_{n+3}\right] \tag{11}$$

$$y_{n+\frac{8}{3}} = \frac{5}{9}y_n - \frac{16}{9}y_{n+1} + \frac{20}{9}y_{n+2} + \frac{h^3}{8748}\left[31f_n + 2268f_{n+1} + 2436f_{n+2} - 675f_{n+\frac{8}{3}} + 260f_{n+3}\right] \tag{12}$$

In particular, to start the initial value problem for n = 0, we obtain the following equations from (9):

$$h\delta_0 = -\frac{3}{2}y_0 + 2y_1 - \frac{1}{2}y_2 + \frac{h^3}{16800}\left[975f_0 + 5596f_1 - 2300f_2 + 2349f_{\frac{8}{3}} - 1020f_3\right] \quad (13)$$

$$h^2\gamma_0 = y_0 - 2y_1 + y_2 + \frac{h^3}{1440}\left[-451f_0 - 1308f_1 + 732f_2 - 729f_{\frac{8}{3}} + 316f_3\right] \quad (14)$$

It is worth noting that the derivatives are provided by

$\delta(x_{n+\tau}) = \delta_{n+\tau}\ ;\ \gamma(x_{n+\tau}) = \gamma_{n+\tau}\ , \tau = 1, 2, \frac{8}{3}\ and\ 3$ as follows:

$$h\delta_{n+1} = -\frac{1}{2}y_n + \frac{1}{2}y_{n+2} - \frac{h^3}{6720}\left[51f_n + 1032f_{n+1} - 4f_{n+2} + 81f_{n+\frac{8}{3}} - 40f_{n+3}\right]$$

$$h\delta_{n+2} = \frac{1}{2}y_n - 2y_{n+1} + \frac{3}{2}y_{n+2} + \frac{h^3}{16800}$$
$$\left[65f_n + 3748f_{n+1} + 2460f_{n+2} - 1053f_{n+\frac{8}{3}} + 380f_{n+3}\right]$$

$$h\delta_{n+\frac{8}{3}} = \frac{7}{6}y_n - \frac{10}{3}y_{n+1} + \frac{13}{6}y_{n+2} + \frac{h^3}{244940}$$
$$\left[17443f_n + 1351212f_{n+1} + 1780788f_{n+2} - 293463f_{n+\frac{8}{3}} + 137780f_{n+3}\right]$$

$$h\delta_{n+3} = \frac{3}{2}y_n - 4y_{n+1} + \frac{5}{2}y_{n+2} + \frac{h^3}{6720}$$
$$\left[61f_n + 4792f_{n+1} + 7060f_{n+2} - 81f_{n+\frac{8}{3}} + 488f_{n+3}\right]$$

$$h^2\gamma_{n+1} = y_n - 2y_{n+1} + y_{n+2} + \frac{h^3}{1800}\left[40f_n + 132f_{n+1} - 315f_{n+2} + 243f_{n+\frac{8}{3}} - 100f_{n+3}\right]$$

$$h^2\gamma_{n+2} = y_n - 2y_{n+1} + y_{n+2} + \frac{h^3}{7200}$$
$$\left[25f_n + 3636f_{n+1} + 4620f_{n+2} - 1701f_{n+\frac{8}{3}} + 620f_{n+3}\right]$$

$$h^2\gamma_{n+\frac{8}{3}} = y_n - 2y_{n+1} + y_{n+2} + \frac{h^3}{194400}$$

$$\left[1195 f_n + 93756 f_{n+1} + 190980 f_{n+2} + 36369 f_{n+\frac{8}{3}} + 1700 f_{n+3}\right]$$

$$h^2\gamma_{n+3} = y_n - 2y_{n+1} + y_{n+2} + \frac{h^3}{1800}\left[10 f_n + 876 f_{n+1} + 1725 f_{n+2} + 729 f_{n+\frac{8}{3}} + 260 f_{n+3}\right]$$

4 Analysis and Implementation of the Method

Following Fatunla [13] and Lambert [4] we define the local truncation error associated with the conventional form of (2) to be the linear difference operator

$$L[y(x);h] = \sum_{j=0}^{k}\{\alpha_j y(x+jh) - h^3\beta_j y'''(x+jh)\} + h^3\beta_v y'''(x+vh) \tag{15}$$

Assuming that y(x) is sufficiently differentiable, we can expand the terms in (15) as a Taylor series about the point x to obtain the expression

$$L[y(x);h] = C_0 y(x) + C_1 hy' + ..., + C_q h^q y^q(x) + ..., \tag{16}$$

where the constant coefficients $C_q, \quad q = 0,1,...$ are given as follows:

$$C_q, \quad q = 0,1,...$$

$$C_0 = \sum_{j=0}^{k}\alpha_j,$$

$$C_1 = \sum_{j=1}^{k} j\alpha_j,$$

.

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$$.C_q = \left[\frac{1}{q!}\sum_{j=1}^{k} j^q\alpha_j - q(q-1)\sum_{j=1}^{k} j^{q-2}\beta_j\right].$$

According to Henrici [14], method (5) has order p if

$$C_0 = C_1 = ... = C_P = C_{P+1} = 0, \quad C_{P+2} \neq 0$$

Our calculations reveal that the methods (11) to (14) have order p = 5 and error constants given by the vector $C_8 = \left(-\frac{7}{7200}, -\frac{85}{157464}, -\frac{811}{18}, -\frac{197}{14} \right)^T$

In order to analyze the methods for zero-stability, we normalize (11) to (14) and write them as a block method given by the matrix difference equation

$$A^0 Y_{\mu+1} = A^1 Y_\mu + h^2 \left[B^0 F_{\mu+1} + B^1 F_\mu \right] \tag{17}$$

Where

$$Y_{\mu+1} = (y_{n+1}, \ldots, y_{n+3})^T,\ Y_\mu = (y_{n-3} \ldots, y_n)^T\ ,F_{\mu+1} = (f_{n+1}, \ldots, f_{n+3})^T,\ F_\mu = (f_{n-3} \ldots, f_n)^T\ \textit{and}\ n = 0,3,\ldots$$

and matrices A^0 and A^1 are defined as follows:

A^0 is an identity matrix of dimension four and

$$A^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

It is worth noting that zero-stability is concerned with the stability of the difference system in the limit as h tends to zero. Thus, as $h \to 0$, the method (17) tends to the difference system

$A^0 Y_{\mu+1} - A^1 Y_\mu = 0$ whose first characteristic polynomial $\rho(R)$ is given by

$$\rho(R) = \det\left(RA^0 - A^1\right) = R^3 (R-1) \tag{18}$$

Following Fatunla [13], the block method (17) is zero-stable, since from (18), $\rho(R) = 0$ satisfy $|R_j| \le 1\ j = 1\ldots, k$ and for those roots with $|R_j|$ =1, the multiplicity does not exceed 2. The block method (17) is consistent as it has order $P > 1$. According to Henrici [14], we can safely assert the convergence of the block method (17).

It is vital to note that the main method given by (10) can be used as a numerical integrator directly and singly in the conventional way on non-overlapping sub-intervals. However, our method is implemented more efficiently by combining methods (11) to (14), each of order five with relatively small error constants, as simultaneous integrators for IVPs without looking for any other methods to provide the starting values. We proceed by explicitly obtaining initial conditions at x_{n+3}, $n = 0,3,\ldots, N-5$ using the computed values $y(x_{n+3}) = y_{n+3}, \delta(x_{n+3}) = \delta_{n+3}\ and \lambda(x_{n+3}) = \lambda_{n+3}$ over sub-intervals $[x_0, x_3], \ldots [x_{n-3}, x_N]$ which do not overlap (see [10]). For instance, for $n = 0$, $(y_1, y_2, y_3)^T$ are simultaneously obtained over the sub-interval $[x_0, x_3]$ as y_0 is known from the IVP; for $n = 3, (y_4, y_5, y_6)^T$ are simultaneously obtained over the sub-interval $[x_3, x_6]$, as y_3 is known from the previous block, and so on. Hence, the sub-intervals do not over-lap and the solutions obtained in this manner are more accurate that those obtained in the conventional way.

4.1 Region of Absolute Stability

The absolute stability region of the newly constructed hybrid linear multi-step methods (8)-(10) is plotted using Chollom [15] by reformulating the methods as general linear methods and is shown in Figure 1 below.

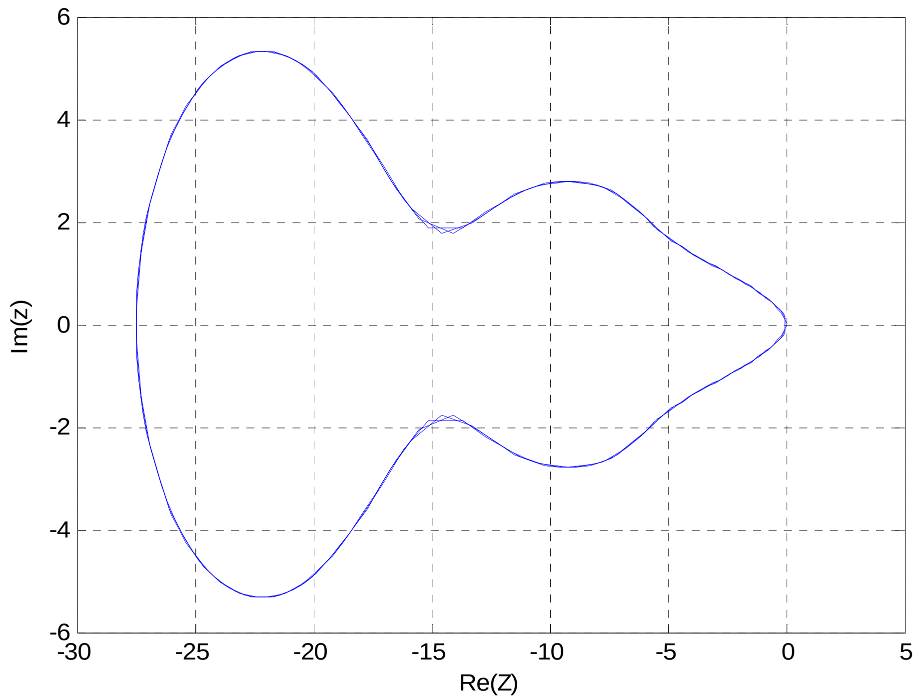

Fig. 1: Region of Absolute Stability Region of Hybrid Linear Multi-Step Method (HLMM)

4.2 Numerical Examples

We report here a few numerical examples on some problems taken from the literature.

Problem 1 (Olabode and Yusuph [12])

$$y''' = e^x$$
$$y(0) = 3, \quad y'(0) = 1, \quad y''(0) = 5$$

Exact Solution is $y(x) = 2 + 2x^2 + e^x$

Table 1: Error of methods for problem 1

X	Exact Solution	Numerical Solution	Error in Proposed Method	Olabode and Yusuph [12]
0	3	3	0.000000000E+00	0.000000000E+00
0.1	3.125170918	3.125170918	0.000000000E+00	-7.5647E-11
0.2	3.301402758	3.301402758	0.000000000E+00	1.83983E-09
0.3	3.529858808	3.529858807	1.000000083E-09	4.42400E-09
0.4	3.811824698	3.811824697	1.000000083E-09	1.03587E-08
0.5	4.148721271	4.148721270	1.000000083E-09	1.12999E-08
0.6	4.542118800	4.542118799	1.000000083E-09	1.46095E-08
0.7	4.993752707	4.993752706	9.999991946E-10	2.05295E-08
0.8	5.505540928	5.505540927	1.000000083E-09	1.95075E-08
0.9	6.079603111	6.079603109	2.000000165E-09	1.08431E-08
1.0	6.718281830	6.718281831	1.000000083E-09	1.54095E-08

Problem 2 (Awoyemi et al [16])

$$y''' + 4y' = x$$
$$y(0) = 0, \quad y'(0) = 0, \quad y''(0) = 1, h = 0.1$$

Exact Solution is $y(x) = -\frac{3}{16}\cos 2x + \frac{5}{16}$

Table 2: Error of methods for problem 2

X	Exact Solution	Numerical Solution	Error in Proposed Method	Error in [16]
0.1	0.004987516700	0.004987517661	9.61000E-10	1.1899E-11
0.2	0.01980106360	0.01980107010	6.50000E-09	3.0422E-09
0.3	0.04399957220	0.04399958817	1.59700E-08	7.7796E-08
0.4	0.07686749200	0.07686750864	1.66400E-08	1.5559E-07
0.5	0.1174433176	0.1174433379	2.03000E-08	3.0541E-07
0.6	0.1645579210	0.1645579476	2.66000E-08	4.6102E-07
0.7	0.2168811607	0.2168811874	2.67000E-08	3.138E-07
0.8	0.2729749104	0.2729749375	2.71000E-08	7.0374E-07
0.9	0.3313503928	0.3313504205	2.77000E-08	1.0177E-06
1.0	0.3905275319	0.3905275591	2.72000E-08	1.6528E-06

Problem 3 (Sagir [17])

$$y''' + 5y'' + 7y' + 3y = 0$$
$$y(0) = 1, \quad y'(0) = 0, \quad y''(0) = -1, h = 0.1$$

Exact Solution is $y(x) = e^{-x} + xe^{-x}$

Table 3: Error of methods for problem 3

X	Exact Solution	Numerical Solution	Error in Proposed Method	Error in [17]
0.1	0.9953211598	0.9953211599	1.00000E-10	6.4300E-08
0.2	0.9824769037	0.9824769040	3.00000E-10	2.7200E-08
0.3	0.9630636869	0.9630636876	7.00000E-10	3.0500E-08
0.4	0.9384480644	0.9384480651	7.00000E-10	8.9800E-08

0.5	0.9097959895	0.9097959901	6.00000E-10	4.4260E-07
0.6	0.8780986178	0.8780986180	2.00000E-10	7.7260E-07
0.7	0.8441950165	0.8441950174	9.00000E-10	1.9523E-06
0.8	0.8087921354	0.8087921382	2.80000E-09	1.0274E-06
0.9	0.7724823534	0.7724823588	5.40000E-09	1.3509E-06
1.0	0.7357588824	0. 7357588824	3.50000E-09	1.3470E-05

Problem 4 (Badmus and Yahaya [18])

$$y''' - y'' + y' - y = 0$$
$$y(0) = 1, \quad y'(0) = 0, \quad y''(0) = -1, h = 0.01$$

Exact Solution is $y(x) = \cos x$

Table 4: Error of methods for problem 4

X	**Exact Solution**	**Numerical Solution**	**Error in Proposed Method**	**Error in [18]**
0.01	0.9999500004	0.9999506724	6.72000E07	1.60168E-05
0.02	0.9998000067	0.9998013508	1.34410E06	1.100991E-04
0.03	0.9995500337	0.9995520507	2.01700E06	5.567153E-04
0.04	0.9992001067	0.9992027951	2.68840E06	1.6332403E-03
0.05	0.9987502604	0.9987536198	3.35940E06	3.62018361E-03

5 Conclusion

We have derived a three-step continuous HLMM from which MFDMs are obtained and applied to solve third order ordinary differential equations (ODE) without first adapting the ODE to an equivalent first order system. The MFDMs are applied as simultaneous numerical integrators over sub-intervals which do not overlap and hence they are more accurate than SFDMs which are generally applied as single formulas over overlapping intervals. We have shown that the methods are convergent and have large intervals of absolute stability, which make them suitable candidates for computing solutions on wider intervals. In addition to providing additional methods and derivatives, the continuous HLMM can be used to obtain global error estimates. Our future research will be focused on adapting the MFDMs to solve third order partial differential equations.

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