Gen. Math. Notes, Vol. 27, No. 2, April 2015, pp.92-106
ISSN 2219-7184; Copyright ©ICSRS Publication, 2015
www.i-csrs.org
Available free online at http://www.geman.in

A New Generalization of s-Weakly Regular Rings

Abdullah M. Abdul-Jabbar

Department of Mathematics, College of Science
Salahaddin University-Erbil , Kurdistan Region, Iraq
E-mail: m1abdullah.math71@gmail.com; abdullah.abduljabbar@su.edu.krd

(Received: 5-3-15 / Accepted: 8-4-15)

Abstract

Right (resp. left) s-weakly regular rings was first introduced by V. Gupta in 1984. Now, in the present paper we introduce and study a new generalization of right (resp. left) s-weakly regular rings, which is called right (resp. left) gs-weakly regular rings as for each $a \in R$, there exists a positive integer $n = n(a)$, depending on a such that $a \in aRa^nR$ (resp. $a \in Ra^nRa$). Moreover, giving several characterizations, properties, main results of it and related it with other types of modules such as flat modules and GP-injective modules.

Keywords: *Regular Rings, s-Weakly Regular Rings, gs-Weakly regular rings, GP-Injective modules.*

1 Introduction

Throughout this paper rings are associative ring with identity and all modules are unitary. For a subset X of R, the right annihilator of X in a ring R is defined by $r(X) = \{y \in R : xy = 0, \text{for all } x \in X\}$. Similarly, define the left annihilator of X in a ring R as $\ell(X) = \{y \in R : yx = 0, \text{for all } x \in X\}$. If $X = \{a\}$ we usually abbreviation $r(a)$(resp. $\ell(a)$). An ideal I of a ring R is said to be *essential* if I has a non-zero intersection with every non-zero ideal of R.

An element a of a ring R is said to be *regular*[18] if there exists an element $b \in R$ such that $a = aba$. A ring R is said to be *von Neumann regular* (briefly, *regular*) if every element of R is regular. A ring R is said to be *strongly regular* [11] if for every $a \in R$, there exists $b \in R$ such that $a = a^2b$.

A ring R is called *right(resp. left) weakly regular*[16] if $I^2 = I$, for each right (resp. left) ideal I of R, equivalently if $x \in xRxR$ (resp. $x \in RxRx$), for every $x \in R$. R is called *weakly regular* if it is both right and left weakly regular. A ring R is said to be *right(resp. left) s-weakly regular*[8] if for every $a \in R$, then $a \in aRa^2R$ (resp. $a \in Ra^2Ra$). (a is called *right(resp. left)s-weakly regular element*).

A ring R is said to be *s-weakly regular* if it is both right and left s-weakly regular. A ring R is called *right(resp. left) weakly π-regular*[7] if $a^n \in a^nRa^nR$ (resp. $a^n \in Ra^nRa^n$). R is called *reduced* if it has no non-zero nilpotent element.

According to Cohn[6], a ring R is called reversible if $ab = 0$ implies $ba = 0$, for $a, b \in R$. A ring R is called periodic [3] if for each $x \in R$, the set $\{x, x^2, x^3, ...\}$ is finite, or equivalently, for each $x \in R$, there are positive integers $m(x)$, $n(x)$ such that $x^{m(x)} = x^{m(x)+n(x)}$, or also equivalently for each $a \in R$, some power of a is idempotent. A right R-module M is called Generalized P-injective (briefly, GP-injective) [13] if for each $a \in R$, there exists a positive integer n such that $a^n \neq 0$ and any right R-homomorphism of a^nR into M extends to one of R into M. The ring R is called right (resp. left) GP-injective if the right (resp. left) R-module R_R(resp. $_R$R) is GP-injective module.

2 Generalized s-Weakly Regular Rings (gs-Weakly Regular Rings)

In this section, we define a new generalization of right (resp. left), s-weakly regular rings, which is called *right(resp. left) gs-weakly regular rings* and we find several characterizations, properties and compare it with other types of rings.

We start this section with the following definition.

Definition 2.1 *A ring R is said to be right(resp. left) generalized s-weakly regular (briefly, gs-weakly regular) if for each $a \in R$, there exists a positive integer $n = n(a)$, depending on a such that $a \in aRa^nR$(resp. $a \in Ra^nRa$).*

A ring R is said to be *gs-weakly regular* if it is both right and left gs-weakly regular. An element a of a ring R is said to be *right(resp. left)gs-weakly regular* if there exists a positive integer n and an element b in Ra^nR such that $a = ab$ (resp. $a = ba$).

The proof of the following lemma is not hard, therefore it is omitted.

Lemma 2.2 *Every right s-weakly regular ring is a right gs-weakly regular.*

The converse of the above lemma is not true in general as it is shown in the following example (1)(ii).

Example 2.3 *(i) Periodic ring is a gs-weakly regular ring.*

(ii) If R is a ring with identity satisfies the property $a = a^{n+1}$, for each $a \in R$ and a positive integer n, then R is a gs-weakly regular rings, but it is not s-weakly regular.

The following example for non-commutative right gs-weakly regular rings.

Example 2.4 *Let* $W = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} : a, b \in R \right\}$, *where R is the set of all real numbers. Then W is a right gs-weakly regular rings, since for any positive integer $n > 1$,*

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}^n \begin{pmatrix} \frac{1}{a^n} & 0 \\ 0 & \frac{1}{b^n} \end{pmatrix}.$$

.

Theorem 2.5 *Let R be a right gs-weakly regular ring. Then, $R = Ra^nR$, for all right non-zero divisor element a in R and a positive integer n.*

Proof: Let a be a right non-zero divisor element of R. Then, $aR = aRa^nR$. Hence,

$$a(R - Ra^nR) = 0 \text{ and } R - Ra^nR \subseteq r(a).$$

Since a is a non-zero divisor. Therefore, $r(a) = \ell(a) = 0. Thus$, R=Ra^nR. □

The proof of the following results are obvious, therefore they are omitted.

Theorem 2.6 *(1) A homomorphic image of a right gs-weakly regular ring is a right gs-weakly regular.*

(2) If R is a right gs-weakly regular ring and I is a two-sided ideal of R, then R/I is a right gs-weakly regular.

Lemma 2.7 *[6] If R is a reversible ring, then $r(a) = \ell(a)$, for each $a \in R$.*

Theorem 2.8 *Let R be a reversible ring. Then R is a right gs-weakly regular if and only if R is a left gs-weakly regular.*

Proof: Let R be a right gs-weakly regular ring and $x \in R$. Then $xR = xRx^nR$ for some positive integer n. Hence $x = \sum_{i=1}^{k} xt_ix^ns_i$ for some $t_i \in R, s_i \in R$. This implies $x(1 - \sum_{i=1}^{k} t_ix^ns_i) = 0$. Then $(1 - \sum_{i=1}^{k} t_ix^ns_i) \in r(x)$. As R is reversible, then by Lemma 2.7, $(1-\sum_{i=1}^{k} t_ix^ns_i) \in \ell(x)$, we get $(1-\sum_{i=1}^{k} t_ix^ns_i)x = 0$. Hence $x = \sum_{i=1}^{k} t_ix^ns_i\ x$. Therefore, $Rx = Rx^nRx$ and hence R is left gs-weakly regular. The converse part can be proved similarly. □

Theorem 2.9 *A ring R is a right(resp. left) gs-weakly regular if and only if $Ra^nR + r(a) = R$ (resp. $Ra^nR + \ell(a) = R$), for each $a \in R$ and a positive integer n.*

Proof: Let R be a right gs-weakly regular ring. Then, for each $a \in R$, there exists $ba^nc \in Ra^nR$ such that $a = aba^nc$ for some $b, c \in R$ and a positive integer n. Then, $a(1 - ba^nc) = 0$. This implies that $(1 - ba^nc) \in r(a)$. Now $1 = ba^nc + (1 - ba^nc)$. Then, $R = Ra^nR + r(a)$.

Conversely, assume that $Ra^nR + r(a) = R$, for each $a \in R$ and a positive integer n. Then, $1 = b + d$, for some $b \in Ra^nR, d \in r(a)$ and a positive integer n. Set $b = ta^ns$, for some $t, s \in R$ and a positive integer n. Therefore, $a.1 = ab + ad = ab$. Then, $a = ata^ns$. Thus, R is a right gs-weakly regular ring. Likewise for the left gs-weakly regular ring. □

Theorem 2.10 *Let $r(a) = r(b)$, for each $a \in R$ and $b \in Ra^nR$. Then, R is a right gs-weakly regular if and only if $r(a)$ is a direct summand, for each $a \in R$ and a positive integer n.*

Proof: Let R be a right gs-weakly regular, and let $a \in R$, then there exists $b = ta^ns \in Ra^nR$ such that $a = ata^ns$, for some $t, s \in R$ and a positive integer n. Then, $a(1-ta^ns) = 0$, so $(1-ta^ns) \in r(a)$. Therefore, $1 = ta^ns+(1-ta^ns)$. Whence, $R = Ra^nR + r(a)$. Now, let $x \in Ra^nR \cap r(a)$, then $x = s_1a^ns_2$, for some $s_1, s_2 \in R$ and $ax = 0$. Thus, $as_1a^ns_2$, so $s_1a^ns_2 \in r(a) = r(b)$. Therefore, $bs_1a^ns_2 = 0$, so $bx = 0$ and hence $x = 0$. Therefore , $Ra^nR \cap r(a) = (0)$. Hence $R = Ra^nR \oplus r(a)$.

Conversely, assume that $r(a)$ is a direct summand, for every $a \in R$ .Then, there exists a right ideal I of R such that $R = r(a) + I$ and $r(a) \cap I = (0)$. In particular, there exist $i \in I$ and $d \in r(a)$ such that $1 = d + i$, multiply by a from the right we obtain $a = ad + ai$. So, $a = ai$. Since Ra^nR is a two-sided ideal of R, then $i \in Ra^nR$.Thus, R is a right gs-weakly regular ring. □

Lemma 2.11 *[1] If R is a reduced ring, then for each $a \in R$ and a positive integer n,*
(i) $r(a) = \ell(a)$.
(ii) $r(a) = r(a^2)$.
(iii) $r(a) = r(a^n)$.

The following result is a relation between gs-weakly regular ring and right weakly π-regular under a condition that R is a reduced ring.

Theorem 2.12 *If R is a reduced ring, then every right gs-weakly regular ring is a right weakly π-regular.*

Proof: Let R be a right gs-weakly regular ring. Then, for each $a \in R$, there exists $ta^n s \in Ra^n R$ such that $a = ata^n s$, for some $t, s \in R$ and a positive integer n. Then, $a(1 - ta^n s) = 0$. Therefore, $(1 - ta^n s) \in r(a)$. Since R is reduced, then by Lemma 2.11(iii), $(1 - ta^n s) \in r(a^n)$. So, $a^n = a^n ta^n s$. Thus, R is a right weakly π-regular. □

Recall that a ring R is called *right(resp. left) duo*[4] if every right (resp. left) ideal of R is two-sided.

Lemma 2.13 *[1] If R is a duo ring, then every idempotent element of R is central.*

Recall that a ring R is said to be π-biregular [15] if for any $a \in R$, $Ra^n R$ is generated by a central idempotent, for some positive integer n.

The following result is a relation between π-biregular ring with gs-weakly regular ring under a condition that R is a duo ring.

Theorem 2.14 *If R is a duo ring, then every π-biregular ring is a gs-weakly regular.*

Proof: Let R be a π-biregular, and $a \in R$. Then, there exists a central idempotent $e \in R$ such that $Ra^n R = eR$, for some positive integer n. This implies that $a^n = et = te$ and $e = ba^n c$, where $t, b, c \in R$. Since R is a duo ring. Therefore, by Lemma 2.13, $a = et = te$. Now, $a = e^2 t = ete = ae = aba^n c$. Thus, R is a right gs-weakly regular. Likewise, we show that R is a left gs-weakly regular. Whence R is a gs-weakly regular. □

The following result is a relation between the ring R is right gs-weakly regular ring with the quotient ring $R/r(a)$ is a right gs-weakly regular ring under a condition that R is a reduced ring.

Theorem 2.15 *Let R be a reduced ring. If $R/r(a)$ is a right gs-weakly regular ring, for all $a \in R$, then R is a right gs-weakly regular.*

Proof: Let $R/r(a)$ be a right gs-weakly regular, for each $a \in R$. Then, there exist $b, c \in R$ and a positive integer n such that

$$\begin{aligned} a + r(a) &= (a + r(a)).(b + r(a)).(a + r(a))^n.(c + r(a)) \\ &= aba^n c + r(a). \end{aligned}$$

This implies that $(a - aba^n c) \in r(a)$. Then, $a(a - aba^n c) = 0$. Therefore, $a^2(1 - ba^n c) = 0$. This implies that, $(1 - ba^n c) \in r(a^2)$. Since R is reduced, then by Lemma 2.11(ii), $(1 - ba^n c) \in r(a)$ and hence $a(1 - ba^n c) = 0$. Therefore, $a = aba^n c$. Thus, R is a right gs-weakly regular ring. □

Theorem 2.16 *Let R be a ring without identity. If for every $a \in R$ and a positive integer n; $a^{n+3} = a$, then R is a gs-weakly regular ring.*

Proof: It is obvious that, for every $a \in R$, set $a = a^{n+3} = a.a.a^n.a \in aRa^n R$. Whence R is a gs-weakly regular ring. □

Now, a necessary and sufficient condition for right gs-weakly regular rings to be right s-weakly regular.

Theorem 2.17 *Let R be a ring. If $a^n R = a^2 R$, for every $a \in R$ and a positive integer n. Then, every gs-weakly regular rings is s-weakly regular.*

Proof: Let R be a right gs-weakly regular ring. Then, for every $a \in R$, there exists a positive integer n such that $a = aba^n c$, for some $b, c \in R$. But, $a^n c \in a^n R = a^2 R$. Therefore, $a^n c = a^2 t$, for some $t \in R$. Now, we obtain $a = aba^2 t$ and hence R is a right s-weakly regular ring. □

Recall that an ideal P of a ring R is said to be *completely prime* [2] if for each $a, b \in R$ such that $a.b \in P$, then either $a \in P$ or $b \in P$.

Theorem 2.18 *If R is a right gs-weakly regular ring, then every completely prime ideal of R is maximal.*

Proof: Let P be a completely prime ideal of R. By contradiction, if P is not maximal, then there exists at least a maximal ideal M such that $P \subset M$. Now, we take $a \in M$, but $a \notin P$. Since R is a right gs-weakly regular ring, then there exist $t, s \in R$ and a positive integer n such that $a = ata^n s$. This implies that $a(1 - ta^n s) = 0 \in P$. Then, $(1 - ta^n s) \in P \subset M$. But $ta^n s \in M$, then $1 \in M$, this is contradiction. Thus, P is a maximal ideal. □

Recall that, an ideal I of a ring R is said to be *completely semi-prime* [12] if for every positive integer n and $a \in R$ such that $a^n \in I$ implies $a \in I$.

Theorem 2.19 *Let R be a commutative ring. If every ideal of R is completely semi-prime, then R is a gs-weakly regular ring if and only if for each ideal I of R, $I = \sqrt{I}$ holds.*

Proof: Let R be a gs-weakly regular ring. It is obvious that $I \subseteq \sqrt{I}$. Now, let $b \in \sqrt{I}$, then $b^n \in I$, for some positive integer n. Now, $b = btb^n s$, for some $t, s \in R$. But $b^n \in I$ and every ideal of R is completely semi-prime, then $b \in I$. Thus, $\sqrt{I} \subseteq I$. Whence, $I = \sqrt{I}$.
Conversely, assume that $I = \sqrt{I}$, for each ideal I of R. We take $I = aRa^nR$, for some positive integer n. Therefore, $aRa^nR = \sqrt{aRa^nR}$.Now, $a^{n+1} \in aRa^nR$, then $a \in \sqrt{aRa^nR} = aRa^nR$. Thus, $a \in aRa^nR$. Whence, R is a gs-weakly regular ring. □

Theorem 2.20 *If $a = ab$ and $r(a) = r(b)$, where $b \in Ra^nR$, for some positive integer n, then b is idempotent.*

Proof: Let $a = ab$, take $b = ta^n s$, for some $t, s \in R$ and a positive integer n. Then, $a = ata^n s$. This implies that $a(1 - ta^n s) = 0$. Therefore, $(1 - ta^n s) \in r(a) = r(ta^n s)$. This implies that, $ta^n s(1 - ta^n s) = 0$. Then, $ta^n s = (ta^n s)^2$. Therefore, $b = b^2$. Whence b is an idempotent. □

Theorem 2.21 *Let R be a duo ring. If a is a right gs-weakly regular element of R with $b \in Ra^nR$, for some positive integer n such that $a = ab$. Then*

(i) $aR \subseteq Ra^{n+1}R$.

(ii) If there exists $c \in Ra^nR$ such that $a = ac$ and $r(a) = r(c)$, then $c = b$.

Proof: (i) Let $at \in aR$, for some $t \in R$. Then, $at = aca^n d$, for some $c, d \in R$ and a positive integer n. But $ac \in aR = Ra$, since R is a duo ring, then $ac = sa$, for some $s \in R$. Thus , $at = saa^n d = sa^{n+1}d \in Ra^{n+1}R$. Whence, $aR \subseteq Ra^{n+1}R$.
(ii) $a = ab = ac$, gives $a(b - c) = 0$, which implies that $(b - c) \in r(a) = r(b) = r(c)$, so $b(b - c) = 0$ and $c(b - c) = 0$. Then, $b^2 = bc$ and $c^2 = cb$. Therefore by Theorem 2.20, $b = bc$ and $c = cb$. Since R is a duo ring, then by Lemma 2.13, b and c are central. Whence, $b = bc = cb = c$. □

Theorem 2.22 *If R is a right gs-weakly regular ring, then R is reduced.*

Proof: Let $a \in R$ and $a^n = 0$, for some positive integer n. Now, we can write a^{n-1} as follows, $a^{n-1} = a^{n-1}b(a^{n-1})^m c$, for some $b, c \in R$ and a positive integer m. Therefore, $a^{n-1} = a^{n-1}b(a^n)^m a^{-m}c$. If $a^n = 0$, then $a^{n-1} = 0$. If we repeat this process n-times, we obtain $a = 0$. Whence R is reduced. □

Theorem 2.23 *If R is a gs-weakly regular ring, then the center of R, $C(R)$ is regular ring.*

Proof: Let $a \in C(R)$. Since R is a gs-weakly regular ring, then $aR = aRa^nR$, for some positive integer n. Then, there exist $b, c \in R$ such that

$$\begin{aligned} a &= aba^nc \\ &= aba^{n-1}ac. \end{aligned}$$

Since $a \in C(R)$, then $ac = ca$. Therefore, $a = aba^{n-1}ca$. If we set $d = ba^{n-1}c \in R$, then $a = ada$ and hence $C(R)$ is a regular ring. □

Recall that, a ring R is called *weakly right duo* [19], if for each a in R, there exists a positive integer n such that $a^nR = Ra^nR$.

Theorem 2.24 *Let R be a right gs-weakly regular ring.*
(i) If R is a weakly right duo ring, then $J(R) = N(R)$.
(ii) If R is a reduced ring, then $Y(R) = (0)$.

Proof: (i) Let $0 \neq x \in J(R)$. Then, there exist $b, c \in R$ and a positive integer n such that $x = xbx^nc$. Since R is a weakly right duo ring, then $x^nc = dx^n$, for some $d \in R$. Therefore, $x = xbdx^n$. If we set $h = bd$, then $x = xhx^n$. So $x(1 - hx^n) = 0$. Since $x \in J(R)$, then $1 - hx^n$ is left invertible .Therefore, there exists an element u such that $(1 - hx^n)u = 0$. Multiply from the left by x^n, we obtain $(x^n - x^nhx^n)u = x^n$. Whence it follows that $x^n = 0$, so $x \in N(R)$ and hence $J(R) \subseteq N(R)$. But $N(R) \subseteq J(R)$. Whence $J(R) = N(R)$.

(ii) Let a be a non-zero element in $Y(R)$. Then $r(a)$ is an essential right ideal of R. Since R is a right gs-weakly regular ring, there exist $b, c \in R$ and a positive integer n such that $a = aba^nc$. Consider $r(a) \cap ba^nR$, let $x \in (r(a) \cap ba^nR)$. Then, $ax = 0$ and $x = ba^nt$, for some $t \in R$. So $aba^nt = 0$, then $aba^ncy = 0$ (let $t = cy$).Thus , $ay = 0$. Therefore, $a^nyc = 0$, so $ba^ncy = 0$ and hence $ba^nt = 0$, yields $x = 0$. Therefore, $r(a) \cap ba^nR = (0)$. Since $r(a)$ is a non-zero essential right ideal of R, then $ba = 0$ and hence $a = 0$. So, $Y(R) = (0)$. □

Theorem 2.25 *If I is a proper ideal of a right gs-weakly regular ring, then each element of I is a left zero divisor.*

Proof: Suppose $x \in I$ such that x is not a left zero divisor. Since R is right gs-weakly regular, then $xR = xRx^nR$, for some positive integer n. Therefore, if $y \in R$, then $xy = \sum_{i=1}^{k} xt_ix^ns_i$, for some $t_i \in R$, $s_i \in R$. This gives $xy - \sum_{i=1}^{k} xt_ix^ns_i = 0 = x(y - \sum_{i=1}^{k} t_ix^ns_i) = 0$. As x is not a left zero divisor, then $y - \sum_{i=1}^{k} t_ix^ns_i = 0$. This implies $y = \sum_{i=1}^{k} t_ix^ns_i \in I$. Hence $I = R$, which contradicts that I is a proper ideal of R. Thus, each element of I is a left zero divisor. □

Recall that a ring R is called *semi primitive* if $J(R) = (0)$.

Theorem 2.26 *A right(resp. left) gs-weakly regular ring is semi primitive.*

Proof: Let $x \in J(R)$. Now, R is right gs-weakly regular implies $xR = xRx^nR$, for some positive integer n. Hence, $x = \sum_{i=1}^{k} xt_ix^ns_i$, for some $t_i \in R$, $s_i \in R$. Hence $x(1 - \sum_{i=1}^{k} t_ix^ns_i) = 0$. Since $x \in J(R)$, then $1 - \sum_{i=1}^{k} t_ix^ns^n$ is a unit. Therefore, $x = 0$. As x is arbitrary, $J(R) = (0)$. Whence, R is semi primitive ring. □

Corollary 2.27 *A right (resp. left) s-weakly regular ring is semi primitive ring.*

Recall that a ring R is called *right(resp. left) non-singular* if $Y(R) = (0)(Z(R) = (0))$.

Finally, we give the following result.

Theorem 2.28 *A left(resp. right) gs-weakly regular ring is right(resp. left) non-singular.*

Proof: Let R be a left gs-weakly regular ring and $x \in Y(R_R)$. Then $r(x)$ is an essential right ideal of R. Since R is left gs-weakly regular, then $Rx = RxRx^n$, for some positive integer n.So, $x = \sum_{i=1}^{k} t_ix^ns_ix$, for some $t_i \in R, s_i \in R$. Let $y = \sum_{i=1}^{k} t_ix^ns_i$, so that $x = yx$ and therefore $yx - x = 0$.This implies that $(y-1)x = 0$. That is, $x \in r(y-1)$.Hence $xR \subseteq r(y-1)$. Now, $r(y)$ is essential right ideal of R and $r(y) \cap r(y-1) = (0)$ implies $r(y-1) = (0)$. Hence, $xR = 0$. Thus, $x = 0$. Hence $Y(R_R) = (0)$ and thus R is right non-singular. If R is right gs-weakly regular ring we can similarly prove that R is left non-singular. □

3 Main Results

In this section, we give some main results of gs-weakly regular rings and its relations with GP-injectivity, flatness, simple ring and strongly regular rings.

We start this section with the following theorem.

Theorem 3.1 *Let R be a reduced ring. If every simple right R-module is GP-injective, then R is a right gs-weakly regular rings.*

Proof: Let $a \in R$. If $Ra^nR + r(a) \neq R$, for all positive integer n, then there exists a maximal ideal M of R such that $Ra^nR + r(a) \subseteq M$. Now, define $f : a^nR \to R/M$ as $f(a^nt) = t + M$, for each $t \in R$ and a positive integer n. First we show that f is a well-defined. Let $a^nt = a^ns$, for some $t, s \in R$, then $a^n(t - s) = 0$. Therefore, $(t - s) \in r(a^n)$. Since R is reduced, then by Lemma 2.11(iii) , $(t - s) \in r(a) \subseteq M$. Therefore, $t + M = s + M$. It means that $f(a^nt) = f(a^ns)$. Whence f is a well-defined. Since R/M is GP-injective, then there exists $h \in R$ such that $f(a^nt) = (h+M)a^nt$, for each $t \in R$. Indeed, $1 + M = f(a^n) = (h + M)a^n = ha^n + M$. Therefore, $(1 - ha^n) \in M$. But $ha^n \in Ra^nR \subseteq M$, then $1 \in M$. This is contradiction . Whence, R is a right gs-weakly regular ring. □

Theorem 3.2 *[5] Let R be a ring, then R is reduced and right weakly regular if and only if for each $a \in R$, $RaR \oplus r(a) = R$.*

Theorem 3.3 *The ring R is a right gs-weakly regular if and only if R is reduced and right weakly regular.*

Proof: Let R be a reduced and right weakly regular, then by Theorem 3.2, for each $a \in R$, $RaR + r(a) = R$. Therefore, it is true for a^n since every weakly regular ring is weakly π-regular, it means that $Ra^nR + r(a^n) = R$, for all $a \in R$ and a positive integer n. Since R is reduced, then by Lemma 2.11(iii), $r(a^n) = r(a)$. Therefore, $Ra^nR+r(a) = R$. Thus, there exist $t, s \in R$ and $b \in r(a)$ such that $ta^ns + b = 1$. Multiply the previous equation from the left by a, we obtain $a = ata^ns$. Whence, R is a right gs-weakly regular ring.

Conversely, let R be a right gs-weakly regular ring, then for each $a \in R$, there exist $t_1, t_2 \in R$ and a positive integer n such that $a = at_1a^nt_2$. This implies that $a = at_1aa^{n-1}t_2$. if we set $t = a^{n-1}t_2 \in R$, then $a = at_1at$. Therefore, R is weakly regular ring. Also, by Theorem 2.22, R is reduced. □

Theorem 3.4 *[14] Let R be a ring. If $\ell(a) \subseteq r(a)$, for each $a \in R$.Then, $RaR + r(a)$ is an essential right ideal of R.*

Theorem 3.5 *[9] Let R be a ring. If $\ell(a) \subseteq r(a)$,for each $a \in R$ and every simple singular right R-module is GP-injective, then R is reduced.*

Theorem 3.6 *Let R be a ring, if $\ell(a) \subseteq r(a)$ for each $a \in R$ and every simple singular right R-module is GP-injective, then R is a right gs-weakly regular ring.*

Proof: By Theorem 3.5, R is reduced. Also, by using Theorem 3.3, it means to prove that R is a right gs-weakly regular ring, we need prove that R is a right weakly regular ring. To prove R is a weakly regular ring, we prove that $RaR+r(a)=R$, for each $a \in R$. Let $b \in R$, where $RbR+r(b) \neq R$. Then, there exists a maximal essential right ideal M of R, which contains $RbR+r(b)$. Since R/M is GP-injective, then every homomorphism from bR to R/M can be extended to one of R into R/M. Define $f: bR \to R/M$ as $f(bt)=t+M$, for each $t \in R$.Since R is reduced, then f is a well-defined. Now, since R/M is GP-injective, then there exists $c \in R$ such that $1+M=f(b)=cb+M$. Therefore, $(1-cb) \in M$. But $cb \in M$, then $1 \in M$. This is a contradiction. Therefore, $RaR+r(a)=R$, for each $a \in R$. Then, R is a right weakly regular ring. Since R is reduced, then by Theorem 3.3, R is a right gs-weakly regular ring. □

Lemma 3.7 [17] *Let I be a right(resp. left) ideal of R. Then R/I is a flat right(resp. left) R-module if and only if for each $a \in I$, there exists $b \in I$ such that $a=ba$(resp. $a=ab$).*

Theorem 3.8 *Let R be a reduced ring such that every essential ideal I of R, R/I is a flat, then R is a gs-weakly regular ring.*

Proof: Let $a \in R$ and $I=Ra^nR+r(a)$, for some positive integer n. Now, we show that I is an essential ideal of R, if it is not, then there exists a non-zero ideal K of R such that $I \cap K=(0)$. This implies that $Ra^nR \cap K=(0)$. Since $a^nR \subseteq Ra^nR$, then $a^nR \cap K=(0)$. But $a^nRK \subseteq a^nR \cap K=(0)$ and R is reduced, then $K \subseteq r(a^nR)=r(a^n)=r(a)$ by Lemma 2.11(iii). Therefore, $K \subseteq r(a)$, but $r(a) \subseteq I$. Therefore, $K \subseteq I$. This implies that $K=K \cap I=(0)$. This is a contradiction to that $K \neq (0)$. Therefore, I is an essential ideal of R. Since R/I is a flat, then by Lemma 3.7, for each $a \in I$, there exists $b \in I$ such that $a=ab=ba$. Since $b \in I$, then $I=r(a)+Ra^nR$. Also , $b=ca^nd+h$, for some $c,d \in R$ and $h \in r(a)$.Then, $a=ab=a(ca^nd+h)=aca^nd$. Also, $a=ba=(ca^nd+h)a=ca^nda$. Whence, R is a gs-weakly regular ring. □

Recall that a ring R is prime [10] in case a product of non-zero ideals is non-zero.

Theorem 3.9 *Let R be a prime ring. If R is a right gs-weakly regular ring, then R is a simple ring.*

Proof: Since R is a right gs-weakly regular ring, then by Theorem 3.3, R is reduced and by Theorem 3.2, for each $a \in R$, $R=RaR+r(a)$. Therefore, $r(a)RaR \subseteq r(a) \cap RaR=(0)$. Since R is a prime ring and by assumption $a \neq 0$, then $r(a)=0$. Therefore, $R=RaR$. □

Theorem 3.10 *let R be a right gs-weakly regular ring. If every principal right ideal of R is essential, then R is a simple ring.*

Proof: Since R is a right gs-weakly regular ring, then by Theorem 3.3, R is reduced and by Theorem 3.2, for each $a \in R$, $R = r(a) \oplus RaR$. Let $r(a) \neq 0$ and every principal right ideal of R is essential. Then, $aR \cap r(a) \neq (0)$. Then, there exists a non-zero element x such that $x \in aR \cap r(a)$. It means that $x = at$ and $ax = 0$. Therefore, $ax = a^2t = 0$. It means that $t \in r(a^2) \subseteq r(a)$. This implies that $x = at = 0$. Therefore, $aR \cap r(a) = (0)$. Since aR is an essential, then $r(a) = 0$, for each $a \in R$. Therefore, $R = RaR$, for each $0 \neq a \in R$ and hence R is a simple ring. □

Theorem 3.11 *If R is a right gs-weakly regular ring and every principal left ideal of R is a left annihilator of an element of R, then R is a strongly regular ring.*

Proof: Let $0 \neq a \in R$. Since R is a right gs-weakly regular ring, then by Theorem 2.22, R is reduced. Therefore by Lemma 2.11(i), $r(a) = \ell(a)$, for each $a \in R$. If a is not divisible by zero in R, there exists $s \in R$ such that $Ra = \ell(a)$ by assumption. This implies that $as = 0$. Then, $s = 0$. It means that $Ra = \ell(s) = R$. Then, there exists $t \in R$ such that $ta = 1$. Multiply by a from the right we obtain $a = ta^2$. Therefore, R is a strongly regular ring.

If a is divisible by zero in R, then there exists $0 \neq b \in R$ such that $a.b = 0$. If $a + b$ is divisible by zero, then there exists $0 \neq c \in R$ such that $(a + b).c = 0$. This implies that $ac = -bc$. Since $b \in r(a)$ and $-b \in r(a)$ and $a \in \ell(b) = r(b)$, also $ac \in r(b)$, then $ac = -bc \in r(b) \cap r(a)$. Also, we have $Ra = \ell(b) = r(b)$.

Let $w \in r(b) \cap r(a)$, then there exists $t \in R$ such that $w = ta$ and $aw = ata = 0$. This implies that $(ta)^2 = tata = 0$. Then, $w = ta = 0$. Therefore, $r(b) \cap r(a) = 0$. Since $ac = -bc \in r(b) \cap r(a)$, then $ac = -bc = 0$. It means that $c \in r(b) \cap r(a)$, then $c = 0$. This is a contradiction. Therefore, $(a + b)$ is not divisible by zero, then there exists $d \in R$ such that $d(a + b) = 1$. Then, $a = da^2$. Whence, R is a strongly regular ring. □

Theorem 3.12 *If R is a right gs-weakly regular ring and $Ra^n = a^nR$, for each $a \in R$ and a positive integer n, then $a = ae$ and $\ell(a) = \ell(e)$, where e is an idempotent element.*

Proof: Since R is a right gs-weakly regular ring, then for each $a \in R$, then exist $b, c \in R$ and a positive integer n such that $a = aba^nc$. Since $ba^n \in Ra^n = a^nR$, then there exists $h \in R$ such that $ba^n = a^nh$. It means that $a = aa^nhc$. If we set $t = hc$, then $a = a^{n+1}t$. This implies that $a(1 - a^nt) = 0$. Since R is a right gs-weakly regular ring, then by Theorem 2.22, R is reduced. Therefore, $(1 - a^n t) \in r(a) = \ell(a)$ by Lemma 2.11(i). Then, $(1 - a^n t)a = 0$. Therefore, $a = a^n ta$. Put $e = a^n t$, then

$$e^2 = a^n ta^n t = a^n taa^{n-1}t = aa^{n-1}t = a^n t = e.$$

Therefore, e is an idempotent.Then, $a(1 - a^n t) = 0$. Therefore $a = a^{n+1}t = aa^n t = ae$.

Let $x \in \ell(a)$, then $xa = 0$. Therefore, $xa^n = 0$, then $xa^n t = 0$, whence $xe = 0$. Thus, $x \in \ell(e)$. Therefore, $\ell(a) \subseteq \ell(e)$. Let $y \in \ell(e)$, then $ye = 0$. Therefore, $ya^n t = 0$ and hence $ya^n ta = 0$, whence $ya = 0$. Therefore, $y \in \ell(a)$.

Then, $\ell(e) \subseteq \ell(a)$. Whence, $\ell(a) = \ell(e)$. □

Finally the following main result will be given.

Theorem 3.13 *If R is a right gs-weakly regular ring, which has no zero divisor and $a^n R = Ra^n$, for each $a \in R$ and a positive integer n, then there exists a unit element u of R and an idempotent e of R such that $a = eu = ue$ and $a = (1 - e) + u$.*

Proof: Since R is a right gs-weakly regular ring and $0 \neq a \in R$. Then,

$$a = a^{n+1}t = a^n ta = ta^n a = ta^{n+1}.$$

Since R has no zero divisor, then $ata^{n-1} = a^n t = ta^n$. If we set $e = ata^{n-1} = a^n t = ta^n$ and since R is a right gs-weakly regular ring, then by Theorem 2.22, R is reduced, then $a = ae = ea$. Let $u = a + e - 1$, then

$$\begin{aligned} eu &= e(a + e - 1) = ea + e^2 - e = ea + e - e = ea = a. \\ ue &= (a + e - 1)e = ae + e^2 - e = ae + e - e = ae = a \end{aligned}$$

If we take $v = a^{n-1}t + e - 1$, then

$$\begin{aligned} uv &= (a + e - 1)(a^{n-1}t + e - 1) \\ &= a^n t + a(e - 1) + (e - 1)a^{n-1}t + (e - 1)^2 \\ &= e + ae - a + ea^{n-1}t - a^{n-1}t + e^2 - 2e + 1 \\ &= e + a - a + ea^{n-1}t - a^{n-1}t + 1 - e \\ uv &= 1 \\ vu &= (a^{n-1}t + e - 1)(a + e - 1) \\ &= a^{n-1}ta + a^{n-1}t(e - 1) + (e - 1)a + (e - 1)^2 \\ &= e + a^{n-1}te - a^{n-1}t + ea - a + e^2 - 2e + 1 \\ &= e + a^{n-1}t - a^{n-1}t + a - a + e - 2e + 1 \\ vu &= 1. \end{aligned}$$

Therefore, $a = eu = ue$ and $(1 - e) + u = 1 - e + a + e - 1 = a$. It means that $a = (1 - e) + u$. □

References

[1] M.R.M. Al-Kouri, On strongly π-regular rings, *Ph.D. Thesis*, (2000), Mosul University.

[2] V.A. Andrunakievich, On completely prime ideals of a ring, *Mat. sb. (N.S.)*, 12(3)(163) (1984), 291-296.

[3] H.E. Bell, Some commutativity results for periodic rings, *Acta. Math. Acad. Sci. Hungar.*, 28(1976), 279-283.

[4] S.H. Brown, Rings over which every simple module is rationally complete, *Canad. J. Math. Bull.*, 25(1973), 693-701.

[5] V. Camillo. and Y. Xiao, Weakly regular rings, *Comm. Algebra.*, 22(10) (1994), 4095-4112.

[6] P.M. Cohn, Reversible rings, *Bull. London Math. Soc.*, 31(1999), 641-648.

[7] V. Gupta, Weakly π-regular rings and group rings, *Math. J. Okayama Univ.*, 19(1977), 123-127.

[8] V. Gupta, A generlization of strongly regular rings, *Acta. Math. Hung.*, 43(1-2) (1984), 57-61.

[9] N.K. Kim, S.B. Nam and J.Y. Kim, On simple singular GP-injective modules, *Comm. Algebra*, 27(5) (1999), 2087-2096.

[10] L. Tsit-Yuen, *A First Course in Non Commutative Rings (2nd ed.)*, Berlin, New York: Springer-Verlag, (2001).

[11] J. Luh, A note on strongly regular rings, *Proc. Japan Acad*, 40(1964), 74-75.

[12] R.D. Mahmood, On pure ideals and pure submodules, *Ph.D. Thesis*, Mosul University, (2000).

[13] R.Y.C. Ming, On regular rings and Artinian rings II, *Riv. Math. Univ. Parma*, 4(11) (1985), 101-109.

[14] S.B. Nam, A note on simple singular GP-injective modules, *Kangween-Kyungki Math. Jour.*, 7(2) (1999), 215-218.

[15] F.S. Naoum, On π-regular rings, *Ph.D. Thesis*, Mosul University, (2004).

[16] V.S. Ramamurthi, Weakly regular rings, *Canad. Math. Bull*, 16(3) (1973), 317-322.

[17] M.B. Rege, On von Neumann regular rings and SF-rings, *Math. Japonica*, 31(6) (1986), 927-936.

[18] J. von Neumann, On regular rings, *Proc. Nat. Scad. Sci. U. S. A.*, 22(1936), 707-713.

[19] Y. Xue, Weakly right duo rings, *Pure Appl. Math. Sci.*, XXI(1-2) (1985), 19-24.

[20] A.M. Younis, On s-weakly regular rings, *M. Sc. Thesis*, Mosul University, (2003).