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On a Z-Transformation Approach to a Continuous-Time Markov Process with Non-fixed Transition Rates

T.A. Anake[1], S.O. Edeki[2], P.E. Oguntunde[3] and O.A. Odetunmibi[4]

[1,2,3,4]Department of Mathematics, School of Natural & Applied Sciences
College of Science & Technology, Covenant University, Ota, Nigeria
[2]E-mail: soedeki@yahoo.com

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Abstract

The paper presents z-transform as a method of functional transformation with respect to its theory and properties in dealing with discrete systems. We therefore obtain the absolute state probabilities as a solution of a differential equation corresponding to a given Birth-and–Death process via the z-transform, and deduce the equivalent stationary state probabilities of the system.

Keywords: *Markov Processes, Birth-Death Process, Z-Transform, Generating Function, Characteristic Differential Equation.*

1 Introduction

In mathematical sciences, engineering, physics and other fields of applied sciences, various forms of transforms such as integral transforms, Laplace transform, Fourier transform, etc are used, depending on the problem-whether discrete or continuous case. Discrete systems cannot be studied using the Laplace or the Fourier transform because they are continuous functions. Even when the continuous Fourier transform can be converted to its equivalent in discrete form by first finding the Discrete Fourier Transform (DFT), the Fourier transform itself does not suit the discrete systems. Such systems can easily be modeled using the z-transform [1].

The z-transform converts a sequence of real or complex numbers into a complex frequency domain representation. It can be considered as a discrete-time equivalent of the Laplace transform. Z-transforms are to difference equations what Laplace transforms are to differential equations. The idea of z-transform was first known to Laplace and was later introduced by W. Hurewicz as a controllable way of solving linear, constant-coefficient difference equations [2] & [3].

In mathematical literature, the idea contained in z-transform is also referred to as a method of generating functions as introduced by de Moivre with regards to probability theory [4].

Birth-and-death processes are examples of Markov processes that have been widely studied. They are used in the analysis of systems whose states involve changes in the size of some population- such as the study of population extinction times in biological systems, the evolution of genes in living things [5], and also have been used in the characterization of information storage and flow in computer systems [6].

Kendall in [7] gives a complete solution of the equations governing the generalized birth-and-death process, in which the birth and death rates are any specified functions of the time *t*. Yechiali [8] considers a queuing-type birth-and-death process defined on a continuous time Markov chain with emphasis on the steady state regime, and he recommends numerical methods for obtaining limiting probability since closed-form solutions are difficult to obtain; if they even exist.

Kundaeli in [9] used the z-transform to analyze a finite state birth-death Markov process in deriving the performance metrics of the system and their variation with the system parameters. In [10], he extended the results in [9] to analyze a birth and death process in which the birth and death transition probabilities can vary from state to state.

In this paper, the z-transform is studied and applied to a birth-and-death process with transition rates, and absolute state probabilities are therefore obtained as solutions of the corresponding differential equation. The paper is structured as follows: section 2 deals with the theory and concept of z-transform, section 3 is on basic processes, and section 4 handles the applications and conclusion.

2 Theory and Concept of Z-Transform

Definition 2.1: *Let* $g=\{g(n)\}_{n=-\infty}^{\infty}$ or $g=\{g_n\}_{n=-\infty}^{\infty}$ *be a sequence of terms with* $n\in\mathbb{Z}$ *and* z *a complex number such that:*

$g=\{\cdots,g(-3),g(-2),g(-1),g(0),g(1),g(2),g(3),g(4),\cdots\}$ *, then the* z *-transform of* g *is defined as:*

$$G(z)=Z_T\{g(n)\}_{n=-\infty}^{\infty}=Z_T\{g_n\}_{n=-\infty}^{\infty}=\cdots g(-3)\,z^3+g(-2)\,z^2+g(-1)\,z^1+g(0)\,z^0$$
$$+g(1)\,z^{-1}+g(2)\,z^{-2}+g(3)\,z^{-3}+g(4)\,z^{-4}+\cdots$$

$$\therefore \qquad Z_T\{g(n)\}=\sum_{n=-\infty}^{\infty}g(n)z^{-n}=\sum_{n=-\infty}^{-1}g(n)z^{-n}+\sum_{n=0}^{\infty}g(n)z^{-n} \tag{1}$$

Equation (1) is referred to as a two-sided or a bilateral z-transform of g .
Suppose g is defined only for $n\geq 0$, then (1) becomes:

$$G(z)=Z_T\{g(n)\}=\sum_{n=0}^{\infty}g(n)z^{-n} \tag{2}$$

We refer to (2) as the unilateral z -transform of g .

Definition 2.2: *Let* $g(n)$ *be the probability that a discrete random variable takes the value* n *, and the function* $G(z)$ *re-written as* $G(\mathrm{s})$ *with* $\mathrm{s}z=1$ *, then (2) becomes*

$$G(\mathrm{s})=\sum_{n=0}^{\infty}g(n)\,\mathrm{s}^n \tag{3}$$

where (3) is referred to as the corresponding probability generating function.

Definition 2.3: *Region of Convergence (ROC). The Region of convergence (ROC) is the set of points in the complex plane for which the z-transform summation converges. Every z-transform is defined over a ROC.*
Thus;

$$ROC=\left\{z:\left|\sum_{n=-\infty}^{\infty}g(n)\,z^{-n}\right|<\infty\right\} \tag{4}$$

Remark: The sequence notation $g=\{g_n\}_{n=-\infty}^{\infty}$ is used in mathematics to study difference equations while $g=\{g(n)\}_{n=-\infty}^{\infty}$ is used by engineers for signal processing.

2.1 Properties of the z-Transform

Let $h=\{h_n\}$ and $b=\{b_n\}$ be two sequences such that $H(z)=Z\{h_n\}$ and $B(z)=Z\{b_n\}$ with k_1 and k_2 as constants, then:

i. Linearity Property:

$$\begin{aligned} Z\{k_1h_n \pm k_2b_n\} &= Z\{k_1h_n\} \pm Z\{k_2b_n\} \\ &= k_1Z\{h_n\} \pm k_2Z\{b_n\} \\ &= k_1H(z) \pm k_2B(\mathrm{z}) \end{aligned}$$

ii. Scaling Property (Change of Scale):

$$Z\{k^n h(n)\} = H\left(\frac{z}{k}\right)$$

Remark: Suppose the ROC of $\{h(n)\}$ is $|z|<R$, then the ROC of $Z\{k^n h(n)\}$ is: $|z|<|k|R$

iii. Shifting Theorem (Delay or Advance Shift):

$$Z\{h(n-k)\} = z^{-k}H(z) \qquad \text{(delay shift) and}$$
$$Z\{h(n+k)\} = z^{k}H(z) \qquad \text{(Advance shift)}$$

iv. Argument as Multiplier (Multiplication by n):

$$Z\{h(n)\} = -z\frac{d}{dz}H(z)$$

Hence,

$$Z\{n^k h(n)\} = \left[-z\frac{d}{dz}\right]^k H(z),\ k \geq 0$$

We remark here that $\left[-z\frac{d}{dz}\right]^k \neq (-z)^k \frac{d^k}{dz^k}$ but a repetitive operation of $\left[-z\frac{d}{dz}\right]$ in k-times.

The proofs of the stated properties, theorem and other concepts such as convolution and inverse transform are found in [1] & [4].

3 Basics Processes

This section deals with the introduction of some basic processes and concepts needed in the remaining part of the work.

3.1 Markov Process

Definition 3.1: *A stochastic process* $\{X(n), n \in \mathbb{N}\}$ *is called a Markov chain if, for all time* $n \in \mathbb{N}$ *and for all states* $(i_0, i_1, i_2, i_3, \cdots, i_n)$ *with* $\underset{r}{P}\{(\bullet)\}$ *a probability function;*

$$\underset{r}{P}\{X_{n+1} = i_{n+1} | X_n = i_n, X_{n-1} = i_{n-1}, X_{n-2} = i_{n-2}, X_{n-3} = i_{n-3}, \cdots, X_0 = i_0\}$$
$$= \underset{r}{P}\{X_{n+1} = i_{n+1} | X_n = i_n\} \quad (5)$$

That is, the future state of the system depends only on the present state (and not on the past states). Condition (5) is referred to as Markovian property. Any stochastic process with such property is called a Markov process [11] & [12].

Definition 3.2: *Continuous-time Markov chain. A stochastic process* $\{X(t), t \geq 0\}$ *with a parameter set* τ *and a discrete state space* $\mathbb{Z}$ *is called a continuous time Markov chain or a Markov chain in continuous time if for any* $n \geq 1$*,*

$t_0 < t_1 < t_2 < t_3 < t_4 < t_5 < \cdots t_n < t_{n+1}$ and $(i_0, i_1, i_2, i_3, \cdots, i_n, i_{n+1})$, $i_k \in \mathbb{Z}$ we have that:

$$\underset{r}{P}\{X(t_{n+1}) = i_{n+1} | X(t_n) = i_n, X(t_{n-1}) = i_{n-1}, X(t_{n-2}) = i_{n-2}, X(t_{n-3}) = i_{n-3}, \cdots, X(t_0) = i_0\}$$
$$= \underset{r}{P}\{X(t_{n+1}) = i_{n+1} | X(t_n) = i_n\} \quad (6)$$

Note 3.1: The conditional probabilities;

$\underset{r}{P}_{ij}(s,t) = \underset{r}{P}\{X(t) = j | X(s) = i\}$, $s < t$, $i, j \in \mathbb{Z}$ are the transition probabilities of the Markov chain.

One example of the continuous time Markov chain is the Birth-Death process. Hence, the following concepts.

3.2 Birth-and- Death Processes

Let $X(t) = n$ be the population size in an infinitesimal time interval $(t, t+\Delta t)$ with a population net change $\Delta X(t, t+\Delta t) = X(t+\Delta t) - X(t)$ in $(t, t+\Delta t)$. We suppose $\lambda \Delta t$ as the probability that an individual gives birth in $(t, t+\Delta t)$ and $\mu \Delta t$ as the probability of a death in $(t, t+\Delta t)$.

A birth-death process has the property that the net change across an infinitesimal time interval Δt is either 1 (for birth), -1 (for death) or 0 (for neither birth nor death). We therefore make the following assumptions:

$$A_1: \quad \underset{r}{P}\{X(t)=n\to 1\}=\lambda_n\Delta t+0\Delta t=\lambda n\Delta t+0\Delta t$$

$$A_2: \quad \underset{r}{P}\{X(t)=n\to -1\}=\mu_n\Delta t+0\Delta t=\mu n\Delta t+0\Delta t$$

Therefore, for a case of neither birth nor death, we have:

$$A_3: \quad \underset{r}{P}\{X(t)=n\to 0\}=1-(\lambda_n\Delta t+\mu_n\Delta t)+0\Delta t=1-n(\lambda\Delta t+\mu\Delta t).$$

Denote $P_n(t)$ as the probability of n individuals in a time interval of length t. Hence, by applying the law of total probability, we have:

$$\begin{aligned} P_n(t+\Delta t) = &\underset{r}{P}\{n \text{ individuals in } t \text{ and neither birth nor death in } \Delta t\} \\ &+\underset{r}{P}\{n-1 \text{ individuals in } t \text{ and a birth in } \Delta t\} \\ &+\underset{r}{P}\{n+1 \text{ individuals in } t \text{ and a death in } \Delta t\} \end{aligned}$$

Thus,

$$P_n(t+\Delta t)=P_n(t)\left[1-\left(\lambda_n\Delta t+\mu_n\Delta t\right)\right]+P_{n-1}(t)\lambda_{n-1}\Delta t+P_{n+1}(t)\mu_{n+1}\Delta t+0\left(\Delta t\right) \qquad (7)$$

For $\mu_i=i\mu$ and $\lambda_i=i\lambda$, $i\geq 0$, such that $P_1(0)=\underset{r}{P}\{X(0)=1\}=1$, (7) becomes:

$$P_n(t+\Delta t)=P_n(t)\left[1-n\left(\lambda\Delta t+\mu_n\Delta t\right)\right]+(n-1)P_{n-1}(t)\lambda\Delta t+(n+1)P_{n+1}(t)\mu\Delta t+0\left(\Delta t\right)$$

Showing that;

$$P_n(t+\Delta t)-P_n(t)=(n-1)\lambda P_{n-1}(t)\Delta t+(n+1)\mu P_{n+1}(t)\Delta t-n(\lambda+\mu)P_n(t)+0\left(\Delta t\right) \qquad (8)$$

Dividing both sides of (8) by Δt and taking limit as $\Delta t\to 0$ yields:

$$P_n'(t)=(n-1)\lambda P_{n-1}(t)+(n+1)\mu P_{n+1}(t)-n(\lambda+\mu)P_n(t), \quad n\geq 1 \qquad (9)$$

$P_0(0)=0$ makes no sense since 0 is absorbing, hence

$$P_0'(t)=\mu P_1(t) \qquad (10)$$

Therefore, a linear birth-and-death process satisfies the system of differential equations (9)-(10).

Remark 3.1:

From (9), if $\mu = 0$, then we have:

$$P_n'(t) = (n-1)\lambda P_{n-1}(t) - \lambda n P_n(t), \quad n \geq 1 \qquad (11)$$

Similarly, $\lambda = 0$ in (9) gives:

$$P_n'(t) = (n+1)\mu P_{n+1}(t) - n\mu P_n(t), \quad n \geq 1 \qquad (12)$$

Equations (11) and (12) are referred to as *linear-birth-process* and *linear-death-process* respectively.

4 Applications: The Z-Transform on a Birth-Death Process

In this section, we consider a birth-and-death process with transition rates [7], [12] & [13]:

$\lambda_i = \lambda$ and $\mu_i = i\mu$, $i = 0,1,2,3,4,\cdots$

and initial distribution:

$P_0(0) = P_r\{X(0) = 1\} = 1$

We therefore subject the system to a z-transform in order to obtain the absolute state probabilities, $P_n(t)$ and the stationary state probabilities, Π_n where

$$\Pi_n = \lim_{t \to \infty^+} P_n(t) \qquad (13)$$

This is done as follows; using the transition rates $\lambda_i = \lambda$ and $\mu_i = i\mu$, $i \geq 0$ in (9) yields a corresponding system of differential equation for $n \geq 1$:

$$P_n'(t) = \lambda P_{n-1}(t) - (\lambda + n\mu)P_n(t) + (n+1)\mu P_{n+1}(t) \qquad (14)$$

and

$$P_0'(t) = -\lambda P_0(t) + \mu P_1(t) \qquad (15)$$

We invoke the z-transform (probability generating function) in (3) re-defined as:

$$\Phi(t,s) = \sum_{n=0}^{\infty} P_n(t) s^n \qquad (16)$$

with $\Phi(0,s)=1$ such that:

$$\frac{\partial\Phi(t,s)}{\partial t}=\sum_{n=0}^{\infty}P_n'(t)s^n \text{ and } \frac{\partial\Phi(t,s)}{\partial s}=\sum_{n=0}^{\infty}nP_n(t)s^{n-1} \tag{17}$$

To subject (14) to (16) and (17), we multiply both sides of (14) by s^n and take the summation from 0 to ∞ , hence:

$$\sum_{n=0}^{\infty}P_n'(t)s^n=\lambda\sum_{n=0}^{\infty}P_{n-1}(t)s^n+\sum_{n=0}^{\infty}(n+1)\mu P_{n+1}(t)s^n-\sum_{n=0}^{\infty}(\lambda+n\mu)P_n(t)s^n \tag{18}$$

Adjusting the limit (index) of the first two series (with $n=m+1$ and $n=k-1$ respectively) in the RHS of (18) gives:

$$\frac{\partial\Phi(t,s)}{\partial t}=\lambda\sum_{m+1=0}^{\infty}P_m(t)s^{m+1}+\mu\sum_{k-1=0}^{\infty}kP_k(t)s^{k-1}-\lambda\sum_{n=0}^{\infty}P_n(t)s^n-\sum_{n=1}^{\infty}n\mu P_n(t)s^n$$

Since $P_{-v}(t)=0$ for $v\geq 1$ and $s^n=s^{n-1}s$, we therefore write:

$$\frac{\partial\Phi(t,s)}{\partial t}=\lambda s\sum_{m=0}^{\infty}P_m(t)s^m+\mu\sum_{k=1}^{\infty}kP_k(t)s^{k-1}-\lambda\sum_{n=0}^{\infty}P_n(t)s^n-\mu\sum_{n=1}^{\infty}nP_n(t)s^{n-1}s$$

$$=\lambda(s-1)\sum_{m=0}^{\infty}P_m(t)s^m-\mu(s-1)\sum_{k=1}^{\infty}kP_k(t)s^{k-1}$$

$$=\lambda(s-1)\Phi(t,s)-\mu(s-1)\frac{\partial\Phi(t,s)}{\partial s} \quad (\text{for } m=n=k)$$

$$\therefore \quad \frac{\partial\Phi(t,s)}{\partial t}+\mu(s-1)\frac{\partial\Phi(t,s)}{\partial s}=\lambda(s-1)\Phi(t,s) \tag{19}$$

Equation (19) has a corresponding system of characteristics differential equations given below:

$$\frac{ds}{dt}=\mu(s-1) \tag{20}$$

and

$$\frac{d\Phi(t,s)}{dt}=\lambda(s-1)\Phi(t,s) \tag{21}$$

In order to obtain $\Phi(t,s)$, (20) and (21) will be solved simultaneously, thus, from (20),

$$\frac{ds}{s-1}=\mu dt \text{ and } (s-1)=\frac{1}{\mu}\frac{ds}{dt} \text{ , such that } \ln(s-1)-\mu t=c_1 \tag{22}$$

As such (21) becomes:

$$\frac{d\Phi(t,s)}{\Phi(t,s)}=\lambda(s-1)dt=\frac{\lambda}{\mu}ds$$

$$\Rightarrow \qquad \ln\Phi(t,s)-\frac{\lambda}{\mu}s=c_2 \text{, but for } \zeta=\frac{\lambda}{\mu} \text{ ,}$$

$$c_2=\ln\Phi(t,s)-\zeta s \tag{23}$$

Since c_1 and c_2 are arbitrary constants, we assume $h(\bullet)$ an arbitrary continuous function satisfied by $\Phi(t,s)$ such that:

$$h(c_1)=c_2 \tag{24}$$

Hence,

$$h(\ln(s-1)-\mu t)=\ln\Phi(t,s)-\zeta s$$

Showing that:

$$\ln\Phi(t,s)=h(\ln|s-1|-\mu t)+\zeta s$$

$$\therefore \qquad \Phi(t,s)=\exp\left[h(\ln|s-1|-\mu t)+\zeta s\right]$$

$$=e^{\left[h(\ln|s-1|-\mu t)\right]}e^{\zeta s} \tag{25}$$

Applying the initial condition $\Phi(0,s)=1$ on (25), yields:

$$e^{\left[h(\ln|s-1|)\right]}=e^{-\zeta s}$$

such that:

$$h(\ln|s-1|)=-\zeta s \tag{26}$$

Letting $\eta=\ln|s-1|$ in (26) implies that $(s-1)=e^{\eta}$ or $s=1+e^{\eta}$

$$\therefore \qquad h(\eta)=-\eta(1+e^{\eta}) \tag{27}$$

So,

$$h(\ln|s-1|-\mu t)=-\zeta(1+e^{\ln|s-1|}e^{-\mu t})$$

$$= -\zeta(1+(s-1)e^{-\mu t})$$

$$= -\zeta(1+se^{-\mu t} - e^{-\mu t})$$

Showing that:

$$\Phi(t,s) = e^{\left[-\zeta(1+se^{-\mu t} - e^{-\mu t}) + s\zeta\right]}$$

$$= e^{(-\zeta - \zeta se^{-\mu t} + \zeta e^{-\mu t})e^{s\zeta}}$$

$$= e^{-\zeta} e^{-\zeta se^{-\mu t}} e^{\zeta e^{-\mu t}} e^{s\zeta}$$

$$= e^{-\zeta(1-e^{-\mu t})} e^{\zeta s(1-e^{-\mu t})} \tag{28}$$

Setting $v = \zeta(1-e^{-\mu t})$ in (28) gives:

$$\Phi(t,s) = e^{-v} e^{sv}$$

$$= e^{-v}\left[1 + \frac{(sv)}{1!} + \frac{(sv)^2}{2!} + \frac{(sv)^3}{3!} + \frac{(sv)^4}{4!} + \frac{(sv)^5}{5!} + \cdots\right]$$

$$= e^{-v}\sum_{n=0}^{\infty}\frac{(sv)^n}{n!}$$

Showing that

$$\Phi(t,s) = \sum_{n=0}^{\infty}\frac{(v)^n e^{-v}}{n!}s^n \tag{29}$$

Comparing (29) with (16) gives:

$$P_n(t) = \frac{(v)^n e^{-v}}{n!} = \frac{(\zeta(1-e^{-\mu t}))^n e^{-\zeta(1-e^{-\mu t})}}{n!}, \ n \ge 0 \tag{30}$$

In addition,

$$\Pi_n = \lim_{t\to\infty^+} P_n(t) = \frac{\zeta^n e^{-\zeta}}{n!} \tag{31}$$

Equations (30) and (31) are the absolute state probabilities and the stationary state probabilities of the system respectively. Equations (30) is a Poisson distribution with intensity (trend) function $\lambda(t) = (\zeta(1-e^{-\mu t}))^n$.

Note: In some cases, $\Phi(t,s)$ cannot be easily expanded as a power series in s; hence, the absolute state probabilities can be computed by differentiating $\Phi(t,s)$.

4.1 Discussion of Result

For the purpose of discussion of result, we studied the governing parameters ζ, μ, and t for some numerical calculations with $\zeta = 9.5$, $\mu = 0.2$, and $t > 0$. The results are shown in Table 1 and Fig.1 below:

Table 1: The probabilities at time t

t	Absolute state probabilities	Stationary state probabilities
0.1	0.828521	7.49E-05
0.2	0.256656	0.000711
0.3	0.088008	0.003378
0.4	0.031283	0.010696
0.5	0.011270	0.025403
0.6	0.004072	0.048266
0.7	0.001467	0.076421
0.8	0.000525	0.103714
0.9	0.000187	0.123160
1.0	6.56E-05	0.130003

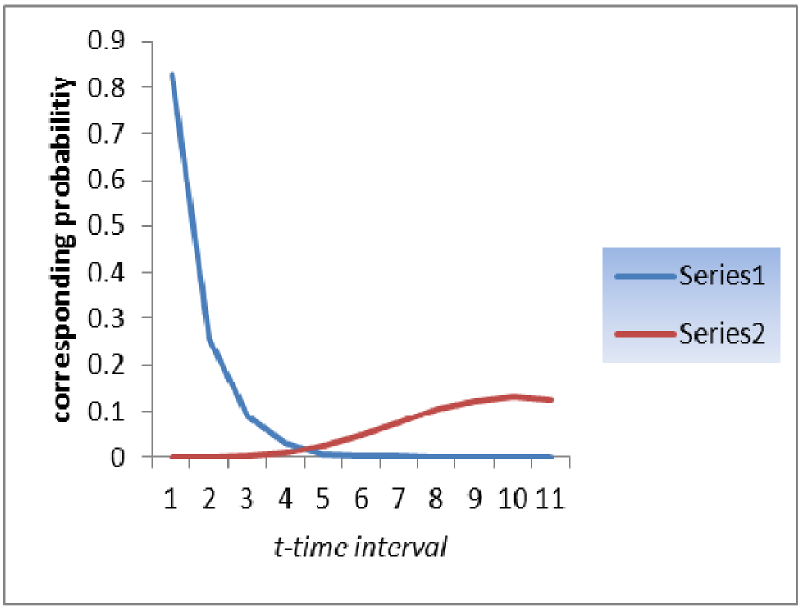

Figure 1: Graphical representation of the probabilities
Key: Series 1- Absolute state probabilities and **Series 2-** Stationary state probabilities

4.2 Concluding Remarks

We have explicitly analyzed the effectiveness of the z-transform and its properties in handling discrete systems. It is shown that the absolute state probability decreases as time increases. The considered birth-and-death process is of great importance in queuing theory, biological sciences, and in the analysis of systems whose states involve changes in population sizes.

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References

[1] K. Asher, Z transform, *International Journal of Applied Mathematics & Statistical Sciences*, 2(2) (2013), 35-42.

[2] E.R. Kanasewich, *Time Sequence Analysis in Geophysics (3rd ed)*, University of Alberta, (1981), 185-186.

[3] J.R. Ragazzini and L.A. Zadeh, The analysis of sampled-data systems, *Trans. Am. Inst. Elect. Eng*., 71(11) (1952), 225-234.

[4] E.I. Jury, *Theory and Application of z-Transform Method*, John Wiley & Sons P1, (1964).

[5] G.P. Karev, Y.I. Wolf, F.S. Berezovskaya and E.V. Koonin, Gene family evolution: An in-depth theoretical and simulation analysis of non-linear birth-death-innovation models, *BMC Evol. Biol.*, 4(2004), 32.

[6] L. Kleinrock, *Queuing Systems Theory (Vol. 1)*, John-Wiley & Sons, New York, (1975).

[7] D.G. Kendall, On the generalized birth-and-death process, *Ann. Math. Statistics*, 19(1) (1948), 1-15.

[8] U. Yechiali, A Queuing-type birth-and-death process defined on a continuous-time Markov chain, *Operation Research*, 21(2) (1973), 604-609.

[9] H.N. Kundaeli, The z-transform applied to a birth-death markov process, *Journal of Science and Technology*, 28(2) (2008), 75-84.

[10] H.N. Kundaeli, The z-transform applied to a birth-death process having varying birth and death rates, *Journal of Science and Technology*, 30(1) (2010), 129-140.

[11] K.A. Borovkov, *Elements of Stochastic Modelling*, World Scientific, Singapore, (2003).

[12] M. Kijima, *Markov Processes for Stochastic Modeling*, Chapman & Hall London, New York, (1996).

[13] F. Beichelt, *Stochastic Processes in Science, Engineering and Finance*, Chapman and Hall/CRC, (2006).