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Application of Sumudu Transform in Fractional Kinetic Equations

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Abstract

In view of the great importance of fractional kinetic equations, we derive the solution of a generalized fractional kinetic equation using Sumudu transform. The fractional kinetic equation discussed here can be used to investigate a wide class of known (may be new also) fractional kinetic equations, hitherto scattered in the literature. The results presented are in compact forms suitable for numerical computation. Some special cases, involving the generalized Mittag-Leffler function and Lorenzo-Hartley function are considered. The obtained results imply more precisely the known results.

Keywords: *Fractional kinetic equations, Mittag-Leffler function, Lorenzo-Hartley function, Riemann-Liouville fractional integral operator, Sumudu transform.*

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1 Introduction

Fractional kinetic equations have gained importance during the last decade due to their occurrence in certain problems in science and engineering. A spherically symmetric non-rotating, self-gravitating model of star like the Sun is assumed to be in thermal equilibrium and hydrostatic equilibrium. The star is characterized by its mass, luminosity, effective surface temperature, radius, central density and central temperature. The stellar structures and their mathematical models are investigated on the basis of above characters and some additional information related to the equation of state, nuclear energy generation rate and the opacity. The assumptions of thermal equilibrium and hydrostatic equilibrium imply that there is no time dependence in the equations describing the internal structure of the star (Kourganoff [20], Perdang [26] and Clayton [12]. Energy in such stellar structures is being produced by the process of chemical reactions (thermonuclear reactions). Computation of such chemical reactions is of the prime importance as it plays the central role in the evolution of such stellar structures. Two most important nuclear reactions (cycles) in stars, during their evolution, are pp chain (proton-proton chain) and CNO cycle (involves nuclei of carbon, nitrogen and oxygen). The total energy production and luminosity of the star is based on the pp chain and the composition of stellar plasma described by CNO cycle. The production and destruction of nuclei in such chemical reactions can be described by the reaction-type (kinetic) equations. Solutions of such reaction-type (linear/nonlinear) equations determine distribution functions of the dynamical states of a single particle. The linear reaction-type equation $\frac{\mathrm{dy}}{\mathrm{dx}} = \mathrm{y}$ can be used to describe the fundamental principles of standard Boltzmann-Gibbs statistical mechanics. The nonlinear generalization of the reaction-type equation $\frac{\mathrm{dy}}{\mathrm{dx}} = \mathrm{y}^{\mathrm{q}}$,leads to new insights into generalized Boltzmann-Gibbs statistical mechanics which is also called nonextensive statistical mechanics. In a recent work Ferro, Lavagro and Quarati [14] showed that a very small deviation from the Maxwell-Boltzman particle distribution and the use of nonextensive statistical mechanics can be applied to describe the modified nuclear reaction rates in stellar plasmas which is consistent with the need of the modification of the nuclear reaction rates of stellar plasma and their chemical composition.

If an arbitrary reaction characterized by a time dependent quantity N =N (t), then it is possible to calculate rate of change dN/dt to a balance between the destruction rate d and the production rate p of N, that is dN/dt = −d+p. In general, through feedback or other interaction mechanism, destruction and production depend on the quantity N itself: d = d(N) or p= p(N). This dependence is complicated since the destruction or production at time t depends not only on N(t) but also on the past history N(τ), $\tau <$ t, of the variable N. This may be formally represented by (Haubold and Mathai [16])

$$\frac{\mathrm{dN}}{\mathrm{dt}} = -\,\mathrm{d}\,(\mathrm{N_t}) + \mathrm{p}(\mathrm{N_t}), \tag{1}$$

where N_t denotes the function defined by $N_t(\mathrm{t}^*) = \mathrm{N}(\mathrm{t} - \mathrm{t}^*)$, $\mathrm{t}^* > 0$.

Haubold and Mathai [16] studied a special case of this equation, when spatial fluctuation or inhomogenities in quantity N(t) are neglected, is given by the equation

$$\frac{d\,\mathrm{N_i}}{dt} = -\,\mathrm{c_i}\,\mathrm{N_i(t)} \tag{2}$$

with the initial condition that N_i $(t = 0) = N_0$ is the number density of species i at time $t = 0$; constant $c_i > 0$, known as standard kinetic equation. A detailed discussion of the above equation is given in Kourganoff [20]. The solution of the above standard kinetic equation (2) is given by

$$\mathrm{N_i(t)} = \mathrm{N_0}\,e^{-c_i t} \tag{3}$$

An alternative form of the same equation can be obtained on integration:

$$\mathrm{N(t)} - \mathrm{N_0} = c\,{}_0D_t^{-1}\,\mathrm{N(t)}, \tag{4}$$

where ${}_0D_t^{-1}$is the standard integral operator. Haubold and Mathai [16] have given the fractional generalization of the standard kinetic equation (2) as

$$\mathrm{N(t)} - \mathrm{N_0} = c^{v}\,{}_0D_t^{-\nu}\,\mathrm{N(t)}, \tag{5}$$

where ${}_0D_t^{-\nu}$is the well known Riemann-Liouville fractional integral operator (Oldham and Spanier [25]; Samko, Kilbas and Marichev [29]; Miller and Ross [23]) defined by

$${}_0D_t^{-\nu} = \frac{1}{\Gamma(\nu)}\int_0^t (t-u)^{\nu-1}\,f(u)\,du, \quad R(\nu) > 0. \tag{6}$$

The solution of the fractional kinetic equation (6) is given by (see Haubold and Mathai [16])

$$\mathrm{N(t)} = \mathrm{N_0}\sum_{k=0}^{\infty}\frac{(-1)^k}{\Gamma(\nu k+1)}\,(ct)^{\nu k}. \tag{7}$$

Fractional kinetic equations are studied by many authors notably Hille and Tamarkin [17], Glöckle and Nonnenmacher [15], Saichev and Zaslavsky [28], Saxena et al. [31-33], Zaslavsky [41], Saxena and Kalla [30], Chaurasia and Pandey [9-10], Chaurasia and Kumar[8] among others, for their importance in the solution of certain physical problems.

Recently, Saxena et al. [34] investigated the solutions of the fractional reaction equation and the fractional diffusion equation. Laplace transform technique is used.

In a recent paper, Watugala [39] introduced a new integral transform, called the Sumudu transform defined for functions of exponential order. Over the set of functions,

$$A = \{f(t)\,|\,\exists M, \tau_1, \tau_2 > 0\,|f(t)| < M\,e^{|t|/\tau_j}, \text{ if } t \in (-1)^j\,[0,\infty)\}, \tag{8}$$

the Sumudu transform is defined by

$$G(u) \;=\; S\,[f(t)] \;=\; \int_0^{\infty} f\,(ut)\,e^{-t}\,dt,\ u\,\in\,(\,-\,\tau_1, \tau_2)\,. \tag{9}$$

For further detail and properties of this transform (see [2], [4], [5] and [11]). The Riemann-Liouville fractional integral of order ν is defined by (Miller and Ross [23], p.45; Kilbas et al. [18])

$$_0D_t^{-\nu}\,N(x,t) \;=\; \frac{1}{\Gamma(\nu)}\int_0^t (t-u)^{\nu-1}\,N(x,u)\,du, \tag{10}$$

where $Re(\nu) > 0$.

From Saxena et al. ([31], eqn. (16)) and Belgacem, Karaballi and Kalla ([5], p.106, eqn (2.1)) it follows that the Sumudu transform of the Riemann-Liouville fractional integral is given by

$$S\,\{_0D_t^{-\nu} f(t)\,;\,u\} \;=\; u^{\nu}\,\bar{f}(u). \tag{11}$$

In view of the results Kilbas et al. ([18], eq.(12)) and Belgacem, Karaballi and Kalla ([5], p.106, eq.(2.1)), we can easily obtain

$$S^{-1}[u^{\gamma-1}(1-\omega u^{\beta})^{-\delta}] \;=\; t^{\gamma-1}\,E^{\delta}_{\beta,\gamma}(\omega t^{\beta}). \tag{12}$$

2 Generalized Fractional Kinetic Equations

In this section, we solve a generalized fractional kinetic equation by using Sumudu transform.

Theorem 1. *If $Re(\nu_i) > 0$, $a_i > 0$, $i \in N$ and, $f(t)$ be a given function defined on R_+, then the equation*

$$N(t)\;-\;N_0\,f(t)\;=\;-\sum_{i=1}^{n} a_i\,{}_0D_t^{-\nu_i}\,N(t), \tag{13}$$

is solvable and its solution is given by

$$N(t) \;=\; N_0\sum_{\ell=0}^{\infty}(-1)^{\ell}\sum_{r_1+\ldots+r_{n-1}=\ell}\frac{(\ell)\,!}{(r_1)\,!\ldots\,(r_{n-1})\,!}\prod_{\mu=1}^{n-1}(a_{\mu+1})^{r_\mu}$$

$$\times\int_0^t f(\xi)\,(t-\xi)^{\sum_{\mu=1}^{n-1} r_\mu\nu_{\mu+1}}\,E^{(\ell+1)}_{\nu_1,\sum_{\mu=1}^{n-1} r_\mu\nu_{\mu+1}}[-a_1(t-\xi)^{\nu_1}]\,d\xi, \tag{14}$$

where the summation (14) is taken over all non-negative integers $r_1,\ldots,r_{n-1}$ such that $r_1 + \ldots + r_{n-1} = \ell$, and provided that the series and integral in (14) are convergent.

Proof. Applying the Sumudu transform both the sides of equation (13), we get

$$\bar{N}(u) - N_0\,\bar{f}(u) \;=\; -\,[a_1u^{\nu_1} + a_2u^{\nu_2} + ... + a_nu^{\nu_n}]\;\bar{N}(u)\,. \tag{15}$$

Solving for $\bar{N}(u)$,it gives

$$\bar{N}(u) \;=\; \frac{N_0\;\bar{f}(u)}{[1 + a_1u^{\nu_1} + a_2\,u^{\nu_2} + ... + a_nu^{\nu_n}]}$$

$$=\; N_0\,\bar{f}(u)\,\sum_{\ell=0}^{\infty}(-1)^{\ell}\,\frac{\left(\sum_{i=1}^{n-1} a_{i+1}\,u^{\nu_{i+1}}\right)^{\ell}}{(1 + a_1\,u^{\nu_1})^{\ell+1}}\,. \tag{16}$$

If we employ the identity (Abramowitz and Stegun, [1], p.823)

$$(x_1 + ... + x_m)^{\ell} = \sum_{r_1+...+r_m=\ell}\frac{(\ell)\,!}{(r_1)\,!\,...\,(r_m)\,!}\,\prod_{\mu=1}^{m} x_{\mu}^{r_\mu}\,, \tag{17}$$

where the summation is taken over all non-negative integers $r_1,\ldots,r_m$, such that $r_1 + ... + r_m = \ell$,then for $|\,a_1\,u^{\nu_1}| < 1$, (16) transform into the form

$$\bar{N}(u) \;=\; N_0u\,\bar{f}(u)\,\sum_{\ell=0}^{\infty}(-1)^{\ell}\sum_{r_1+...+r_{n-1}=\ell}\frac{(\ell)\,!}{(r_1)\,!\,...\,(r_{n-1})\,!}$$

$$\times\left\{\prod_{\mu=1}^{n-1}(a_{\mu+1})^{r_\mu}\right\}\,\frac{u^{\sum_{\mu=1}^{n-1} r_\mu\nu_{\mu+1}-1}}{(1 + a_1\,u^{\nu_1})^{\ell+1}}\,. \tag{18}$$

Now, taking inverse Sumudu transform both the sides of (18) and making use of the formula (12) and applying the convolution theorem of the Sumudu transform, we obtain the desired result (14).

3 Special Cases

When $\nu_i = i\,\nu$, $a_i = \begin{pmatrix} n \\ i \end{pmatrix} c^{i\nu}$ $(i \in N)$, we obtain the following result given by Saxena et al. [34]

Corollary 1. *If Re (ν) > 0, c > 0 and f(x) ∈ R+, then for the solution of the equation*

$$N(t) \;-\; N_0\,f(t) \;=\; -\sum_{r=1}^{n}\begin{pmatrix} n \\ r \end{pmatrix} c^{\nu r}D_t^{-\nu r}N(t), \tag{19}$$

there holds the formula

$$N(t) \;=\; N_0 \frac{d}{dt} \int_0^t f(u)\, E^{n}_{\nu,1}[-c^{\nu}(t-u)^{\nu}]\, du. \qquad (20)$$

If we set $\nu_i = i\,\nu$, $a_i = \binom{n}{i} c^{i\nu}$ and $f(t) \;=\; G_{\nu,\mu,\delta}(-c^{\nu}, b, t)$, we arrive at the following result recently obtained by Chaurasia and Pandey [10].

Corollary 2. *If $c > 0$, $b \geq 0$, $\delta > 0$, $\nu > 0$, $\mu > 0$, $(\delta\nu - \mu) > 0$, then for the solution of*

$$N(t) \;-\; N_0\, G_{\nu,\mu,\delta}(-c^{\nu}, b, t) \;=\; -\sum_{r=1}^{n} \binom{n}{r} c^{r\nu}\, {}_0D_t^{-r\nu} N(t), \qquad (21)$$

there holds the formula

$$N(t) \;=\; N_0\, G_{\nu,(\mu+\nu n),(\delta+n)}\,(-c^{\nu}, b, t). \qquad (22)$$

If we take $n \;=\; 1$, $a_1 = c^{\nu}$, $\nu_1 = \nu$ and $f(t) \;=\; G_{\nu,\mu,\delta}(-c^{\nu}, b, t)$,we obtain the result derived by Chaurasia and Pandey [9].

Corollary 3. *If $c > 0$, $b \geq 0$, $\delta > 0$, $\nu > 0$, $\mu > 0$, $(\delta\nu - \mu) > 0$, then for the solution of*

$$N(t) \;-\; N_0\, G_{\nu,\mu,\delta}(-c^{\nu}, b, t) \;=\; -\, c^{\nu}\, {}_0D_t^{-\nu} N(t), \qquad (23)$$

there holds the formula

$$N(t) \;=\; N_0\, G_{\nu,(\mu+\nu),(\delta+1)}\,(-c^{\nu}, b, t). \qquad (24)$$

If we set $n \;=\; 1$, $a_1 \;=\; c^{\nu}$, $\nu_1 \;=\; \nu$ and $f(t) \;=\; t^{\gamma-1} E^{\delta}_{\nu,\gamma}[-(ct)^{\nu}]$,it yields the results obtained by Saxena et al. [34].

Corollary 4. *If* $\mathrm{Re}\,(\nu) \;>\; 0$, $\mathrm{Re}\,(\gamma) \;>\; 0$, $c \;>\; 0$,*then for the solution of the equation*

$$N(t) \;-\; N_0\, t^{\gamma-1}\, E^{\delta}_{\nu,\gamma}[(-ct)^{\nu}] \;=\; -\sum_{r=1}^{n} \binom{n}{r} c^{r\nu}\, {}_0D_t^{-r\nu},\; n \in N, \qquad (25)$$

holds the relation

$$N(t) \;=\; N_0\, t^{\gamma-1}\, E^{\delta+n}_{\nu,\gamma}[-(ct)^{\nu}],\; n \in N. \qquad (26)$$

For $n = 1$, equation (25) reduces to a result given by Saxena et al. [32].

Finally, on taking $n \;=\; 1$, $a_1 \;=\; c^{\nu}$, $\nu_1 = \nu$ and $f(t) \;=\; t^{\rho-1}$,we arrive at the following result given by Saxena et al. [31].

Corollary 5. *If $\nu > 0$, $\rho > 0$, $c > 0$, then the solution of the integral equation*

$$N(t) \;-\; N_0\, t^{\rho-1} = -\, c^{\nu}\, {}_0D_t^{-\nu}\, N(t), \qquad (27)$$

is given by

$$N(t) = N_0\, t^{\rho-1}\Gamma(\rho)\, E_{\nu,\rho}[-(ct)^{\nu}]. \tag{28}$$

4 Conclusion

In this paper, we have presented a solution of generalized fractional kinetic equation. The solution has been developed in terms of the Mittag-Leffler function in a compact and elegant form with the help of Sumudu transform and it's inverse. Most of the results obtained are in a form suitable for numerical computation. Fractional kinetic equation can be used to compute the particle reaction rate and describes the statistical mechanics associated with the particle distribution function. The generalized fractional kinetic equation discussed in this article, contains a number of known (may be new also) fractional kinetic equations involving various special functions (Mittag-Leffler function and Lorenzo-Hartley function). The result obtained in the present paper provides an extension of the results given by Haubold and Mathai [16], Saxena, Mathai and Haubold ([31],[32] and [33]) Chaurasia and Pandey ([9] and [10]) and Chaurasia and Kumar [8].

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