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Resonance Problem of a Class of Quasilinear Parabolic Equations

Wang Zhong-Xiang[1], Jia Gao[2] and Zhang Xiao-Juan[3]

[1,2,3]College of Science, University of Shanghai for Science and Technology
Shanghai, China
[1]E-mail: wzhx5016674@126.com
[2]E-mail: gaojia89@163.com
[3]E-mail: zhangxiaojuan0609@163.com

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Abstract

In this paper, we study the resonance problem of a class of singular quasilinear parabolic equations with respect to its higher near-eigenvalues. Under a generalized Landesman-Lazer condition, it is proved that the resonance problem admits at least one nontrivial solution in weighted Sobolev spaces. The proof is based upon applying the Galerkin-type technique, the Brouwer's fixed-point theorem and a compact embedding theorem of weighted Sobolev spaces by Shapiro.

Keywords: *Weighted Sobolev Space, Quasilinear Parabolic Equation, Resonance.*

1 Introduction

Resonance problems of quasilinear elliptic (or parabolic) partial differential equations have been studied extensively in the usual Sobolev spaces. Since the celebrated paper by Landesman and Lazer [8], many existence results were obtained under various nonlinearity growth conditions and the Landesman-Lazer conditions (see [1–4, 6, 7, 9, 11–15] and references therein). However, there has been very limited existence results for the case of singular quasilinear elliptic(or parabolic) equations in the existing literature.

In 2001, Shapiro published a paper [12] on the resonance problems of singular quasilinear equations. An important element of that paper is the existence of a complete orthonormal basis in the weighted Sobolev space associated with singular coefficients of the differential operator. In that paper, a new concept of near-eigenvalues for singular quasilinear elliptic operators was introduced, a new compact embedding theorem in the weighted Sobolev spaces was established, and some new existence results for the resonance problems were obtained.

In 2002, Chung-Cheng Kuo [7] applied Galerkin-type techniques and Brouwer's fixed point theorem to obtain existence theorems of time-periodic solutions for quasilinear parabolic partial differential equations with respect to its first eigenvalue in which the Landesman-Lazer condition may be excluded.

In 2005, Rumbos and Shapiro [11] introduced a generalized Landesman-Lazer condition and studied the resonance problem of the semilinear elliptic equations with respect to its first eigenvalue by using the linking argument and a deformation theorem in weighted Sobolev spaces.

Inspired by papers [9, 10, 12, 14], we have studied the resonance problem of quasilinear or singular quasilinear elliptic(or parabolic) equations in weighted Sobolev spaces with respect to their first eigenvalues by using the Galerkin-type technique and the Brouwer's fixed-point theorem [2–4].

Motivated by [10–12], in this paper, we show the existence of solutions for a class of singular quasilinear parabolic equations with respect to its higher near-eigenvalue in the Hilbert space $\widetilde{H}(\widetilde{\Omega},\Gamma)$:

$$\begin{cases}\rho D_t u + \mathcal{M}u = (\lambda_{j_0}u + b(x,t,u)u^- + f(x,t,u))\rho - G, & (x,t)\in\widetilde{\Omega},\\ u\in\widetilde{H}(\widetilde{\Omega},\Gamma),\end{cases} \tag{P}$$

where

$$\mathcal{M}u = -\sum_{i,j=1}^{N} D_i[p_i^{\frac{1}{2}}(x)p_j^{\frac{1}{2}}(x)s_i^{\frac{1}{2}}(u)s_j^{\frac{1}{2}}(u)a_{ij}(x)D_j u] + a_0(x)s_0(u)qu, \tag{1.1}$$

and λ_{j_0} is an eigenvalue of $\mathcal{L}$.

As in paper [3], we assume the existence of a linear uniformly elliptic operator which is close to the original singular quasilinear operator in a certain sense, and hence the existence of a complete orthonormal basis in the weighted Sobolev space associated with singular coefficients of the differential operator. However, unlike the case of the first near-eigenvalue which is simple and whose eigenfunction is of one-sign, the case of higher near-eigenvalue is challenging to study due to the fact that the multiplicity of higher near-eigenvalue is greater than 1 and their corresponding eigenfunctions are sign-changing. By using a space decomposition technique, we are able to prove that the resonance problem has at least one solution under a generalized Landesman-Lazer condition.

The proof method is similar to [12] and [4], which is based also upon applying the Galerkin-type technique, the Brouwer's fixed-point theorem and the compact embedding theorem of weighted Sobolev spaces by Shapiro [12].

This paper is organized as follows. In Section 2, we describe the resonance problem of a class of singular quasilinear parabolic equations to be studied, and state the main result. In Section 3, we prove the main theorem.

2 Statement of the Problem and Main Result

Let $\Omega \subset R^N (N \geq 1)$, be an open set(possibly unbounded) and let $\rho(x), p_i(x) \in C^0(\Omega)$ be positive functions with the property that

$$\int_\Omega \rho(x)dx < \infty, \int_\Omega q(x)dx < \infty, \int_\Omega p_i(x)dx < \infty, \quad i = 1, 2, \cdots, N. \tag{2.1}$$

Let $q(x) \in C^0(\Omega)$ be a nonnegative function and $\Gamma \subset \partial\Omega$ be a fixed closed set. Note that Γ may be an empty set and $q(x)$ may be zero. On the other hand, $q(x)$ will satisfy: there exists $K > 0$, such that

$$0 \leq q(x) \leq K\rho(x), \text{ for all } x \in \Omega. \tag{2.2}$$

Here $\mathcal{A}$ is a set of real-valued functions defined as

$$\mathcal{A} = \{u : u \in C^0(\bar{\Omega} \times R), u(x, t + 2\pi) = u(x, t), \text{ for all } (x, t) \in \bar{\Omega} \times R\}.$$

Setting $\widetilde{\Omega} = \Omega \times T, T = (-\pi, \pi), p = (p_1, \cdots, p_N)$ and $D_i = \frac{\partial u}{\partial x_i} (i = 1, 2, \cdots, N)$, we consider the following pre-Hilbert spaces (see [12]):

$$\widetilde{C}^0_\rho(\widetilde{\Omega}) = \left\{ u \in C^0(\widetilde{\Omega}) : \int_{\widetilde{\Omega}} |u(x,t)|^2 \rho(x) dxdt < \infty \right\},$$

with inner product $\langle u, v\rangle^\sim_\rho = \int_{\widetilde{\Omega}} u(x,t)v(x,t)\rho(x)dxdt$, and the space

$$\widetilde{C}^1_{p,\rho}(\widetilde{\Omega}, \Gamma) = \{u \in \mathcal{A} \cap C^1(\Omega \times R) \Big| u(x,t) = 0, \text{ for all } (x,t) \in \Gamma \times R;$$

$$\int_{\widetilde{\Omega}} [\sum_{i=1}^N |D_i u|^2 p_i + (u^2 + |D_t u|^2)\rho] < \infty\}$$

with inner product

$$\langle u, v\rangle_{\widetilde{H}} = \int_{\widetilde{\Omega}} \left[\sum_{i=1}^N p_i D_i u D_i v + (uv + D_t u D_t v)\rho \right] dxdt.$$

Let $\widetilde{L}^2_\rho \triangleq L^2_\rho(\widetilde{\Omega})$ denote the Hilbert space obtained from the completion of $\widetilde{C}^0_\rho$ with the norm $||u||_\rho = (\langle u,u\rangle_{\widetilde{\rho}})^{\frac{1}{2}}$, and $\widetilde{H} \triangleq \widetilde{H}(\widetilde{\Omega},\Gamma)$ denote the completion of the space $\widetilde{C}^1_{p,\rho}$ with the norm $||u||_{\widetilde{H}} = \langle u,u\rangle^{\frac{1}{2}}_{\widetilde{H}}$. Similarly, we have $\widetilde{L}^2_{p_i}(i = 1,2,\cdots,N)$ and $\widetilde{L}^2_q$.

It is assumed throughout the paper that $s_i(u)(i=0,1,\cdots,N)$ meets:

(S1) $s_i(u)$: $\widetilde{H}\to R$ is weakly sequentially continuous;

(S2) there exist $\eta_0,\eta_1>0$ such that $\eta_0\le s_i(u)\le\eta_1$, and $s_i(u)$ is measurable, for $u\in\widetilde{H}$.

The functions $a_{ij}(i,j=1,2,\cdots,N)$ and $a_0(x)$ satisfy(also $b_{ij}(x)$ and $b_0(x)$):

(A1) $a_0(x), a_{ij}(x)\in C^0(\Omega)\bigcap L^\infty(\Omega), a_{ij}(x)=a_{ji}(x), \forall x\in\Omega$;

(A2) $a_0(x)\ge\beta_0>0,\ \forall x\in\Omega$;

(A3) there exists $c_0>0$, for $x\in\Omega$ and $\xi\in R^N$, such that $\sum\limits_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \ge c_0\mid\xi\mid^2$.

Furthermore, we assume both Caratheodory functions $b(x,t,s)$ and $f(x,t,s)$ satisfy the following conditions.

(B1) There exist constants $\delta>0$ and $k>1$ such that

$$|b(x,t,s)|\le\begin{cases}\delta|s|, & |s|\le\gamma_1,\\ \frac{\delta\gamma_1}{(|s|+1-\gamma_1)^m}, & |s|>\gamma_1,\end{cases}\tag{2.3}$$

and $0<\gamma_1<1$, where $\gamma_1=\frac{\lambda_{j_0+j_1}-\lambda_{j_0}}{k}$ and $m\ge1$.

Conditions on $f(x,t,s)$:

(f1) There exists a nonnegative function $f_0(x,t)\in\widetilde{L}^2_\rho$ such that

$$|f(x,t,s)|\le f_0(x,t),\quad \text{for } a.e.\ \ x\in\Omega \ \text{ and } \forall s\in R;$$

(f2) $\limsup_{s\to+\infty} f(x,t,s) = f^+(x,t)\in L^\infty(\Omega)$, $\liminf_{s\to-\infty} f(x,t,s) = f^-(x,t)\in L^\infty(\Omega)$.

It is, in general, difficult to study the eigenvalues and eigenfunctions of $\mathcal{M}$. Shapiro [12] introduced the concepts of near-related operators and near-eigenvalue of $\mathcal{M}$.

We first introduce some operators related to this paper.

Definition 2.1. *For the quasilinear differential operator $\mathcal{M}$, the two form is*

$$\mathcal{M}(u,v)=\sum_{i,j=1}^{N}\int_{\widetilde{\Omega}}\left[p_i^{\frac{1}{2}}p_j^{\frac{1}{2}}s_i^{\frac{1}{2}}(u)s_j^{\frac{1}{2}}(u)a_{ij}D_juD_iv\right]+\int_{\widetilde{\Omega}}qs_0(u)a_0uv,\ \ u,v\in\widetilde{H}(\widetilde{\Omega},\Gamma).\tag{2.4}$$

Defining

$$\mathcal{L}_xu=-\sum_{i,j=1}^{N}D_i\left[p_i^{\frac{1}{2}}p_j^{\frac{1}{2}}b_{ij}D_ju\right]+b_0qu,\tag{2.5}$$

for $u \in H_{p,q,\rho} = H_{p,q,\rho}(\Omega,\Gamma)$ *(as described in [12]), and*

$$\mathcal{L}u = -\sum_{i,j=1}^{N} D_i\left[p_i^{\frac{1}{2}}p_j^{\frac{1}{2}}b_{ij}D_ju\right] + a_0qu,\ \ u \in \widetilde{H}(\widetilde{\Omega},\Gamma), \tag{2.6}$$

then the bilinear form of $\mathcal{L}_x$ *is*

$$\mathcal{L}_x(u,v) = \sum_{i,j=1}^{N}\int_\Omega p_i^{\frac{1}{2}}p_j^{\frac{1}{2}}b_{ij}(x)D_juD_iv + \int_\Omega b_0uvq,\ \ u,v \in H_{p,q,\rho}(\Omega,\Gamma), \tag{2.7}$$

and the bilinear form of $\mathcal{L}$ *is*

$$\mathcal{L}(u,v) = \sum_{i,j=1}^{N}\int_{\widetilde{\Omega}} p_i^{\frac{1}{2}}p_j^{\frac{1}{2}}b_{ij}(x)D_juD_iv + \int_{\widetilde{\Omega}} b_0uvq,\quad u,v \in \widetilde{H}(\widetilde{\Omega},\Gamma). \tag{2.8}$$

□

We further assume that domain Ω and operator $\mathcal{L}_x$ satisfy the so-called $V_L(\Omega,\Gamma)$ conditions [12, 14]:

(V_L-1) There exists a complete orthonormal sequence of functions $\{\varphi_n\}_{n=1}^{\infty}$ in $L^2_\rho(\Omega)$, such that $\varphi_n \in H^1_{p,q,\rho}(\Omega,\Gamma) \cap C^2(\Omega)$ for all n.

(V_L-2) The uniformly elliptic operator $\mathcal{L}_x$ has a sequence of real eigenvalues $\{\lambda_n\}_{n=1}^{\infty}$ corresponding to the orthonormal sequence $\{\varphi_n\}_{n=1}^{\infty}$, satisfying

$$0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_n \to \infty \ \text{ as } \ n \to \infty,$$

and

$$\mathcal{L}_x(\varphi_n, v) = \lambda_n\langle\varphi_n, v\rangle_\rho,\ \forall v \in H^1_{p,q,\rho}(\Omega,\Gamma) \text{ and } n \geq 1.$$

Also $\varphi_1 > 0$ in Ω.

Here $\langle u,v\rangle_\rho = \int_{\widetilde{\Omega}} uv\rho$. For the sake of simplicity, in the following, we will denote $\langle u,v\rangle_\rho$ as $\langle u,v\rangle$.

Examples of operators and domains for which the $V_L(\Omega,\Gamma)$ conditions hold can be found in [12](pp. 20-26). The $V_L(\Omega,\Gamma)$ conditions play a key role in our study of the resonance problem of singular quasilinear elliptic equations.

Definition 2.2. *Operator* $\mathcal{M}$ *is said to be near-related to operator* $\mathcal{L}$ *(denoted as* $\mathcal{M} \sim \mathcal{L}$ *for convenience), if, for any* $v \in \widetilde{H}$,

$$\lim_{||u||_{\widetilde{H}} \to \infty} \frac{\mathcal{M}(u,v) - \mathcal{L}(u,v)}{||u||_{\widetilde{H}}} = 0. \tag{2.9}$$

□

Definition 2.3. *Assume* $\mathcal{M} \sim \mathcal{L}$ *in* $\widetilde{H}$. λ *is called a near-eigenvalue of* $\mathcal{M}$ *if*

(1) λ *is an eigenvalue of* $\mathcal{L}_x$; *(2)* $\lim_{||u||_{\widetilde{H}}\to\infty} \frac{\mathcal{M}(u,P_\lambda u)-\mathcal{L}(u,P_\lambda u)}{||u||_{\widetilde{H}}} = 0$,
where P_λ *is the orthogonal projection from* $L^2_\rho(\Omega)$ *onto the eigenspace of* $\mathcal{L}_x$ *corresponding to the eigenvalue* λ. □

We now state the main result of this paper:

Theorem 2.4. *Let* $\Omega \subset R^N (N \geq 1)$, $T = (-\pi, \pi)$, $\widetilde{\Omega} = \Omega \times T$, $p = (p_1, \cdots, p_N)$, ρ *and* $p_i (i = 1, \cdots, N)$ *be positive functions in* $C^0(\Omega)$ *satisfying (2.1),* $q \in C^0(\Omega)$ *be a nonnegative function satisfying (2.2), and* $\Gamma \subset \partial\Omega$ *be a closed set. Let* $\mathcal{M}$ *and* $\mathcal{L}$ *be given by (1.1) and (2.6) satisfying (S1)-(S2), (A1)-(A3) respectively and* $\mathcal{L}_x$ *satisfies the conditions of* $V_L(\Omega, \Gamma)$. *If* $\mathcal{M} \sim \mathcal{L}$, λ_{j_0} *is a near-eigenvalue of* $\mathcal{M}$ *of multiplicity* j_1, *(B1) and (f1)-(f2) hold, and* $G \in (\widetilde{H})^*$, *then the problem (P) has at least one weak solution; i.e., there exits* $u^* \in \widetilde{H}$ *such that*

$$\langle D_t u^*, v\rangle_\rho + \mathcal{M}(u^*, v) = \lambda_{j_0}\langle u^*, v\rangle_\rho + \langle f(x,t,u^*) + g(x,t,u^*), v\rangle_\rho - G(v), \ \forall v \in \widetilde{H}. \tag{2.10}$$

Here, we will introduce some lemmas and concepts which will be used later. If (A1)-(A3) and the conditions of $V_L(\Omega, \Gamma)$ hold, we have

$$\{\widetilde{\varphi}^c_{jk}\}^{\infty,\infty}_{j=1,k=0} \cup \{\widetilde{\varphi}^s_{jk}\}^{\infty,\infty}_{j=1,k=1} \ \text{ is a CONS for } \ \widetilde{L}^2_\rho, \tag{2.11}$$

where

$$\widetilde{\varphi}^c_{jk}(x,t) = \begin{cases} \frac{\varphi_j(x)}{\sqrt{2\pi}}, & k = 0, j = 1, 2, \cdots, \\ \frac{\varphi_j(x)\cos(kt)}{\sqrt{\pi}}, & k, j = 1, 2, \cdots, \end{cases} \tag{2.12}$$

and

$$\widetilde{\varphi}^s_{jk}(x,t) = \frac{\varphi_j(x)\sin(kt)}{\sqrt{\pi}}, \ k, j = 1, 2, \cdots. \tag{2.13}$$

Obviously, both $\widetilde{\varphi}^c_{jk}$ and $\widetilde{\varphi}^s_{jk}$ are in $\widetilde{H}(\widetilde{\Omega}, \Gamma)$.

Lemma 2.5. *If* $\{\widetilde{\varphi}^c_{jk}\}^{\infty,\infty}_{j=1,k=0} \cup \{\widetilde{\varphi}^s_{jk}\}^{\infty,\infty}_{j=1,k=1}$ *is a* CONS *for* $L^2_\rho(\widetilde{\Omega})$ *defined by (2.11), setting*

$$\tau_n(v) = \sum_{j=1}^n \widehat{v}^c(j,0)\widetilde{\varphi}^c_{j0} + \sum_{j=1}^n \sum_{k=1}^n \left[\widehat{v}^c(j,k)\widetilde{\varphi}^c_{jk} + \widehat{v}^s(j,k)\widetilde{\varphi}^s_{jk}\right], \tag{2.14}$$

we have

$$\lim_{n\to\infty} ||\tau_n(v) - v||_{\widetilde{H}} = 0, \quad \text{for all } v \in \widetilde{H}. \tag{2.15}$$

Lemma 2.6. (i) *If* $v \in \widetilde{H}$, *then*

$$\mathcal{L}_1(v,v) + ||D_t v||_\rho^2 = \sum_{j=1}^{\infty} |\widehat{v}^c(j,0)|^2 (\lambda_j + 1) + \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \left[|\widehat{v}^c(j,k)|^2 + |\widehat{v}^s(j,k)|^2\right]\left(\lambda_j + 1 + k^2\right). \tag{2.16}$$

(ii) *If* $v \in L_\rho^2(\widetilde{\Omega})$ *and* $\mathcal{L}_1(v,v) + ||D_t v||_\rho^2 < \infty$, *then* $v \in \widetilde{H}$. *Here* $\mathcal{L}_1(v,v) = \mathcal{L}(v,v) + < v, v >$.

Lemma 2.7. *Let* $\widetilde{\Omega}, \rho, p, q$, *and* $\mathcal{L}$ *be as in the hypothesis of Theorem 2.1 and assume that* (Ω, Γ) *is a* $V_L(\Omega,\Gamma)$. *Then* $\widetilde{H}$ *is compactly imbedded in* $L_\rho^2(\widetilde{\Omega})$.

The proofs of Lemmas 2.1-2.3 can be found in [12]. We define the set

$$S_n = \left\{ v \in \widetilde{H} : v = \sum_{j=1}^{n} \eta_{j0}^c \widetilde{\varphi}_{j0}^c + \sum_{j=1}^{n}\sum_{k=1}^{n} \eta_{jk}^c \widetilde{\varphi}_{jk}^c + \eta_{jk}^s \widetilde{\varphi}_{jk}^s,\ \eta_{jk}^c, \eta_{jk}^s \in R \right\}. \tag{2.17}$$

Remark 2.8. *(1) If* $u_n \in S_n$, *then* $\mathcal{M}(u_n, D_t u_n) = 0$; *(2)* $\langle D_t(\alpha\widetilde{\varphi}_{jk}^c + \beta\widetilde{\varphi}_{jk}^s), \alpha\widetilde{\varphi}_{jk}^c + \beta\widetilde{\varphi}_{jk}^s\rangle = 0,\ j,k \geq 1, \alpha, \beta \in R$.

3 Proof of Theorem 2.1

The proof of Theorem 2.1 can be divided into three steps. The first step is to construct a set of approximate solutions $\{u_n\}$ of (2.10) in $\widetilde{H}$, where $u_n \in S_n$ and S_n is defined as in (2.17). Then we show in the second step that $\{u_n\}$ is bounded in $\widetilde{H}$. Finally, we show $\{u_n\}$ converges to a weak solution $u^* \in \widetilde{H}$ of (2.10).

Lemma 3.1. *Assume that all the conditions in the hypothesis of Theorem 2.1 hold. Let* S_n *be the subspace of* $\widetilde{H}$ *defined by (2.17). Taking* $n_0 = j_0 + j_1$ *and* $\gamma_0 = \frac{1}{2}(\lambda_{j_0+j_1} - \lambda_{j_0})$, *then for* $n \geq n_0$, *there is a function* $u_n \in S_n$ *with the property that*

$$\begin{aligned}\langle D_t u_n, v\rangle + \mathcal{M}(u_n, v) =& (\lambda_{j_0} + \gamma_0 n^{-1})\langle u_n, v\rangle + \langle b(x,t,u_n)(u_n)^-, v\rangle \\ &+ (1-n^{-1})\langle f(x,t,u_n), v\rangle - G(v),\ \forall v \in S_n.\end{aligned} \tag{3.1}$$

Proof. Let $\{\psi_i\}_{i=1}^{2n^2+n}$ be an enumeration of $\{\widetilde{\varphi}_{jk}^c\}_{j=1,k=0}^{n,n} \cup \{\widetilde{\varphi}_{jk}^s\}_{j=1,k=1}^{n,n}$, and set

$$n^* = (j_0 + j_1 - 1)(2n+1). \tag{3.2}$$

So $\{\psi_i\}_{i=1}^{n^*}$ is an enumeration of $\{\widetilde{\varphi}_{jk}^c\}_{j=1,k=0}^{j_0+j_1-1,n} \cup \{\widetilde{\varphi}_{jk}^s\}_{j=1,k=1}^{j_0+j_1-1,n}$, where $n \geq n_0$.

With this enumeration defined, for $\alpha = (\alpha_1, \cdots, \alpha_{2n^2+n})$, we set

$$u = \sum_{i=1}^{2n^2+n} \alpha_i \psi_i, \qquad \widetilde{u} = \sum_{i=1}^{2n^2+n} \delta_i \alpha_i \psi_i, \tag{3.3}$$

where $\delta_i = -1$, if $1 \le i \le n^*$; $\delta_i = 1$, if $n^* + 1 \le i \le 2n^2 + n$, and define

$$\begin{aligned} F_i(\alpha) = \langle D_t u, \delta_i \psi_i \rangle + \mathcal{M}(u, \delta_i \psi_i) - (\lambda_{j_0} + \gamma_0 n^{-1}) \langle u, \delta_i \psi_i \rangle \\ -\langle b(x,t,u)u^-, \delta_i \psi_i \rangle - (1 - n^{-1}) \langle f(x,t,u), \delta_i \psi_i \rangle + G(\delta_i \psi_i). \end{aligned} \tag{3.4}$$

It is clear from orthogonality that $\langle D_t u, \widetilde{u} \rangle = 0$. From (3.3) and (3.4) we get

$$\begin{aligned} \sum_{i=1}^{2n^2+n} F_i(\alpha)\alpha_i = \mathcal{M}(u, \widetilde{u}) - (\lambda_{j_0} + \gamma_0) \langle u, \widetilde{u} \rangle \\ -\langle b(x,t,u)u^-, \widetilde{u} \rangle - (1 - n^{-1}) \langle f(x,t,u) - \gamma_0 u, \widetilde{u} \rangle + G(\widetilde{u}). \end{aligned} \tag{3.5}$$

Then

$$\sum_{i=1}^{2n^2+n} F_i(\alpha)\alpha_i = I(\alpha) + II(\alpha), \tag{3.6}$$

where

$$\begin{aligned} I(\alpha) =& \mathcal{L}(u, \widetilde{u}) - (\lambda_{j_0} + \gamma_0) \langle u, \widetilde{u} \rangle - \langle b(x,t,u)u^-, \widetilde{u} \rangle \\ & - (1 - n^{-1}) \langle f(x,t,u) - \gamma_0 u, \widetilde{u} \rangle + G(\widetilde{u}), \\ II(\alpha) =& \mathcal{M}(u, \widetilde{u}) - \mathcal{L}(u, \widetilde{u}). \end{aligned}$$

Consider $I(\alpha)$ in (3.6) first. Note that $\gamma_0 = \frac{1}{2}(\lambda_{j_0+j_1} - \lambda_{j_0})$ and $\delta_j(\lambda_j - \lambda_{j_0} - \gamma_0) \ge \gamma_0 (j = 1, 2, \cdots, n)$, then

$$\mathcal{L}(u, \widetilde{u}) - (\lambda_{j_0} + \gamma_0) \langle u, \widetilde{u} \rangle > \gamma_0 |\alpha|^2. \tag{3.7}$$

By condition (B1), we have

$$\begin{aligned} |\langle b(x,t,u)u^-, \tilde{u} \rangle_\rho| \le & \int_{\widetilde{\Omega} \cap \{|u| \le \gamma_1\}} |u|^2 |\tilde{u}| \rho + \delta \gamma_1 \int_{\widetilde{\Omega} \cap \{|u| > \gamma_1\}} \frac{|u||\tilde{u}|\rho}{(|u| + 1 - \gamma_1)^m} \\ \le & c|\alpha|. \end{aligned} \tag{3.8}$$

From (f1), Hölder inequality and Minkowski inequality, we have

$$|\langle f(x,t,u) - \gamma_0 u, \ \tilde{u} \rangle| \le \gamma_0 |\alpha|^2 + ||f_0||_\rho |\alpha|. \tag{3.9}$$

Note that $G \in (\widetilde{H})^*$. It follows from Lemma 2.3 that, for each given $n \ge j_0 + j_1$,

$$|G(\tilde{u})| \le c|\alpha|. \tag{3.10}$$

Thus, it follows from (3.7)-(3.10) that

$$I(\alpha) > \frac{1}{n}\gamma_0|\alpha|^2 - c|\alpha|. \tag{3.11}$$

By $\mathcal{M} \sim \mathcal{L}$ and $||u||_\rho^2 = ||\widetilde{u}||_\rho^2 = |\alpha|^2$, we have

$$\lim_{|\alpha|\to\infty} \frac{II(\alpha)}{|\alpha|^2} = \lim_{|\alpha|\to\infty} \frac{\mathcal{M}(u,\tilde{u}) - \mathcal{L}(u,\tilde{u})}{|\alpha|^2} = 0. \tag{3.12}$$

Thus it follows from (3.6), (3.11) and (3.12) that, for any given $n \geq j_0 + j_1$, there exists $A_0 > 0$ such that $\sum_{i=1}^n F_i(\alpha)\alpha_i > 0$ for $|\alpha| \geq A_0$. Under the assumptions of Theorem 2.1, it is straightforward to verify that $F_i : \mathbb{R}^n \to \mathbb{R}$ is continuous for $1 \leq i \leq n$. By applying the Brouwer's fixed-point theorem [5], there exists $\alpha^* = (\alpha_1^*, \alpha_2^*, \cdots, \alpha_n^*) \in \mathbb{R}^n$ such that $F_i(\alpha^*) = 0$ for $1 \leq i \leq n$. Let $u_n^* = \sum_{i=1}^n \alpha_i^*\varphi_i \in S_n$. It follows from (3.4) that u_n^* is a solution of (3.1). □

In next step, we will prove that $\{u_n^*\}_{n=j_0+j_1}^\infty$ is bounded in $\widetilde{H}$.

Lemma 3.2. *Assume the conditions in Lemma 3.1 hold, and $\{u_n^*\}_{n=j_0+j_1}^\infty \subset \widetilde{H}$ is the sequence of solutions obtained in Lemma 3.1. Assume further $G \in (\widetilde{H})^*$ satisfies the following generalized Landesman-Lazer condition:*

$$G(w) < \int_{\widetilde{\Omega}_1} f^+(x,t)w(x)\rho + \int_{\widetilde{\Omega}_2} f^-(x,t)w(x)\rho(x), \tag{3.13}$$

for every nontrivial λ_{j_0}-eigenfunction w of $\mathcal{L}_x$, where $\widetilde{\Omega}_i = \Omega_i \times (-\pi,\pi)(i = 1,2)$, $\Omega_1 = \{x \in \Omega; w(x) > 0\}$ and $\Omega_2 = \{x \in \Omega; w(x) < 0\}$. Then $\{u_n^*\}$ is bounded in $\widetilde{H}$.

Proof. For simplicity of notation, we denote $\{u_n^*\}_{n=j_0+j_1}^\infty$ by $\{u_n\}_{n=j_0+j_1}^\infty$. It follows from Lemma 3.1 that $u_n \in S_n$ and u_n satisfies

$$\begin{aligned}\langle D_t u_n, v\rangle + \mathcal{M}(u_n, v) = (\lambda_{j_0} + \gamma_0 n^{-1})\langle u_n, v\rangle + \langle b(x,t,u_n)(u_n)^-, v\rangle \\ +(1-n^{-1})\langle f(x,t,u_n), v\rangle - G(v), \quad \forall v \in S_n,\end{aligned} \tag{3.14}$$

where $\gamma_0 = (\lambda_{j_0+j_1} - \lambda_{j_0})/2$, and $n \geq n_0 = j_0 + j_1$.

In order to prove Lemma 3.2, we only need to prove that there exists a constant such that $\{u_n\}$ obtained by Lemma 3.1 satisfies

$$\|u_n\|_{\widetilde{H}} \leq K. \tag{3.15}$$

Assume that (3.15) dose not hold. Then there exists a subsequence of $\{u_n\}$, denoted again by $\{u_n\}$, such that

$$\lim_{n\to\infty} \|u_n\|_{\widetilde{H}} = \infty. \tag{3.16}$$

Letting $v = D_t u_n$ in (3.14), by (f2), $\langle D_t u_n, u_n\rangle = 0$ and $\mathcal{M}(D_t u_n, u_n) = 0$, we have

$$\begin{aligned}|\langle b(x,t,u_n)u_n^-, D_t u_n\rangle| &\leq \int_{\widetilde{\Omega}\cap\{|u_n|\leq\gamma_1\}} |u_n|^2 |D_t u_n|\rho \\ &\quad +\delta\gamma_1 \int_{\widetilde{\Omega}\cap\{|u_n|>\gamma_1\}} \frac{|u_n|\cdot|D_t u_n|\rho}{(|u_n|+1-\gamma_1)^m} \\ &\leq c(\delta,\gamma_1,|\widetilde{\Omega}|)\|D_t u_n\|_\rho\ ,\end{aligned}$$

and we can conclude that there exists $K > 0$ such that

$$\|D_t u_n\|\rho \leq K. \tag{3.17}$$

Under conditions (B1) and (S2), it follows from (1.1) that

$$\mathcal{M}(u_n,u_n) \geq c_0\Big(\sum_{i=1}^{N} \|D_i u_n\|_{p_i}^2 + \|u_n\|_q^2\Big),$$

where c_0 is a positive constant. Then we have

$$c_1\|u_n\|_{\widetilde{H}}^2 \leq \mathcal{M}(u_n,u_n) + c_2(\|u_n\|_\rho^2 + \|D_t u_n\|_\rho^2). \tag{3.18}$$

Now by letting $v = u_n$ in (3.14), and the proof of (3.9), we have

$$|\langle f(x,t,u_n) - \gamma_0 u_n, u_n\rangle| \leq \gamma_0\|u_n\|_\rho^2 + K\|u_n\|_\rho. \tag{3.19}$$

From (B1) and Hölder inequality, we have

$$\begin{aligned}|\langle b(x,t,u_n)u_n^-, u_n\rangle| &\leq \int_{\widetilde{\Omega}\cap\{|u_n|\leq\gamma_1\}} \delta|u_n|^3\rho + \delta\gamma_1\int_{\widetilde{\Omega}\cap\{|u_n|>\gamma_1\}} \frac{|u_n|^2\rho}{(|u_n|+1-\gamma_1)^m} \\ &\leq c_2^*(\delta,\gamma_1,|\widetilde{\Omega}|)\|u_n\|_\rho^{2-m} + c_3^*(\delta,\gamma_1,|\widetilde{\Omega}|).\end{aligned} \tag{3.20}$$

Then by (3.19), (3.20) and $\langle D_t u_n, u_n\rangle = 0$, we have

$$\begin{aligned}c_1\|u_n\|_{\widetilde{H}}^2 &\leq (\lambda_{j_0}+\gamma_0)\langle u_n,u_n\rangle + \langle b(x,t,u_n)u_n^-, u_n\rangle \\ &\quad + (1-n^{-1})\langle f(x,t,u_n)-\gamma_0 u_n, u_n\rangle - G(u_n) + c_1(\|u_n\|_\rho^2 + \|D_t u_n\|_\rho^2) \\ &\leq K_4\|u_n\|_\rho^2 + K\|u_n\|_{\widetilde{H}} + c_2^*(\delta,\gamma_1,|\widetilde{\Omega}|)\|u_n\|_\rho^{2-m} + c_3^*(\delta,\gamma_0,|\widetilde{\Omega}|),\end{aligned}$$

where $K_4 = \lambda_{j_0} + 2\gamma_0 + c_1$, and $m > 1$. Dividing both sides of the above inequalities by $\|u_n\|_{\widetilde{H}}^2$ and then by (3.16), we know that there exists $n_1(n_1 \geq n_0)$ such that

$$0 < \frac{c_1}{K_4} \leq \frac{\|u_n\|_\rho^2}{\|u_n\|_{\widetilde{H}}^2} \leq 1, \quad \forall n \geq n_1.$$

Noticing (3.16), the above inequalities establish if and only if

$$\lim_{n\to\infty} \|u_n\|_\rho = \infty, \tag{3.21}$$

that is, there exists $K > 0$ such that

$$\|u_n\|_{\widetilde{H}} \le K\|u_n\|_\rho, \quad \forall n \ge n_1. \tag{3.22}$$

Rewrite u_n as $u_n = u_{n1} + u_{n2} + u_{n3}$, and let $\tilde{u}_n = -u_{n1} - u_{n2} + u_{n3}$, where

$$\begin{cases} u_{n1} = \sum_{j=1}^{j_0-1} \widehat{u}_n^c(j,0)\widetilde{\varphi}_{j0}^c + \sum_{j=1}^{j_0-1}\sum_{k=1}^{n}(\widehat{u}_n^c(j,k)\widetilde{\varphi}_{jk}^c + \widehat{u}_n^s(j,k)\widetilde{\varphi}_{jk}^s), \\ u_{n2} = \sum_{j=j_0}^{j_0+j_1-1} \widehat{u}_n^c(j,0)\widetilde{\varphi}_{j0}^c + \sum_{j=j_0}^{j_0+j_1-1}\sum_{k=1}^{n}(\widehat{u}_n^c(j,k)\widetilde{\varphi}_{jk}^c + \widehat{u}_n^s(j,k)\widetilde{\varphi}_{jk}^s), \\ u_{n3} = \sum_{j=j_0+j_1}^{n} \widehat{u}_n^c(j,0)\widetilde{\varphi}_{j0}^c + \sum_{j=j_0+j_1}^{n}\sum_{k=1}^{n}(\widehat{u}_n^c(j,k)\widetilde{\varphi}_{jk}^c + \widehat{u}_n^s(j,k)\widetilde{\varphi}_{jk}^s). \end{cases} \tag{3.23}$$

First, for given any $n \ge n_1$, we can prove the following conclusion

$$\lim_{n\to\infty} \frac{\|u_{n1}\|_{\widetilde{H}} + \|u_{n3}\|_{\widetilde{H}}}{\|u_n\|_\rho} = 0. \tag{3.24}$$

As a result, from (3.14) with $v = \widetilde{u}_n$, we have

$$\begin{aligned} &\langle b(x,t,u_n)(u_n)^-, \tilde{u}_n\rangle + (1-n^{-1})\langle f(x,t,u_n) - \gamma_0 u_n, \tilde{u}_n\rangle \\ &-G(\tilde{u}_n) + \mathcal{L}(u_n,\tilde{u}_n) - \mathcal{M}(u_n,\tilde{u}_n) \\ =&\sum_{j=1}^{n} \delta_j(\lambda_j - \lambda_{j_0} - \gamma_0)|\hat{u}_n^c(j,0)|^2 \\ &+\sum_{j,k=1}^{n} \delta_j(\lambda_j - \lambda_{j_0})[|\hat{u}_n^c(j,k)|^2 + |\hat{u}_n^s(j,k)|^2]. \end{aligned} \tag{3.25}$$

Since

$$\begin{aligned} (3.25)_R =&\gamma_0\|u_n\|_\rho^2 + \sum_{j=1}^{j_0+j_1-1}(\lambda_{j_0} - \lambda_j)|\hat{u}_n^c(j,0)|^2 + \sum_{j=j_0+j_1}^{n}(\lambda_j - \lambda_{j_0} - 2\gamma_0)|\hat{u}_n^c(j,0)|^2 \\ &+ \sum_{j=1}^{j_0+j_1-1}\sum_{k=1}^{n}(\lambda_{j_0} - \lambda_j)[|\hat{u}_n^c(j,k)|^2 + |\hat{u}_n^s(j,k)|^2] \\ &+ \sum_{j=j_0+j_1}^{n}\sum_{k=1}^{n}(\lambda_j - \lambda_{j_0} - 2\gamma_0)[|\hat{u}_n^c(j,k)|^2 + |\hat{u}_n^s(j,k)|^2], \end{aligned}$$

by (3.8) and the proof of (3.9), we get

$$(3.25)_L \le \gamma_0\|u_n\|_\rho^2 + c^*(\delta,\gamma_1,|\widetilde{\Omega}|,K)\|u_n\|_\rho + \mathcal{L}(u_n,\tilde{u}_n) - \mathcal{M}(u_n,\tilde{u}_n).$$

In this way, it follows from (3.25) that

$$(3.25)_R \leq \gamma_0\|u_n\|_\rho^2 + c^*(\delta,\gamma_1,|\widetilde{\Omega}|,K)\|u_n\|_\rho + \mathcal{L}(u_n,\tilde{u}_n) - \mathcal{M}(u_n,\tilde{u}_n). \quad (3.26)$$

For fixed n, there exists a constant $\gamma' > 0$ such that

$$\gamma'(1+\lambda_k) \leq \lambda_{j_0} - \lambda_k,\ k = 1,2,\cdots j_0 - 1,$$

$$\gamma'(1+\lambda_k) \leq \lambda_k - \lambda_{j_0} - 2\gamma_0,\ k \geq j_0 + j_1.$$

Since

$$\mathcal{L}_1(u_n,u_n) = \sum_{j=1}^{n}(1+\lambda_j)\hat{u}_n^c(j,0)\widetilde{\varphi}_{j0}^c + \sum_{j=1}^{n}\sum_{k=1}^{n}(1+\lambda_j)[\hat{u}_n^c(j,k)\widetilde{\varphi}_{jk}^c + \hat{u}_n^s(j,k)\widetilde{\varphi}_{jk}^s],$$

by (3.26) and the above inequalities, there exists $\gamma^* > 0$ such that

$$\gamma^*(\|u_{n1}\|_{\widetilde{H}^2} + \|u_{n3}\|_{\widetilde{H}^2}) \leq c^*\|u_n\|_\rho + \mathcal{L}(u_n,\tilde{u}_n) - \mathcal{M}(u_n,\tilde{u}_n) + K.$$

Dividing both sides of the above inequality by $\|u_n\|_\rho^2$ and taking the limit as $n \to \infty$, it follows from (3.21) and $\mathcal{M} \sim \mathcal{L}$ that (3.23) establishes.

Next, taking use of the notation of (3.23) and letting

$$w_n = \frac{u_n}{\|u_n\|_\rho}, \quad w_{ni} = \frac{u_{ni}}{\|u_n\|_\rho},\ i = 1,2,3, \quad (3.27)$$

thus by (3.22), there exists $K > 0$ such that

$$\|w_n\|_{\widetilde{H}} \leq K \text{ and } \|w_{ni}\|_{\widetilde{H}} \leq K,\ i = 1,2,3, \quad \forall n \geq n_1, \quad (3.28)$$

that is, $\|w_n\|_{\widetilde{H}}$ is a bounded sequence in $\widetilde{H}$. As $\widetilde{H}$ is a separable Hilbert space, by Lemma 2.3 and (3.28), there exists a subsequence of w_n(denoted again by w_n) and $w \in \widetilde{H}$ such that

$$\begin{cases} (1) \lim\limits_{n\to\infty} ||w_n - w||_{\widetilde{H}} = 0; \\ (2) \exists w^* \in \widetilde{L}_\rho^2,\ s.t. |w_n(x,t)| \leq w^*(x,t), \quad a.e. \quad (x,t) \in \widetilde{\Omega}; \\ (3) \lim\limits_{n\to\infty} w_n(x,t) = w(x,t), \quad a.e. \quad (x,t) \in \overline{\widetilde{\Omega}}. \end{cases} \quad (3.29)$$

Since $\mathcal{M} \sim \mathcal{L}$, we get from (3.28) that

$$\lim_{n\to\infty} \frac{\mathcal{M}(u_n,w_{ni}) - \mathcal{L}(u_n,w_{ni})}{\|u_n\|_\rho} = 0,\ i = 1,2,3.$$

We observe from (3.24) that $\lim\limits_{n\to\infty}\|w_{n3}\|_\rho = 0$. Hence, if $n \to \infty$, then

$$\langle w_n, \widetilde{\varphi}_{jk}^c\rangle = \langle w_{n3}, \widetilde{\varphi}_{jk}^c\rangle \to 0,\ j \geq j_0 + j_1.$$

Now by (3.29), we get $\hat{w}^c(j,k)=0$, for $j\ge j_0+j_1$ and all k. Similarly, we have $\hat{w}^s(j,k)=0$, for $j\ge j_0+j_1$ and all k. By (3.24), we gain $\lim_{n\to\infty}\|w_{n1}\|_\rho=0$, similarly, we can obtain $\hat{w}^c(j,k)=0$ and $\hat{w}^s(j,k)=0$, for $1\le j\le j_0-1$ and all k. Thus, we get

$$\begin{cases}\hat{w}^c(j,k)=0 \text{ and } \hat{w}^s(j,k)=0, \text{ for } j\ge j_0+j_1 \text{ and all } k;\\ \hat{w}^c(j,k)=0 \text{ and } \hat{w}^s(j,k)=0, \text{ for } 1\le j\le j_0-1 \text{ and all } k.\end{cases}\tag{3.30}$$

Hence, letting $v=D_tu_n$ in (3.14), and by $\mathcal{M}(u_n,D_tu_n)=0$, Schwarz inequality and $G\in(\widetilde{H})^*$, we get

$$\|D_tu_n\|_\rho\le\|f(x,t,u_n)\|_\rho+c(\delta,\gamma_1,|\widetilde{\Omega}|).$$

Therefore, we have

$$\lim_{n\to\infty}\frac{\|D_tu_n\|_\rho^2}{\|u_n\|_\rho^2}=0,$$

that is,

$$\lim_{n\to\infty}\|D_tw_n\|_\rho^2=0.\tag{3.31}$$

On the other hand, for $k\ge1$ and $j_0\le j\le j_0+j_1-1$, from (2.12), (2.13) and (3.31), we know

$$k\hat{w}^c(j,k)=-\lim_{n\to\infty}\int_{\widetilde{\Omega}}D_tw_n(x,t)\varphi_{jk}^s(x,t)\rho(x)dxdt=0.$$

A similar situaion prevails for $k\hat{w}^s(j,k)=0$. So we have

$$\hat{w}^c(j,k)=0\ \text{ and }\ \hat{w}^s(j,k)=0,$$

for $k\ge1$ and $\mathrm{j}_0\le j\le j_0+j_1-1$. Hence, we know that $w(x,t)$ is a function unrelated to t; i.e.,

$$w(x,t)\equiv w(x)=\sum_{j=j_0}^{j_0+j_1-1}\widehat{w}^c(j,0)\widetilde{\varphi}_{j0}^c(x).\tag{3.32}$$

Replacing v by u_{n2} in (3.14), and by (V_L-2), for $\forall n\ge n_1$, we have

$$\begin{aligned}&(1-n^{-1})\langle f(x,t,u_n),u_{n2}\rangle-G(u_{n2})+\mathcal{L}(u_n,u_{n2})-\mathcal{M}(u_n,u_{n2})\\&\le-\gamma_0n^{-1}\|u_{n2}\|_\rho^2+|\langle b(x,t,u_n)(u_n)^-,u_{n2}\rangle|\le\langle|b(x,t,u_n)(u_n)^-,u_{n2}\rangle|.\end{aligned}\tag{3.33}$$

On the other hand, we have

$$\begin{aligned}|\langle b(x,t,u_n)(u_n)^-,u_{n2}\rangle|\le{}& int_{\widetilde{\Omega}}|b(x,t,u_n)|\cdot|u_n|^2\rho\\&+\int_{\widetilde{\Omega}}|b(x,t,u_n)u_n(u_{n1}+u_{n3})|\rho.\end{aligned}\tag{3.34}$$

By (B1) and the computing method of (3.20), we can get

$$\int_{\widetilde{\Omega}} |b(x,t,u_n)| \cdot |u_n|^2 \rho \leq \delta\gamma_1^4 |\widetilde{\Omega}| + (\delta\gamma_1)^2 \int_{\widetilde{\Omega}\cap\{|u_n|>\gamma_1\}} \rho = c_4(\delta, \gamma_1, |\widetilde{\Omega}|). \quad (3.35)$$

So, by (3.35), we can obtain

$$\int_{\widetilde{\Omega}} |b(x,t,u_n) u_n (u_{n1} + u_{n3})| \rho \leq c_4^*(\delta, \gamma_1, \widetilde{\Omega}) \|u_{n1} + u_{n3}\|_\rho. \quad (3.36)$$

By using of (3.34)-(3.36), then it follows from (3.33) that

$$\begin{aligned} &(1 - n^{-1})\langle f(x,t,u_n), u_{n2}\rangle - G(u_{n2}) + \mathcal{L}(u_n, u_{n2}) - \mathcal{M}(u_n, u_{n2}) \\ &\leq c_2^*(\delta, \gamma_1, |\widetilde{\Omega}|) \|u_n\|_\rho^{2-m} + c_3^*(\delta, \gamma_1, |\widetilde{\Omega}|) + c_4^*(\delta, \gamma_1, |\widetilde{\Omega}|) \|u_{n1} + u_{n3}\|_\rho. \end{aligned} \quad (3.37)$$

Dividing by $\|u_n\|_\rho$ on both sides of (3.37), we get

$$\begin{aligned} &(1 - n^{-1})\langle f(x,t,u_n), w_n\rangle - G(w_n) + (\mathcal{L}(u_n, u_{n2}) - \mathcal{M}(u_n, u_{n2}))/\|u_n\|_\rho \\ &\leq c_2^*(\delta, \gamma_1, |\widetilde{\Omega}|) \|u_n\|_\rho^{1-m} + c_3^*(\delta, \gamma_1, |\widetilde{\Omega}|)/\|u_n\|_\rho \\ &+ c_4^*(\delta, \gamma_1, |\widetilde{\Omega}|) \|u_{n1} + u_{n3}\|_\rho / \|u_n\|_\rho. \end{aligned} \quad (3.38)$$

From (f2) and (3.29)(2), there exists K such that

$$\int_{\widetilde{\Omega}} f(x,t,u_n) w_n \rho \leq \|h(x,t)\|_\rho \|w^*(x,t)\|_\rho \leq K. \quad (3.39)$$

Because of $\mathcal{M} \sim \mathcal{L}$, by (3.21) and (3.22), we have

$$\lim_{\|u_n\|_\rho \to \infty} \frac{|\mathcal{L}(u_n, u_{n2}) - \mathcal{M}(u_n, u_{n2})|}{\|u_n\|_\rho} = 0. \quad (3.40)$$

Taking the limit in (3.38) as $n \to \infty$, and by (3.21), (3.24), (3.39), (3.40) and (3.29)(3), we get

$$\limsup_{n\to\infty} \int_{\widetilde{\Omega}} f(x,t,u_n) w_n \rho \leq G(w). \quad (3.41)$$

Setting

$$\widetilde{\Omega}_1 = \{(x,t) \in \widetilde{\Omega} : w(x) > 0\}, \quad \widetilde{\Omega}_2 = \{(x,t) \in \widetilde{\Omega} : w(x) < 0\},$$

it follows from (3.39) and (3.41) that

$$\liminf_{n\to\infty} \int_{\widetilde{\Omega}_1} f(x,t,u_n) w_n \rho + \liminf_{n\to\infty} \int_{\widetilde{\Omega}_2} f(x,t,u_n) w_n \rho \leq G(w). \quad (3.42)$$

By (3.21) and (3.29)(1)(3), we have

$$\lim_{n\to\infty} u_n(x,t) = +\infty, \quad a.e. \ (x,t) \in \widetilde{\Omega}_1;$$

$$\lim_{n\to\infty} u_n(x,t) = -\infty, \quad a.e. \quad (x,t) \in \widetilde{\Omega}_2.$$

Next, it follows from (f2) and (3.29)(3) that

$$\begin{cases} f^+ w\rho = \liminf_{n\to\infty} f(x,t,u_n) w_n \rho, & a.e. \quad (x,t) \in \widetilde{\Omega}_1; \\ f^- w\rho = \liminf_{n\to\infty} f(x,t,u_n) w_n \rho, & a.e. \quad (x,t) \in \widetilde{\Omega}_2. \end{cases} \tag{3.43}$$

And by (3.42), (3.43) and Fatou Lemma, we obtain

$$\int_{\widetilde{\Omega}_1} f^+(x,t) w(x) \rho + \int_{\widetilde{\Omega}_2} f^-(x,t) w(x) \rho \le G(w).$$

By (3.24) and (3.27), we know $\|w\|_\rho = 1$, thus, w is a nontrivial eigenfunction and satisfies (3.13). But it forms a contradiction between (3.13) and the above inequalities. Therefore, (3.15) is established and we complete the proof of Lemma 3.2. □

We are now ready to prove Theorem 2.1.

Proof of Theorem 2.1. Since $\widetilde{H}(\widetilde{\Omega}, \Gamma)$ is a separable Hilbert space, we see from (3.15) and Lemma 2.3 that there exists a subsequence (For the sake of simplicity, we take to be a full sequence$\{u_n\}$) and a function $u^* \in \widetilde{H}(\widetilde{\Omega}, \Gamma)$ with the following properties:

$$\begin{cases} (1) \lim_{n\to\infty} ||u_n - u^*||_\rho = 0; \\ (2) \ \exists k(x,t) \in \widetilde{L}^2_\rho, \quad \text{s.t.} \quad |u_n(x,t)| \le k(x,t), \quad a.e. \ (x,t) \in \widetilde{\Omega}, \quad \forall n; \\ (3) \lim_{n\to\infty} u_n(x,t) = u^*(x,t), \quad \text{a.e.} \ (x,t) \in \widetilde{\Omega}; \\ (4) \lim_{n\to\infty} \langle D_i u_n, v\rangle_{p_i} = \langle D_i u^*, v\rangle_{p_i}, \quad \text{for all} \quad v \in \widetilde{L}^2_{p_i}, \quad i = 1, \cdots, N; \\ (5) \lim_{n\to\infty} \langle a_0(x) u_n, v\rangle_q = \langle a_0(x) u^*, v\rangle_q, \quad \text{for all} \quad v \in \widetilde{L}^2_q. \end{cases} \tag{3.44}$$

Since $s_i(u)$ satisfies (S1), we have

$$\lim_{n\to\infty} s_i(u_n) = s_i(u^*), \quad i = 0, 1, 2, \cdots, N.$$

Let $v \in \widetilde{H}$ and $\tau_J(v)$ be defined by (2.14). Then $\tau_J(v) \in S_J (J \ge n_0)$ and from (3.44)(1)(4)(5) we have that

$$\lim_{n\to\infty} \mathcal{M}(u_n, \tau_J(v)) + \lim_{n\to\infty} \langle D_t u_n, \tau_J(v)\rangle = \mathcal{M}(u^*, \tau_J(v)) + \langle D_t u^*, \tau_J(v)\rangle. \tag{3.45}$$

Next from (f1)-(f2), (3.44)(2)(3) and the Lebesgue dominated convergence theorem, we obtain

$$\lim_{n\to\infty} \langle f(x,t,u_n), \tau_J(v)\rangle = \langle f(x,t,u^*), \tau_J(v)\rangle, \quad a.e. \ (x,t) \in \widetilde{\Omega}. \tag{3.46}$$

And from (B1), (3.44)(2)(3) and the Lebesgue dominated convergence theorem, we get

$$\lim_{n\to\infty}\langle b(x,t,u_n)(u_n)^-,\tau_J(v)\rangle = \langle b(x,t,u^*)(u^*)^-,\tau_J(v)\rangle,\ \ a.e.\ (x,t)\in\widetilde{\Omega}. \quad (3.47)$$

It follows from (3.44)-(3.47) that

$$\begin{aligned}\langle D_t u^*,\tau_J(v)\rangle + \mathcal{M}(u^*,\tau_J(v)) = \lambda_{j_0}\langle u^*,\tau_J(v)\rangle + \langle b(x,t,u^*)(u^*)^-,\tau_J(v)\rangle \\ +\langle f(x,t,u^*),\tau_J(v)\rangle - G(\tau_J(v)).\end{aligned} \quad (3.48)$$

Passing to the limit as $J\to\infty$ on both sides of (3.48), we have

$$\langle D_t u^*,v\rangle + \mathcal{M}(u^*,v) = \lambda_{j_0}\langle u^*,v\rangle + \langle b(x,t,u^*)(u^*)^-,v\rangle + \langle f(x,t,u^*),v\rangle - G(v).$$

Thus we complete the proof of Theorem 2.1. $\square$.

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