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On (θ, θ)-Derivations in Semiprime Rings

Najat M. Muthana

Department of Mathematics, Faculty of Science for Girls
King Abdulaziz University, Jeddah, K.S.A.
E-mail: nmuthana@kau.edu.sa

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Abstract

The main objective of the present paper is to prove the following result: let m and n be positive integers with $m+n \neq 0$, and let R be an $(m+n+2)!-$torsion free semiprime ring with identity element. Let θ be an automorphism of R. Suppose there exists an additive mapping $D: R \to R$ such that $D(x^{m+n+1}) = (m+n+1)\ \theta(x^m)D(x)\theta(x^n)$ for all $x \in R$, then D is a $(\theta,\theta)-$derivation on R.

Keywords: *Semiprime rings, (θ, θ)-derivation, nth power property, Jordan derivation, Commuting maps.*

1 Introduction

This research has been motivated by the work of Herstein [5], and Bridges and Bergen [4]. Throughout, R designates an associative ring with center $Z(R)$. Let θ and ϕ be endomorphisms of R. An additive mapping $d: R \to R$ is called a $(\theta,\phi)-$derivation if $d(xy) = d(x)\theta(y) + \phi(x)d(y)$ for all $x, y \in R$. If 1 denotes the identity mapping on R, then a $(\theta, 1)-$derivation is called simply a $\theta-$derivation , a 1-derivation is an ordinary derivation. An additive mapping $\phi: R \to R$ is called a Jordan $(\theta,\phi)-$derivation if $d(x^2) = d(x)\theta(x) + \phi(x)d(x)$ for all $x \in R$. Jordan $\theta-$derivations and Jordan derivations are defined analogously. There do exist equivalent conditions of a ring R to be called prime, the basic one is that if $aRb = (0), a, b \in R$, implies that $a = 0$ or $b = 0$. A ring R is called a semiprime ring if $aRa = 0, a \epsilon R$ implies that $a \epsilon R$.

A classical result due to Herstein [5] states that every Jordan derivation of prime rings of characteristics not 2 is a derivation. In [1], Bresar and Vukman presented a brief proof of Herstein's result. This result was extended to 2-torsion free semiprime ring in [6]. Further, the above mentioned result was generalized by Bresar and Vukman [2] for Jordan (θ, ϕ) −derivations in the setting of prime rings. It is straightforward to check that if d is a derivation of R and if $n > 1$ is any integer, then

$$d(x^n) = \sum_{j=1}^{n} x^{j-1} d(x) x^{n-j}$$

for any $x \in R$ where $x^0 r = r = r x^0$ for any $x \in R$. This is known as *nth power property*. Assuming only that $d: R \to R$ is additive and satisfies the *nth power property*, must d be a derivation? When $n = 2$, the *nth power property* makes d a Jordan derivation. The result for arbitrary n was proven by Bridges and Bergen in [4] when R is a prime ring with identity and when char $R > n$ or is zero. The author together with Daif [8] extended Bridges' result to nth$-(\theta, \phi)$ power property

$$d(x^n) = \sum_{j=1}^{n} (\theta(x))^{j-1} d(x) (\phi(x))^{n-j}$$

for all $x \in R$ in a semiprime ring. In the year 2007, Lanski [7] generalized Bridges' result to (θ, ϕ) − generalized derivations in semiprime rings.

2 The Results

Another perspective on the derivation of x^n in some rings is to consider some identities on an additive map D:$R \to R$. It is our aim in this paper to prove the following result.

Theorem 1: *Let $m \geq 0, n \geq 0$, and $m + n \neq 0$ be some fixed integers, and let R be an $(m+n+2)! - torsion$ free semiprime ring with identity e. Let θ be an automorphism of R. Suppose there exists an additive mapping $D: R \to R$ such that*

$$D(x^{m+n+1}) = (m+n+1)\theta(x^m) D(x) \theta(x^n)$$

is fulfilled for all $x \in R$. In this case, D is a (θ, θ) −derivation on R.

Let us discuss in some more detail about background of the result mentioned above. An additive mapping $D: R \to R$ is called a left derivation if $D(xy) = xD(y) + yD(x)$ holds for all pairs $x, y \in R$, and is called a left Jordan derivation in case

$$D(x^2) = 2xD(x)$$

is fulfilled for all $x \in R$. The concept of left derivations and left Jordan derivations have been introduced by Bresar and Vukman [3]. Bresar and Vukman [3] have proved that there are no nonzero left Jordan derivation on a noncommutative prime ring R of characteristic different from two and three. In [10], Vukman has established that any left Jordan derivation which maps a 2-torsion free semiprime ring R into itself, is a derivation which maps R into $Z(R)$. In [9], Vukman has also proved the following result. Let R be a noncommutative prime ring with the identity element and of characteristic different from two and three, and let $D: R \to R$ be an additive mapping satisfying the relation

$$D(x^3) = 3xD(x)x$$

for all $x \in R$. In this case $D = 0$. The relations mentioned above lead to the following result proved by Vukman and Ulbl [11]. Let $m \geq 0, n \geq 0$, and $m + n \neq 0$ be fixed integers. Let R be an $(m + n + 2)! -$ torsion free semiprime ring with identity element. Suppose there exists an additive mapping $D: R \to R$ such that

$$D(x^{m+n+1}) = (m + n + 1)x^m D(x)x^n$$

is fulfilled for all $x \in R$. In this case, D is a derivation which maps R into its center. In case R is a noncommutative prime ring we have $D = 0$. Theorem 1 is in the spirit of the result we have just mentioned above. In order to prove Theorem 1, we need the following results.

Theorem 2 [11, Theorem 4]: *Let R be a 2-torsion free semiprime ring. Suppose that an additive mapping $F: R \to R$ satisfies $[[F(x), x], x] = 0$ for all $x \in R$. Then, $[F(x), x] = 0$ holds for all $x \in R$.*

Theorem 3: *Let R be a 2-torsion free semiprime ring. Let θ be an automorphism of R. Suppose that an additive mapping $F: R \to R$ satisfies*

$[[F(x), \theta(x)], \theta(x)] = 0$ *for all* $x \in R$.

Then $[F(x), \theta(x)] = 0$ *holds for all* $\in R$ *, i.e.* F *is* $\theta - commuting$.

Proof: Given that $[[F(x), \theta(x)], \theta(x)] = 0, for\ all\ x \in R$. Since θ is an automorphism θ^{-1} is also an automorphism and hence

$\theta^{-1}([F(x),\theta(x)],\theta(x)]) = 0$. This yields that $[[\theta^{-1}F(x),x],x] = 0$. But if F and θ are additive, then $\theta^{-1}F$ is also additive mapping and hence by Theorem 2, $[\theta^{-1}F(x),x] = 0$ for all $x \in R$. This implies that $[F(x),\theta(x)] = 0$ for all $x \in R$.

Proof of Theorem 1: By the hypothesis, we have

$$D(x^{m+n+1}) = (m+n+1)\,\theta(x^m)D(x)\theta(x^n),\ \ for\ all\ x \in R. \qquad (1)$$

Replacing x by e in (1), we get

$$D(e) = 0 \qquad (2)$$

where e denotes the identity element. Putting $x+e$ for x in the relation (1) and using (2), we obtain

$$\sum_{i=0}^{m+n+1}\binom{m+n+1}{i} D\left(x^{m+n+1-i}\right) = (m+n+1)\left(\sum_{i=0}^{m}\binom{m}{i}\theta\left(x^{m-i}\right)\right)D(x)\left(\sum_{i=0}^{n}\binom{n}{i}\theta\left(x^{n-i}\right)\right), \forall\, x \in R. \qquad (3)$$

Using (1) and collecting together terms of (3) involving the same number of factors of e, we obtain

$$\sum_{i=0}^{m+n} f_i(\theta(x),e) = 0\ \ for\ all\ x \in R\,. \qquad (4)$$

where $f_i(\theta(x),e)$ stands for the expression of terms involving i factors of e.

Replacing x by $x+2e, x+3e, \ldots\ldots, x+(m+n)e$ in turn in (1) and expressing the resulting system of $m+n$ homogeneous equations, we say that the coefficient matrix of the system is a Vander Monde matrix

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ 2 & 2^2 & \cdots & 2^{m+n} \\ \vdots & \vdots & & \vdots \\ m+n & (m+n)^2 & \ldots & (m+n)^{m+n} \end{pmatrix}. \qquad (5)$$

Since the determinant of the matrix is different from zero, it follows that the system has only a trivial solution. In particular, we have

$$f_{m+n-1}(\theta(x),e) = \binom{m+n+1}{m+n-1} D(x^2) - (m+n+1)\left(\binom{m}{m-1}\binom{n}{n}\theta(x)\,D(x) + \binom{m}{m}\binom{n}{n-1} D(x)\theta(x)\right) = 0\ \ for\ all\ x \in R. \qquad (6)$$

And

$$f_{m+n-2}(x,e) = \binom{m+n+1}{m+n-2} D(x^3) - (m+n+1)\left(\binom{m}{m-2}\binom{n}{n}\theta(x^2)D(x) + \binom{m}{m-1}\binom{n}{n-1}\theta(x)D(x)\theta(x) + \binom{m}{m}\binom{n}{n-2}D(x)\theta(x^2)\right) = 0, \ \forall x \in R \quad (7)$$

Since R is an $(m+n+2)!-$ torsion free ring, the above equations reduce to:

$$(m+n)D(x^2) = 2m\theta(x)\,D(x) + 2nD(x)\theta(x), \quad \forall x \in R. \quad (8)$$

And

$$(m+n)(m+n-1)D(x^3) = 3m(m-1)\theta(x^2)D(x) + 6mn\theta(x)D(x)\theta(x) + 3n(n-1)D(x)\theta(x^2), \quad \forall x \in R. \quad (9)$$

Now, substituting $x+y$ for x in (8), we get

$$(m+n)D\,(xy+yx) = 2m\theta(x)D(y) + 2m\theta(y)\,D(x) + 2nD(x)\,\theta(y) + 2nD(y)\,\theta(x), \quad \forall x,y\in R \quad (10)$$

Putting $y=(m+n)x^2$ in the relation above, we obtain

$$(m+n)^2\,D(x^3) = m(m+n)\theta(x)\,D(x^2) + m(m+n)\theta(x^2)\,D(x) + n(m+n)D(x)\theta(x^2) + n(m+n)\,D(x^2)\,\theta(x), \ x\in R. \quad (11)$$

According to (8), the above relation reduces to

$$(m+n)^2\,D(x^3) = (3m^2+mn)\theta(x)^2\,D(x) + 4mn\,\theta(x)D(x)\theta(x) + (3n^2+mn)D(x)\theta(x)^2, \ for\ all\ x \in R. \quad (12)$$

Subtracting (9) from (12), we obtain

$$(m+n)D(x^3) = m(n+3)\theta(x^2)\,D(x) - 2mn\theta(x)D(x)\,\theta(x) + n(m+3)\,D(x)\,\theta(x^2), \ \forall x \in R. \quad (13)$$

From the above relation, we conclude that

$$(m+n)^2 D(x^3) = (m+n)m(n+3)\theta(x^2)\, D(x) - 2(m+n)mn\ \ \theta(x)D(x)\theta(x) + (m+n)n\ (m+3)\ D(x)\theta(x^2)\ \ for\ all\ x \in R. \tag{14}$$

Subtracting (14) from (12), we obtain

$$mn(m+n+2)\theta(x^2)D(x) - 2mn\,(m+n+2)\theta(x)\,D(x)\ \theta(x) + mn(m+n+2)D(x)\theta(x^2) = 0\ for\ all\ \ x \in R. \tag{15}$$

Since R is an $(m+n+2)! -$ torsion free ring, the above relation reduces to

$$D(x)\theta(x^2) + \theta(x^2)D(x) - 2\theta(x)D(x)\,\theta(x) = 0\ for\ all\ \ x \in R \tag{16}$$

This can be written in the form

$$[[D(x),\theta(x)],\theta(x)] = 0\ for\ all\ \ x \in R. \tag{17}$$

In view of Theorem 3, we are forced to conclude that

$$[D(x),\theta(x)] = 0\ for\ all\ \ x \in R. \tag{18}$$

This means D is $\theta -$ commuting on R which makes it possible to replace $D(x)\theta(x)$ in (8) by $\theta(x)D(x)$. The relation (8) reduces to $D(x^2) = 2\theta(x)D(x)\ for\ all\ \ x \in R.$

Also, $D(x^2) = D(x)\theta(x) + \theta(x)D(x)\ for\ all\ \ x \in R.$ In other words, D is $(\theta,\theta) -$ Jordan derivation. Hence by [7, Theorem 2], D is $(\theta,\theta) - derivation$. This completes the proof.

Theorem 4: *Let* R *be a* $2, m, n, m+n, and\ |m-n| - torsion\ free$ *semiprime ring. Let* θ *be an automorphism of* R*. Suppose* $D: R \to R$ *is an additive mapping satisfying the relation*

$$(m+n)D(xy) = 2m\ D(x)\theta(y) + 2n\ \theta(x)D(y) \tag{19}$$

for all $x, y \in R$ and some integers $m \geq 0, n \geq 0, m+n \neq 0$. In case $m \neq n$, then $D = 0$.

Proof: In the relation (19), we compute the expression $(m+n)^2 D(xyx)$ in two ways. First we obtain

$$(m+n)^2 D\big(x(yx)\big) = 2m(m+n)D(x)\theta(y)\theta(x) + 2n(m+n)\theta(x)D(yx)$$
$$= 2m(m+n)D(x)\theta(y)\theta(x)$$
$$+ 2n\theta(x)\big(2mD(y)\theta(x) + 2n\theta(y)D(x)\big),$$
$$for\ all\ x, y \in R. \tag{20}$$

This implies that

$$(m+n)^2 D(xyx)$$
$$= 2m(m+n)D(x)\theta(y)\theta(x) + 4mn\theta(x)D(y)\theta(x)$$
$$+ 4n^2\theta(x)\theta(y)D(x) \text{ for all } x, y \in R. \tag{21}$$

On the other hand, we have

$$(m+n)^2 D\big((xy)x\big) = 2m(m+n)D(xy)\theta(x) + 2n(m+n)\theta(x)\theta(y)D(x)$$
$$= 2m\big(2mD(x)\theta(y) + 2n\theta(x)D(y)\big)\theta(x)$$
$$+ 2n(m+n)\theta(x)\theta(y)D(x), \quad x, y \in R. \tag{22}$$

Thus, we have

$$(m+n)^2 D(xyx)$$
$$= 4m^2 D(x)\theta(y)\theta(x) + 4mn\,\theta(x)D(y)\theta(x)$$
$$+ 2n(m+n)\theta(x)\theta(y)D(x) \text{ for all } x, y \in R. \tag{23}$$

Subtracting the relation (21) from (23), we obtain

$$m(m-n)D(x)\theta(y)\theta(x) + n(m-n)\theta(x)\theta(y)D(x) = 0 \tag{24}$$

Which reduces to

$$m\,D(x)\theta(y)\theta(x) + n\theta(x)\theta(y)D(x) = 0 \text{ for all } x, y \in R. \tag{25}$$

Putting yx for y in (25), we obtain

$$m\,D(x)\theta(y)\theta(x^2) + n\theta(x)\theta(y)\theta(x)D(x) = 0\ , \forall\ x, y \in R \tag{26}$$

Right multiplication of the relation (25) by $\theta(x)$ gives

$$mD(x)\theta(y)\theta(x^2) + n\theta(x)\theta(y)D(x)\theta(x) = 0, \forall\ x, y \in R. \tag{27}$$

Subtracting the relation (26) from (27), we obtain

$$n\Big(\theta(x)\theta(y)\big(D(x)\theta(x) - \theta(x)D(x)\big)\Big) = 0 \text{ for all } x, y \in R. \tag{28}$$

The last relation yields that

$$\theta(x)\theta(y)[D(x),\theta(x)] = 0 \text{ for all } x,y \in R. \quad (29)$$

Substituting $D(x)y$ for y in (29) and using the fact that θ is an automorphism of R, then multiplying the relation (29) by $D(x)$ from the left and comparing the relations so obtained, we get

$$[D(x),\theta(x)]y[D(x),\theta(x)] = 0 \text{ for all } x,y \in R. \quad (30)$$

This implies that

$$[D(x),\theta(x)] = 0 \text{ for all } x \in R. \quad (31)$$

Putting $y = x$ in the relation (19) and using (31), we obtain

D$(x^2) = 2\,D(x)\theta(x)$ for all $x \in R$.

This can be written in the form

$$D(x^2) = D(x)\theta(x) + \theta(x)D(x) \text{ for all } x \in R. \quad (32)$$

In other words, D is a $(\theta,\theta)-$ Jordan derivation. By [7, Theorem 2], we conclude that D is a $(\theta,\theta)-$ derivation. Now, we replace $D(xy)$ with $D(x)\theta(y) + \theta(x)D(y)$ in the left hand side of (19), we obtain

$$D(x)\theta(y) = \theta(x)D(y) \text{ for all } x,y \in R. \quad (33)$$

Substituting zx for x in (33) gives

$$D(z)\theta(x)\theta(y) = 0 \text{ for all } x,y,z \in R. \quad (34)$$

Since θ is an automorphism of R, so it follows $D(z)xD(z) = 0$ for all $x,z \in R$. Thus by the semiprimenss of R, we are forced to conclude that $D = 0$. This completes the proof.

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