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# Approach Spaces for Near Families

**James F. Peters and Somashekhar (Som) A. Naimpally**

Department of Electrical & Computer Engineering
University of Manitoba
Winnipeg, Manitoba R3T 51-V6, Canada
Email: jfpeters@ee.umanitoba.ca
Department of Mathematics
Lakehead University
Thunder Bay, Ontario, P7B 5E1, Canada
Email: somnaimpally@yahoo.ca

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**Abstract**

*This article considers the problem of how to formulate a framework for the study of the nearness of collections of subsets of a set (also more tersely termed families of a set). The solution to the problem stems from recent work on approach spaces, near sets, and a specialised form of gap functional. The collection of all subsets of a set equipped with a distance function is an approach space.*

**Keywords** *Approach space, families, gap functional, merotopy, near sets.*
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## 1 Introduction

The problem considered in this paper is how to formulate a framework for the study of the nearness of families of sets. The solution to the problem stems from recent work on near sets [15, 14, 16, 20] and from the realisation that the nearness of collections of subsets of a set $X$ (denoted $\mathcal{P}X$) can be viewed in the context of approach spaces [7, 10, 11, 19]. The basic approach is to consider a nonempty set $X$ equipped with a distance function $\rho : \mathcal{P}X \times \mathcal{P}X :\rightarrow [0, \infty)$ satisfying certain conditions. In that case, $(X, \rho)$ is an approach space. A collection $\mathcal{A} \subset \mathcal{P}X$ is near when $\nu_B(\mathcal{A}) := \inf_{B \subset \mathcal{P}X} \sup_{A \subset \mathcal{A}} \rho(B, A) = 0$.

# 2 Approach Spaces

The collection of subsets of a nonempty set $X$ is denoted $\mathcal{P}X = 2^X$ (power set). For $A, B \subset \mathcal{P}X$, $A^\varepsilon = \{A \in \mathcal{P}X : \rho(A, B) \leq \varepsilon\}$ for a distance function $\rho : \mathcal{P}X \times \mathcal{P}X :\rightarrow [0, \infty)$. An **approach space** [11, 1] is a nonempty set $X$ equipped with a distance function $\rho$ if, and only if, for all nonempty subsets $A, B, C \subset \mathcal{P}X$, conditions (A.1)-(A.4) are satisfied. [(A.1)]

(A.1) $\rho(A, A) = 0$,

(A.2) $\rho(A, \emptyset) = \infty$,

(A.3) $\rho(A, B \cup C) = min\{\rho(A, B), \rho(A, C)\}$,

(A.4) $\rho(A, B) \leq \rho(A, C) + \underset{C \subset \mathcal{P}X}{sup} \rho(C, B)$.

**Example 1** Sample approach space.
For a nonempty subset $A \subset X$ and a nonempty set $B \subset X$, define a norm-based **gap functional** $D_{\rho_{\|\cdot\|}}(A, B)$, a variation of the gap functional introduced by S. Leader in 1959 [9] (see, also, [5]), where

$$D_{\rho_{\|\cdot\|}}(A, B) = \begin{cases} \inf\{\rho_{\|\cdot\|}(a, b) : a \in A, b \in B\}, & \text{if } A \text{ and } B \text{ are not empty,} \\ \infty, & \text{if } A \text{ or } B \text{ is empty.} \end{cases}$$

Let $\rho_{\|\cdot\|}$ denote $\| \cdot \|$: $X \times X :\rightarrow [0, \infty)$ denote the norm on $X \times X$ defined by $\rho_{\|\cdot\|}(\vec{x}, \vec{y}) = \| \vec{x} - \vec{y} \|_1 = \sum_{i=1,n} |x_i - y_i|$. A gap functional is finite-valued and symmetric. Hyperspace topologies arise from topologies determined by families of gap functionals [2].

**Lemma 2.1.** *Suppose $X$ is a metric space with distance function $\rho$, $x \in X$ and $\mathcal{A} \subset \mathcal{P}X$. Then*

$$\rho(x, \bigcup \mathcal{A}) = inf\{\rho(x, A) : A \in \mathcal{A}\}.$$

*Proof.* The proof appears in [17, p. 25]. □

**Lemma 2.2.** $D_{\rho_{\|\cdot\|}} : \mathcal{P}X \times \mathcal{P}X \rightarrow [0, \infty)$ *satisfies (A.1)-(A.4).*

*Proof.* (A.1)-(A.2) are immediate from the definition of $D_{\rho_{\|\cdot\|}}$. For all $A, B, C \subset \mathcal{P}X$, $D_{\rho_{\|\cdot\|}}$ satisfies (A.3), since, from Lemma 2.1, we have

$$D_{\rho_{\|\cdot\|}}(A, B \cup C) = inf\{D_{\rho_{\|\cdot\|}}(A, B), D_{\rho_{\|\cdot\|}}(A, C)\}.$$

$D_{\rho_{\|\cdot\|}}$ satisfies (A.4), since

$$D_{\rho_{\|\cdot\|}}(A,B) \leq D_{\rho_{\|\cdot\|}}(A,C) + \underset{C\subset\mathcal{P}X}{sup}\ D_{\rho_{\|\cdot\|}}(C,B)\}.$$

□

**Theorem 2.3.** $(X, D_{\rho_{\|\cdot\|}})$ *is an approach space.*

# 3 Descriptively Near Sets

Descriptively near sets are disjoint sets that resemble each other. Feature vectors (vectors of numbers represent feature values extracted from objects) provide a basis for set descriptions (see, *e.g.*, [15, 14, 13]). A feature-based gap functional defined for the norm on a set $X$ is introduced by J.F. Peters in [16]. Let $\Phi_n(x) = (\phi_1(x), \ldots, \phi_n(x))$ denote a **feature vector**, where $\phi_i :\rightarrow \Re$. In addition, let $\Phi_X = \{\Phi_1(x), \ldots, \Phi_{|X|}(x)\}$ denote a set of feature vectors for objects $x \in X$. In this article, a description-based gap functional $D_{\Phi_X,\rho_{\|\cdot\|}}$ is defined in terms of the Hausdorff lower distance [6] of the norm on $\mathcal{P}\Phi_X \times \mathcal{P}\Phi_Y$ for sets $X, Y \subset \mathcal{P}X$, *i.e.*,

$$D_{\Phi_X,\rho_{\|\cdot\|}}(A,B) = \begin{cases} \inf\{\rho(\Phi_X, \Phi_Y)\}, & \text{if } \Phi_X \text{ and } \Phi_Y \text{ are not empty,} \\ \infty, & \text{if } \Phi_X \text{ or } \Phi_Y \text{ is empty.} \end{cases}$$

**Theorem 3.1.** $(X, D_{\Phi_X,\rho_{\|\cdot\|}})$ *is an approach space.*

*Proof.* Immediate from the definition of $D_{\Phi_X,\rho_{\|\cdot\|}}$ and Lemma 2.2. □

Given an approach space $(X, \phi)$, define $\nu : \mathcal{P}(\mathcal{P}X) :\rightarrow [0, \infty]$ by

$$\nu(\mathcal{A}) = \inf_{x\in X} \sup_{A\in\mathcal{A}} \rho(x, A).$$

The collection $\mathcal{A} \subset \mathcal{P}X$ is **near** if, and only if $\nu(\mathcal{A}) = 0$ for some $x \in X$ [11]. The function $\nu$ is called an **approach merotopy** [19]. In the sequel, rewrite $\nu(\mathcal{A})$, replacing $x \in X$ with $B \subset \mathcal{P}X$ and $\rho_{\|\cdot\|}$, then, for a selected $B \subset \mathcal{P}X$,

$$\nu_B(\mathcal{A}) = \inf_{B\subset\mathcal{P}X} \sup_{A\in\mathcal{A}} \rho_{\|\cdot\|}(B, A).$$

Then the collection $\mathcal{A} \subset \mathcal{P}X$ is $B$-**near** if, and only if $\nu_B(\mathcal{A}) = 0$ for some $B \subset \mathcal{P}X$.

**Theorem 3.2.** *Given an approach space* $(X, D_{\Phi_X,\rho_{\|\cdot\|}})$*, a collection* $\mathcal{A} \subset \mathcal{P}X$ *is* $B$*-near if, and only if* $D_{\Phi_X,\rho_{\|\cdot\|}}(A,B) = 0$ *for some* $B \subset \mathcal{P}X$ *and for every* $A \subset \mathcal{A}$*.*

*Proof.*
$\Rightarrow$ Given that a collection $\mathcal{A} \subset \mathcal{P}X$ is $B$-near, then $\nu_B(\mathcal{A}) = 0$. Hence, for some $B \subset \mathcal{P}X$, $D_{\Phi_X,\rho_{\|\cdot\|}}(A, B) = 0$.
$\Leftarrow$ Given that $D_{\Phi_X,\rho_{\|\cdot\|}}(A, B) = 0$ for some $B \subset \mathcal{P}X$ and for every $A \subset \mathcal{A}$, it follows from the definition of $\nu_B(\mathcal{A})$ that the collection $\mathcal{A} \subset \mathcal{P}X$ is $B$-near. □

# 4 Clusters and Filters

A collection $\mathcal{C} \subset \mathcal{P}X$ is a **cluster** if, and only if $\mathcal{C}$ is a maximal near collection, *i.e.*, [(C.1)]

(C.1) $\nu(\mathcal{C}) = 0$,

(C.2) for all $C \subset X, \nu(\mathcal{C} \cup \{C\}) = 0 \Rightarrow C \in \mathcal{C}$.

Filters were introduced by H. Cartan in 1937 [3, 4]. A theory of convergence stems from the notion of a filter. A collection $\mathcal{F} \subset \mathcal{P}X$ is a **filter** if, and only if, for all nonempty $A, B \subset \mathcal{F}$, it satisfies conditions (F.1)-(F.3). [(F.1)]

(F.1) $A, B \in \mathcal{F}$ implies $A \cap B \in \mathcal{F}$,

(F.2) $B \supset A \in \mathcal{F}$ implies $B \in \mathcal{F}$,

(F.3) $\emptyset \notin \mathcal{F}$. A set $A \subset \mathcal{A} \in \mathcal{P}X$ is a **neighbourhood of a point** $x \in X$

(denoted $N_x$) in an approach space $(X, \rho)$ if, and only if there exists a $G \in A$ such that $x \in G \subset A$. For a neighbourhood $N_x$ for $a$ in an approach space $X$, point $x$ is called a **limit of a filter** $\mathcal{F}$. This is a specialisation of the notion of a neighbourhood in a topology [18] in terms of approach spaces. J.L. Kelley [8] observes that a filter $\mathcal{F}$ converges to a point $x \in X$ in an approach space $(X, \rho)$ if, and only if each neighbourhood of $x$ is a member of $\mathcal{F}$.

**Theorem 4.1.** *Let $\mathcal{F}$ be a filter in an approach space $(X, \rho)$. A point $x \in X$ is a limit of the filter if, and only if $N_x \supset \mathcal{F}$.*

*Proof.* See proof in [18]. □

**Corollary 4.2.** *Given an approach space $(X, D_{\Phi_X,\rho_{\|\cdot\|}})$, a filter $\mathcal{F} \subset \mathcal{P}X$ is $B$-near if, and only if $D_{\Phi_X,\rho_{\|\cdot\|}}(A, B) = 0$ for some $B \subset \mathcal{P}X$ and for every $A \subset \mathcal{F}$.*

*Proof.* Symmetric with the proof of Theorem 3.2. □

**Corollary 4.3.** *Given a neighbourhood $N_a \subset \mathcal{A} \in \mathcal{P}X$ an approach space $(X, D_{\Phi_X,\rho_{\|\cdot\|}})$, a filter $\mathcal{F} \subset \mathcal{P}X$ is $N_x$-near if, and only if $D_{\Phi_X,\rho_{\|\cdot\|}}(A, N_x) = 0$ for every $A \subset \mathcal{A}$.*

# 5 Grills and Stacks

A collection $\mathcal{A} \in \mathcal{P}X$ is a **stack** if, and only if

$$\text{for all } A, B \subset X : (A \in \mathcal{A} \text{ and } B \supset A) \Rightarrow B \in \mathcal{A}.$$

There is a particular form of stack called a grill. A **grill** $\mathcal{G} \subset \mathcal{P}X$ on a set $X$ is nonempty stack satisfying

$$\text{for all } G, H \subset X : G \cup H \in \mathcal{G} \Rightarrow (G \in \mathcal{G} \text{ or } H \in \mathcal{G}).$$

The correspondence between grills and filters relies on the sec operator [11] such that

$$\text{for } \mathcal{A} \subset \mathcal{P}X, sec(\mathcal{A}) = \{B \subset X : \forall A \in \mathcal{A}, A \cap B \neq \emptyset\}.$$

**Theorem 5.1.**

*(1) A collection $\mathcal{F}$ is a filter if, and only if $sec(\mathcal{F})$ is a grill.*

*(2) A collection $\mathcal{G}$ is a grill if, and only if $sec(\mathcal{G})$ is a filter.*

**Lemma 5.2.** *Every cluster is a grill.*

The proof appears in [11].

**Corollary 5.3.** *Every cluster is a near grill.*

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