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# On $n$-normed linear space valued strongly $\nabla_r$-Cesàro and strongly $\nabla_r$-lacunary summable sequences

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**Abstract**

*In this article we introduce the spaces $|\sigma_1|(X, \nabla_r)$ and $N_\theta(X, \nabla_r)$ of $X$-valued strongly $\nabla_r$-Cesàro summable and strongly $\nabla_r$-lacunary summable sequences respectively, where $X$, a real linear n-normed space and $\nabla_r$ is a new difference operator, where r is a non-negative integer. This article extends the notion of strongly Cesàro summable and strongly lacunary summable sequences to n-normed linear space valued (n-nls valued) difference sequences. We study these spaces for existence of norm as well as for completeness. Further we investigate the relationship between these spaces.*

**Keywords:** *n-norm; Difference sequence space; Cesàro summable sequence; Lacunary summable sequence; Completeness.*

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# 1 Introduction

Let $w$ denote the space of all real or complex sequences. By $c$, $c_0$ and $\ell_\infty$, we denote the Banach spaces of convergent, null and bounded sequences $x = (x_k)$, respectively normed by $\|x\| = \sup_k |x_k|$.

Let $Z$ be a sequence space, then Kizmaz [8] introduced the following sequence spaces: $Z(\Delta) = \{x = (x_k) \in w : (\Delta x_k) \in Z\}$, for $Z = \ell_\infty$, $c$, $c_0$, where $\Delta x_k = x_k - x_{k+1}$, for all $k \in N$.

The following definitions can be found in [5].

The spaces $|\sigma_1|$ of strongly Cesàro summable sequence is defined as follows:

$$|\sigma_1| = \{x = (x_k) : \text{there exists } L \text{ such that } \frac{1}{p}\sum_{k=1}^{p} |x_k - L| \to 0\},$$

which is a Banach space normed by

$$\|x\| = \sup_p \left(\frac{1}{p}\sum_{k=1}^{p} |x_k|\right).$$

By a lacunary sequence $\theta = (k_p)$, $p = 1, 2, 3, \ldots$, where $k_0 = 0$, we mean an increasing sequence of non-negative integers with $h_p = (k_p - k_{p-1}) \to \infty$ as $p \to \infty$. We denote $I_p = (k_{p-1}, k_p]$ and $\eta_p = \frac{k_p}{k_{p-1}}$ for $p = 1, 2, 3, \ldots$. The space of strongly lacunary summable sequence $N_\theta$ is defined as follows:

$$N_\theta = \{x = (x_k) : \lim_{p\to\infty} \frac{1}{h_p}\sum_{k\in I_p} |x_k - L| = 0, \text{ for some } L\}.$$

The space $N_\theta$ is a Banach space with the norm

$$\|x\|_\theta = \sup_p \frac{1}{h_p}\sum_{k\in I_p} |x_k|.$$

The concept of 2-normed spaces was initially developed by Gähler in the mid of 1960's, which can be found in [6] while that of $n$-normed spaces can be found in [9]. Since then, many others have studied this concept and obtained various results; see for instance [1, 3, 4, 7].

**Definition 1.1** *Let $n \in N$ and $X$ be a real linear space of dimension $d \geq n \geq 2$. A real valued function $\|\bullet, \bullet, \cdots, \bullet\| : X^n \to R$ satisfying the following four properties:*

$(nN_1)$ : $\|x_1, x_2, \ldots, x_n\| = 0$ *if and only if* $x_1, x_2, \ldots, x_n$ *are linearly dependent vectors,*

$(nN_2)$ : $\|x_1, x_2, \ldots, x_n\| = \|x_{j_1}, x_{j_2}, \ldots, x_{j_n}\|$ *for every permutation* $(j_1, j_2, \ldots, j_n)$ *of* $(1, 2, \ldots, n)$ *i.e.,* $\|x_1, x_2, \ldots, x_n\|$ *is invariant under any permutation of* $x_1, x_2, \ldots, x_n$.

$(nN_3)$ : $\|\alpha x_1, x_2, \ldots, x_n\| = |\alpha|\ \|x_1, x_2, \ldots, x_n\|$ *for all* $\alpha \in R$

$(nN_4)$ : $\|x + x', x_2, \ldots, x_n\| \leq \|x, x_2, \ldots, x_n\| + \|x', x_2, \ldots, x_n\|$ *for all* $x, x', x_2, \ldots, x_n \in X$, *is called an $n$-norm on $X$ and the pair $(X, \|\bullet, \bullet, \cdots, \bullet\|)$ is called linear $n$-normed space.*

The standard $n$-norm on $X$, a real inner product space of dimension $d \geq n$ is as follows:

$$\|x_1, x_2, \ldots, x_n\|_S = \begin{vmatrix} < x_1, x_1 > & \cdots & < x_1, x_n > \\ \vdots & \vdots & \vdots \\ < x_n, x_1 > & \cdots & < x_n, x_n > \end{vmatrix}^{\frac{1}{2}}$$

where $< ., . >$ denotes the inner product on $X$. If $X = R^n$, then this $n$-norm is exactly the same as the Euclidean $n$-norm, $\|x_1, x_2, \ldots, x_n\|_E$ as mention below. For n=1, this $n$-norm is the usual norm $\|x_1\| =< x_1, x_1 >^{\frac{1}{2}}$.

A trivial example of an $n$-normed space is $X = R^n$ equipped with the following Euclidean $n$-norm:

$$\|x_1, x_2, \ldots, x_n\|_E = abs \left( \begin{vmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \vdots & \vdots \\ x_{n1} & \cdots & x_{nn} \end{vmatrix} \right)$$

where $x_i = (x_{i1}, \ldots, x_{in}) \in R^n$ for each $i = 1, 2, \ldots, n$.

**Definition 1.2** *A sequence $(x_k)$ in a linear $n$-normed space $(X, \|\bullet, \bullet, \cdots, \bullet\|)$ is said to convergent to $L \in X$ if $\lim_{k\to\infty} \|x_k - L, w_2, w_3, \ldots, w_n\| = 0$, for every $w_2, w_3, \ldots, w_n \in X$.*

**Definition 1.3** *A sequence $(x_k)$ in a linear $n$-normed space $(X, \|\bullet, \bullet, \cdots, \bullet\|)$ is called Cauchy sequence if $\lim_{k,m\to\infty} \|x_k - x_m, w_2, w_3 \ldots, w_n\| = 0$, for every $w_2, w_3, \ldots, w_n \in X$.*

**Definition 1.4** *A linear $n$-normed space $X$ is said to be complete if every Cauchy sequence in $X$ is convergent. A complete $n$-normed space is called an $n$-Banach space.*

Now we state the following important result [7] on $n$-norms as Lemma.

**Lemma 1.5** *A standard $n$-normed space is complete if and only if it is complete with respect to usual norm $\|.\|_S =< ., . >^{\frac{1}{2}}$.*

# 2 The Spaces $|\sigma_1|(X, \nabla_r)$ and $N_\theta(X, \nabla_r)$

Throughout this section $(X, \|\bullet, \bullet, \cdots, \bullet\|_X)$ will be a real linear $n$-normed space and $w(X)$ will denotes $X$-valued sequence space. The $n$-norm $\|\bullet, \bullet, \cdots, \bullet\|_X$ on $X$ is either a standard $n$-norm or a non-standard $n$-norm. In general we write $\|\bullet, \bullet, \cdots, \bullet\|_X$ and for standard case we write $\|\bullet, \bullet, \cdots, \bullet\|_S$.

Let $r$ be a non-negative integer. Then we define the following sequence space:
$|\sigma_1|(X,\nabla_r) = \{x \in w(X) : \lim_{p\to\infty} \frac{1}{p}\sum_{k=1}^{p} \|\nabla_r x_k - L, z_1, z_2, \ldots, z_{n-1}\|_X = 0$, for every $z_1, z_2, \ldots, z_{n-1} \in X$ and for some $L\}$, where $\nabla_r x_k = x_k - x_{k-r}$ with $\nabla_0 x_k = x_k$, for all $k \in N$ (See for details in [2]). In this expansion, we take $x_k = 0$, for all non-positive values of $k$. For $L = 0$, we write this space as $|\sigma_1|^0(X,\nabla_r)$.

We call $|\sigma_1|(X,\nabla_r)$, the set of all $X$-valued strongly $\nabla_r$-Cesàro summable sequences.

Let $\theta$ be a lacunary sequence and $r$ be a non-negative integer. Then we define the following space:
$N_\theta(X,\nabla_r) = \{x \in w(X) : \lim_{p\to\infty} \frac{1}{h_p}\sum_{k\in I_p} \|\nabla_r x_k - L, z_1, z_2, \ldots, z_{n-1}\|_X = 0$, for every $z_1, z_2, \ldots, z_{n-1} \in X$ and for some $L\}$. For $L = 0$, we write this space as $N_\theta^0(X,\nabla_r)$.

We call $N_\theta(X,\nabla_r)$, the set of all $X$-valued strongly $\nabla_r$-lacunary summable sequences.

In the special case where $\theta = (2^p)$, we have $N_\theta(X,\nabla_r) = |\sigma_1|(X,\nabla_r)$. For $r = 0$, we write the above two spaces as $|\sigma_1|(X)$ and $N_\theta(X)$ respectively.

It is obvious that $|\sigma_1|(X) \subset |\sigma_1|(X,\nabla_r)$ and $N_\theta(X) \subset N_\theta(X,\nabla_r)$. This means that every $X$-valued strongly Cesàro summable sequence is strongly $\nabla_r$-Cesàro summable and every $X$-valued strongly lacunary summable sequence is strongly $\nabla_r$-lacunary summable.

**Theorem 2.1** *(i) If $X$ is an n-Banach space then $|\sigma_1|(X,\nabla_r)$ is a Banach space normed by*

$$\|x\| = \sup_{p\geq 1,\ z_1,z_2,\ldots,z_{n-1}\in X} \left( \frac{1}{p}\sum_{k=1}^{p} \|\nabla_r x_k, z_1, z_2, \ldots, z_{n-1}\|_X \right) \tag{2.1}$$

*(ii) If $X$ is an n-Banach space then $N_\theta(X,\nabla_r)$ is a Banach space normed by*

$$\|x\|_\theta = \sup_{p\geq 1,\ z_1,z_2,\ldots,z_{n-1}\in X} \left( \frac{1}{h_p}\sum_{k\in I_p} \|\nabla_r x_k, z_1, z_2, \ldots, z_{n-1}\|_X \right) \tag{2.2}$$

**Proof.** $(i)$ It is easy to see that $|\sigma_1|(X,\nabla_r)$ is a normed linear space. To prove completeness, one may use same arguments as applied in [1] and [2].

$(ii)$ Proof of this part follows by applying similar arguments as applied to prove part $(i)$.

The following Corollary is due to Lemma 1.5.

**Corollary 2.2** *Let $X$ be equipped with standard $n$-norm. Then*

*(i) If $X$ is a Banach space then $|\sigma_1|(X,\nabla_r)$ is a Banach space normed by*

$$\|x\| = \sup_{p\geq 1,\ z_1,z_2,\ldots,z_{n-1}\in X} \left(\frac{1}{p}\sum_{k=1}^{p} \|\nabla_r x_k, z_1, z_2, \ldots, z_{n-1}\|_S\right)$$

*(ii) If $X$ is a Banach space then $N_\theta(X,\nabla_r)$ is a Banach space normed by*

$$\|x\|_\theta = \sup_{p\geq 1,\ z_1,z_2,\ldots,z_{n-1}\in X} \left(\frac{1}{h_p}\sum_{k\in I_p} \|\nabla_r x_k, z_1, z_2, \ldots, z_{n-1}\|_S\right)$$

**Proposition 2.3** *Let $\theta = (k_p)$ be a lacunary sequence with $\liminf_p \eta_p > 1$, then $|\sigma_1|(X,\nabla_r) \subseteq N_\theta(X,\nabla_r)$.*

**Proof.** Let $\liminf_p \eta_p > 1$. Then there exists $v > 0$ such that $1 + v \leq \eta_p$ for all $p \geq 1$. Let $x \in |\sigma_1|(X,\nabla_r)$. Then there exists some $L \in X$ such that

$$\lim_{t\to\infty} \frac{1}{t}\sum_{k=1}^{t} \|\nabla_r x_k - L, z_1, z_2, \ldots, z_{n-1}\|_X = 0 \text{ for every } z_1, z_2, \ldots, z_{n-1} \in X$$

Now we write $\frac{1}{h_p}\sum_{k\in I_p} \|\nabla_r x_k - L, z_1, z_2, \ldots, z_{n-1}\|_X$

$$= \frac{1}{h_p}\sum_{1\leq i\leq k_p} \|\nabla_r x_i - L, z_1, z_2, \ldots, z_{n-1}\|_X - \frac{1}{h_p}\sum_{1\leq i\leq k_{p-1}} \|\nabla_r x_i - L, z_1, z_2, \ldots, z_{n-1}\|_X$$

$$= \frac{k_p}{h_p}\left(\frac{1}{k_p}\sum_{1\leq i\leq k_p} \|\nabla_r x_i - L, z_1, z_2, \ldots, z_{n-1}\|_X\right)$$

$$-\frac{k_{p-1}}{h_p}\left(\frac{1}{k_{p-1}}\sum_{1\leq i\leq k_{p-1}} \|\nabla_r x_i - L, z_1, z_2, \ldots, z_{n-1}\|_X\right) \qquad (2.3)$$

Now we can have

$$\frac{k_p}{h_p} \leq \frac{1+v}{v} \text{ and } \frac{k_{p-1}}{h_p} \leq \frac{1}{v}, \text{ since } h_p = k_p - k_{p-1}$$

Hence using (2.3), we have $x \in N_\theta(X,\nabla_r)$.

**Proposition 2.4** *Let $\theta = (k_p)$ be a lacunary sequence with $\limsup_p \eta_p < \infty$, then $N_\theta(X,\nabla_r) \subseteq |\sigma_1|(X,\nabla_r)$.*

**Proof.** Let $\limsup_p \eta_p < \infty$. Then there exists $M > 0$ such that $\eta_p < M$ for all $p \geq 1$. Let $x \in N_\theta^0(X, \nabla_r)$ and $\epsilon > 0$. We can find $R > 0$ and $K > 0$ such that that

$$\sup_{i\geq R} S_i = \sup_{i\geq R} \left( \frac{1}{h_i} \sum_{i=1}^{k_i} \|\nabla_r x_i, z_1, z_2, \ldots, z_{n-1}\|_X - \frac{1}{h_i} \sum_{i=1}^{k_{i-1}} \|\nabla_r x_i, z_1, z_2, \ldots, z_{n-1}\|_X \right) < \varepsilon$$

and $S_i < K$ for all $i = 1, 2, \ldots$.. Then if $t$ is any integer with $k_{p-1} < t \leq k_p$, where $p > R$, we can write

$$\frac{1}{t} \sum_{i=1}^{t} \|\nabla_r x_i, z_1, z_2, \ldots, z_{n-1}\|_X \leq \frac{1}{k_{p-1}} \sum_{i=1}^{k_p} \|\nabla_r x_i, z_1, z_2, \ldots, z_{n-1}\|_X$$

$$= \tfrac{1}{k_{p-1}} (\textstyle\sum_{I_1} \|\nabla_r x_i, z_1, z_2, \ldots, z_{n-1}\|_X + \sum_{I_2} \|\nabla_r x_i, z_1, z_2, \ldots, z_{n-1}\|_X$$

$$+ \cdots + \textstyle\sum_{I_p} \|\nabla_r x_i, z_1, z_2, \ldots, z_{n-1}\|_X)$$

$$= \frac{k_1}{k_{p-1}} S_1 + \frac{k_2 - k_1}{k_{p-1}} S_2 + \ldots + \frac{k_R - k_{R-1}}{k_{p-1}} S_R + \frac{k_{R+1} - k_R}{k_{p-1}} S_{R+1} + \ldots + \frac{k_p - k_{p-1}}{k_{p-1}} S_p$$

$$\leq \left( \sup_{i\geq 1} S_i \right) \tfrac{k_R}{k_{p-1}} + \left( \sup_{i\geq R} S_i \right) \tfrac{k_p - k_R}{k_{p-1}}$$

$$< K \tfrac{k_R}{K_{p-1}} + \epsilon M$$

Since $k_{p-1} \to \infty$ as $t \to \infty$, it follows that $x \in |\sigma_1|^0(X, \nabla_r)$. The general inclusion $N_\theta(X, \nabla_r) \subseteq |\sigma_1|(X, \nabla_r)$ follows by linearity.

The following Proposition is the consequence of the above two Propositions.

**Proposition 2.5** *Let $\theta = (k_p)$ be a lacunary sequence with $1 < \liminf_p \eta_p \leq \limsup_p \eta_p < \infty$, then $|\sigma_1|(X, \nabla_r) = N_\theta(X, \nabla_r)$.*

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