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A Note on Convergence in Normed Hypervector Spaces

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Abstract

The aim of this paper is to introduce the concept of convergence of a sequence on hypernormed spaces and establish a few basic properties of convergent sequences and Cauchy sequences on hypernormed spaces. Also we have established a necessary and sufficient condition for a Cauchy sequence to be convergent sequence in this spaces. In fact, also it has been shown that limit of a convergent sequence in such type spaces is not necessarily unique.

Keywords : *Hypernorm, Cauchy sequence, convergent sequence.*

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1 Introduction

The notion of hyperstructure was first introduced by F. Marty [2] in 1934. Then he established the definition of hypergroup in 1935 in the paper [3]. Thereafter many researchers have studied this field in different views. To get an idea, see the references [4, 1, 9, 10] etc. In the definition of hypervector space, many researchers [5] have taken only the multiplication structure as a hyperstructure. In our paper [6], we have generalized this concept by considering all the structures of hypervector spaces as a hyperstructures and established a few important results, theorems there. Later on, this concept has been more generalize than the previous concepts of hypervector spaces[6] in our papers [7, 8]. In fact, only a few algebraic properties have been developed there and no mathematical analysis has been done there. In this present paper, our aim is to develop the mathematical analysis on hypervector space.

In §3 of this paper, first we have defined the norm on a hypervector space and then we have established some important results, propositions and theorems which will be needed to prove the theorems in §4. In §4, after the introduction of convergent sequences and Cauchy sequences on normed hypervector spaces, it is shown that the limit of a convergent sequence is not necessarily unique and a convergent sequence is bounded. Also we have established that all subsequential limits are equal and a necessary and sufficient condition for a Cauchy sequence to be convergent sequence on this spaces.

2 Preliminaries

Definition 2.1 *:[4] A* ***hyperoperation*** *over a non-empty set X is a mapping of $X \times X$ into the set of all non-empty subsets of X.*
A non-empty set X with exactly one hyperoperation $'\#'$ is called a ***hypergroupoid****.*
Let $(X\,,\,\#)$ be a hypergroupoid. For every point $x \in X$ and every non-empty subset A of X, we define $x \,\#\, A = \bigcup_{a \in A}\{x \,\#\, a\}$.

Definition 2.2 *[6] A hypergroupoid $(X\,,\,\#)$ is called a* ***hypergroup*** *if*
(i) $x\#\,(y \,\#\, z) = (x \,\#\, y)\,\#z,\ \forall\ x, y, z \in X$.
(ii) $\exists\ 0 \in X$ such that for every $a \in X$, there is unique element $b \in X$ for which $0 \in a \,\#\, b$ and $0 \in b\#a$. Here b is denoted by $-a$.
(iii) $\forall\ a, b, c \in X$ if $a \in b\#c$, then $b \in a\#(-c)$.

Proposition 2.3 *[4] (i) In a hypergroup $(X\,,\,\#)$, $-(-a) = a$, $\forall\, a \in X$.*
(ii) $0 \,\#\, a = \{a\}$, $\forall\, a \in X$ if $(X\,,\,\#)$ is a commutative hypergroup.
(iii) In a commutative hypergroup $(X\,,\,\#)$, 0 is unique.

Definition 2.4 *[4] A* ***hyperring*** *is a non-empty set equipped with a hyperaddition $'\#'$ and a multiplication $'.'$ such that $(X\,,\,\#)$ is a commutative hypergroup and $(X\,,\cdot)$ is a semigroup and the multiplication is distributive across the hyperaddition both from the left and from the right and $a.0 = 0.a = 0$, $\forall a \in X$, where 0 is the zero element of the hyperring.*

Definition 2.5 *[6] A **hyperfield** is a non-empty set X equipped with a hyperaddition $'\#'$ and a multiplication $'.'$ such that*
(i) $(X\,,\,\#\,,\,\cdot)$ *is a hyperring.*
(ii) $\exists$ *an element* $1 \in X$*, called the identity element such that* $a.1 = a$, $\forall\, a \in X$.
(iii) *For each non zero element* $a \in X$, $\exists$ *an element* a^{-1} *such that* $a.a^{-1}$=1.
(iv) $a.b = b.a,\ \forall\, a, b\, \in X$.

Definition 2.6 *[7] Let* $(F\,,\oplus\,,\cdot)$ *be a hyperfield and* $(V\,,\,\#)$ *be an additive commutative hypergroup. Then* V *is said to be a **hypervector space** over the hyperfield* F *if there exists a hyperoperation* $* :\ F \times V \to P^*(V)$ *such that*
(i) $a * (\alpha \,\#\, \beta) \subseteq a * \alpha \,\#\, a * \beta$, $\quad \forall\, a \in F$ *and* $\forall\, \alpha, \beta \in V$.
(ii) $(a \oplus b) * \alpha \subseteq a * \alpha \,\#\, b * \alpha$, $\quad \forall\, a, b \in F$ *and* $\forall\, \alpha \in V$.
(iii) $(a\,.\,b) * \alpha = a * (b * \alpha)$, $\quad \forall\, a, b \in F$ *and* $\forall\, \alpha \in V$.
(iv) $(-a) * \alpha = a * (-\alpha)$, $\quad \forall\, a \in F$ *and* $\forall\, \alpha \in V$.
(v) $\alpha \in 1_F * \alpha,\ \theta \in 0 * \alpha$ *and* $0 * \theta = \theta$, $\quad \forall\, \alpha \in V$.
where 1_F *is the identity element of* F, 0 *is the zero element of* F *and* θ *is zero vector of* V *and* $P^*(V)$ *is the set of all non-empty subset of* V.

Result 2.7 *In a hypervector space* V*, Show that* $-\alpha \in -1_F * \alpha$.

Proof: Since $\alpha \in 1_F * \alpha,\ \forall\ \alpha \in V \Rightarrow -\alpha \in 1_F * -\alpha,\ \forall\ \alpha \in V$
$\Rightarrow -\alpha \in -1_F * \alpha,\ \forall\ \alpha \in V$, as $(-1_F) * \alpha = 1_F * (-\alpha)$.

Definition 2.8 *[7] Let* $\mathbf{R}$ *be the set of all real numbers. The hyperfield defined on* $\mathbf{R}$ *is called the **real hyperfield**.*

3 Hypernorm spaces

Definition 3.1 *Let* $(V\,,\#\,,*)$ *be a hypervector space over the real hyperfield* $\mathbf{R}$*. A **Hypernorm** on* V *is a mapping* $\|\cdot\| : V \to \mathrm{R}$, *where* R *is a usual real space, such that for all* $a \in \mathbf{R}$ *and* $\alpha, \beta \in V$ *conditions hold*
(i) $\|\alpha\| \geq 0$.
(ii) $\|\alpha\| = 0$ *if and only if* $\alpha = \theta$.
(iii) $\sup \|\alpha \# \beta\| \leq \|\alpha\| + \|\beta\|$, *where* $\|\alpha \,\#\, \beta\| = \{\, \|x\|,\ x \in \alpha \,\#\, \beta \,\}$.
(iv) $\sup \|a * \alpha\| \leq |a| \cdot \|\alpha\|$, *where* $\|a * \alpha\| = \{\, \|x\|,\ x \in a * \alpha \,\}$.

If $\|\cdot\|$ is a hypernorm on V then the tuple $(V, \#, *)$ is said to be a **normed hypervector space** or **hypernormed space**.
Throughout our discussion, instead of normed hypervector space $(V, \#, *)$, V will be consider as a normed hypervector space.

Result 3.2 *Let V be a normed hypervector space, then $\| -\alpha\| = \|\alpha\|$, $\forall\ \alpha \in V$.*

Proof: Let $\alpha \in V$. We have $-\alpha \in -1 * \alpha \Rightarrow \| -\alpha\| \leq \sup \| -1 * \alpha\|$
$\Rightarrow \| -\alpha\| \leq | -1| \|\alpha\| = \|\alpha\|$. That is, $\| -\alpha\| \leq \|\alpha\|$.
Again $\alpha \in (-1) * (-\alpha) \Rightarrow \|\alpha\| \leq \sup \|(-1) * (-\alpha)\| \Rightarrow \|\alpha\| \leq | -1| \| -\alpha\|$.
That is, $\|\alpha\| \leq \| -\alpha\|$.
Hence, $\| -\alpha\| = \|\alpha\|$, $\forall\ \alpha \in V$.

Result 3.3 *Let V be a normed hypervector space and $\alpha, \beta \in V$, then* $\inf \|\alpha\# -\beta\| = \inf \|\beta\# -\alpha\|$.

Proof: Let $\gamma \in \alpha\# -\beta \Leftrightarrow \alpha \in \gamma\#(-(-\beta))$
$\Leftrightarrow \alpha \in \gamma\#\beta \Leftrightarrow \beta \in \alpha\# -\gamma \Leftrightarrow -\gamma \in \beta\# -\alpha$.
Now by the result 3.2, we have $\|\gamma\| = \| -\gamma\|$, $\forall\, \gamma \in V$. Therefore
$\|\alpha\# -\beta\| = \{\|x\| :\ x \in \alpha\# -\beta\} = \{\|y\| :\ y \in \beta\# -\alpha\} = \|\beta\# -\alpha\|$.
Hence $\inf \|\alpha\# -\beta\| = \inf \|\beta\# -\alpha\|$.

Theorem 3.4 *Let V be a normed hypervector space and $\alpha, \beta, \gamma \in V$, then* $\inf \|\alpha\# -\beta\| \leq \inf \|\alpha\# -\gamma\| + \inf \|\gamma\# -\beta\|$.

Proof: Let $x \in \alpha\# -\gamma$ and $y \in \gamma\# -\beta$
$\Rightarrow -\gamma \in x\# -\alpha$ and $\gamma \in y\# -(-\beta)$, (by the last axiom of definition 2.2)
$\Rightarrow -\gamma \in x\# -\alpha$ and $\gamma \in y\#\beta$
$\Rightarrow \gamma\# -\gamma \subseteq x\# -\alpha\#y\#\beta$
$\Rightarrow \theta \in x\#y\# -\alpha\#\beta$, as $\theta \in \gamma\# -\gamma$
$\Rightarrow \theta \in (x\#y)\#z$, for some $z \in -\alpha\#\beta$
$\Rightarrow -z \in x\#y$
$\Rightarrow \| -z\| \leq \sup \|x\#y\| \leq \|x\| + \|y\|$
$\Rightarrow \|z\| \leq \|x\| + \|y\|$, as $\| -z\| = \|z\|$
$\Rightarrow \inf\{\|z\| :\ z \in -\alpha\#\beta\} \leq \|x\| + \|y\|$.
Therefore, $\inf \| -\alpha\#\beta\| \leq \|x\| + \|y\|$, $\forall\ x \in \alpha\# -\gamma$ and $\forall\ y \in \gamma\# -\beta$
$\Rightarrow \inf \|\alpha\# -\beta\| \leq \|x\| + \|y\|$, $\forall\ x \in \alpha\# -\gamma$ and $\forall\ y \in \gamma\# -\beta$,
by result 3.3.
Thus $\inf \|\alpha\# -\beta\| \leq \inf\{\|x\| :\ x \in \alpha\# -\gamma\} + \inf\{\|y\| :\ y \in \gamma\# -\beta\}$.
Hence $\inf \|\alpha\# -\beta\| \leq \inf \|\alpha\# -\gamma\| + \inf \|\gamma\# -\beta\|$.

Theorem 3.5 *Let V be a normed hypervector space and $\alpha, \beta \in V$. If $\inf \|\beta\# -\alpha\| = 0$, then $\|\alpha\| = \|\beta\|$.*

Proof: By the theorem 3.4, we have $\inf \|\beta\#\theta\| \leq \inf \|\beta\# -\alpha\| + \inf \|\alpha\#\theta\|$
$\Rightarrow \|\beta\| \leq \inf \|\beta\# -\alpha\| + \|\alpha\|$, as $\alpha\#\theta = \alpha$, $\forall\ \alpha \in V$
$\Rightarrow \|\beta\| \leq \|\alpha\|$.

Again By the theorem 3.4, we have
$\inf \|\alpha\#\theta\| \leq \inf \|\alpha\# - \beta\| + \inf \|\beta\#\theta\|$
$\Rightarrow \|\alpha\| \leq \inf \|\alpha\# - \beta\| + \|\beta\|$, as $\alpha\#\theta = \alpha$, $\forall\ \alpha \in V$
$\Rightarrow \|\alpha\| \leq \inf \|\beta\# - \alpha\| + \|\beta\|$, by result 3.3
$\Rightarrow \|\alpha\| \leq \|\beta\|$. Hence $\|\alpha\| = \|\beta\|$.

Proposition 3.6 *Let V be a normed hypervector space over a real hyperfield $\mathbf{R}$, $A \subseteq V$ and $a \in \mathbf{R}$. Then* $\sup \|a * A\| \leq |a| \sup \|A\|$.

Proof: Let $\alpha \in A$. Then $\sup \|a * \alpha\| = |a| \cdot \|\alpha\| \leq |a| \sup \|A\|$.
Therefore $\sup \|a * \alpha\| \leq |a| \sup \|A\|$, for all $\alpha \in A$
$\Rightarrow \sup_{\alpha \in A} \sup \|a * \alpha\| \leq |a| \sup \|A\|$.
Hence $\sup \|a * A\| \leq |a| \sup \|A\|$.

Proposition 3.7 *Let V be a normed hypervector space over a real hyperfield $\mathbf{R}$ and $A, B \subseteq V$. Then* $\sup \|A\#B\| \leq \sup \|A\| + \sup \|B\|$.

Proof: Let $\alpha \in A$ and $\beta \in B$.
Then $\sup \|\alpha\#\beta\| \leq \|\alpha\| + \|\beta\| \leq \sup \|A\| + \sup \|B\|$, $\forall\ \alpha \in A$ and $\forall\ \beta \in B$.
Therefore $\sup_{\alpha \in A, \beta \in B} \sup \|\alpha\#\beta\| \leq \sup \|A\| + \sup \|B\|$.
Hence $\sup \|A\#B\| \leq \sup \|A\| + \sup \|B\|$.

4 sequence in normed hypervector spaces

Definition 4.1 *A sequence $\{\alpha_n\}$ in a normed hypervector space V is said to **converge** to a point $\alpha \in V$ if for any $\epsilon\,(> 0)$, there exists a positive integer n_0 such that* $\inf \|\alpha_n\#(-\alpha)\| < \epsilon$, $\forall\ n \geq n_0$.
If a sequence $\{\alpha_n\} \subseteq V$ converges to a point α in V, then we write $\lim_{n\to\infty} \alpha_n = \alpha$ *or $\alpha_n \to \alpha$ as $n \to \infty$ and we call α is a limit of $\{\alpha_n\}$ in V.*

Definition 4.2 *A sequence $\{\alpha_n\}$ in a normed hypervector space V is said to be a **Cauchy sequence** if for any $\epsilon\,(> 0)$, there exists a positive integer n_0 such that* $\inf \|\alpha_n\#(-\alpha_m)\| < \epsilon$, $\forall\ m, n \geq n_0$.
*The normed hypervector space V is said to be **complete** if every cauchy sequence in V converges to some point in V.*

Theorem 4.3 *If $\alpha_n \to \alpha$ and $\alpha_n \to \beta$ as $n \to \infty$ in a normed hypervector space V, then* $\inf \|\alpha\# - \beta\| = 0$.

Proof: Let $\epsilon > 0$, then there exists a positive Integer n_0 such that
$\inf \|\alpha_n\# - \alpha\| < \frac{\epsilon}{2}$ and $\inf \|\beta_n\# - \beta\| < \frac{\epsilon}{2}$, $\forall\ n \geq n_0$.
Now by the theorem 3.4, we have
$\inf \|\alpha\# - \beta\| \leq \inf \|\alpha\# - \alpha_n\| + \inf \|\alpha_n\# - \beta\|$ for all positive integer n.

$\Rightarrow \inf \|\alpha\# - \beta\| \leq \inf \|\alpha_n\# - \alpha\| + \inf \|\alpha_n\# - \beta\|$ for all positive integer n,
by using result 3.3
$\Rightarrow \inf \|\alpha\# - \beta\| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$, this is true for any $\epsilon > 0$.
Therefore $\inf \|\alpha\# - \beta\| = 0$.

Theorem 4.4 *Let V be a normed hypervector space and $\alpha, \beta, \gamma \in V$. If $\inf \|\beta\# - \alpha\| = 0$ and $\inf \|\gamma\# - \alpha\| = 0$, then $\inf \|\beta\# - \gamma\| = 0$.*

Proof: By the theorem 3.4, we have
$\inf \|\beta\# - \gamma\| \leq \inf \|\beta\# - \alpha\| + \inf \|\alpha\# - \gamma\|$
$\Rightarrow \inf \|\beta\# - \gamma\| \leq \inf \|\beta\# - \alpha\| + \inf \|\gamma\# - \alpha\| = 0$.
Again, we know that $\inf \|\beta\# - \gamma\| \geq 0$.
Hence $\inf \|\beta\# - \gamma\| = 0$.

Definition 4.5 *Let V be a normed hypervector space and $\alpha \in V$. Then the set $\{\beta \in V : \inf \|\beta\# - \alpha\| < r\}$, where $r > 0$ is called a **open hyperball** of redious r with centre at α in V.*

Remark 4.6 *By the theorem 4.3, it is clear that if a convergent sequence has many limit points, then every open hyperball with centre at any limit point must contain all the other limit points.*

Remark 4.7 *By the theorems 4.3 and 3.5, it is clear that all the limits of a convergent sequence have the same norm.*

Definition 4.8 *A subset F of a normed hypervector space V is said to be **bounded** if there exists a real number M such that $\|\alpha\| \leq M$, $\forall\ \alpha \in F$.*

Theorem 4.9 *Every convergent sequence in a normed hypervector space V is bounded.*

Proof: Let the sequence $\alpha_n \to \alpha$ as $n \to \infty$ in V.
Then there exists a positive number n_0 such that
$\inf \|\alpha_n\# - \alpha\| < 1$, $\forall\ n \geq n_0$.
Let $x \in \alpha_n\# - \alpha \Rightarrow \alpha_n \in x\#\alpha \Rightarrow \|\alpha_n\| \leq \sup \|x\#\alpha\| \leq \|x\| + \|\alpha\|$.
Therefore for any $x \in \alpha_n\# - \alpha$, we have $\|\alpha_n\| \leq \|x\| + \|\alpha\|$.
So, $\|\alpha_n\| \leq \inf\{\|x\| : x \in \alpha_n\#-\alpha\} + \|\alpha\|$. That is $\|\alpha_n\| \leq \inf \|\alpha_n\#-\alpha\| + \|\alpha\|$.
Thus $\|\alpha_n\| \leq 1 + \|\alpha\|$, $\forall\ n \geq n_0$.
Let $M = \max\{\|\alpha_1\|, \|\alpha_2\|, \cdots, \|\alpha_{n_0-1}\|, 1 + \|\alpha\|\}$.
Therefore, $\|\alpha_n\| \leq M$ for all positive integer n.
This completes the proof.

Theorem 4.10 *Every convergent sequence of a hyperreal space $\mathbf{R}$ is bounded.*

Proof: Since the hyperreal space $\mathbf{R}$ is also a normed hypervector space over itself where the norm of a point is defined by its absolute value. So the theorem follows by the theorem 4.9.

Theorem 4.11 *Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two convergent sequences that converge to α and β respectively. Then for every $\gamma \in \alpha\#\beta$, there exists a sequence $\{\gamma_n\}$, where $\gamma_n \in \alpha_n\#\beta_n \ \forall\ n$, such that $\gamma_n \to \gamma$.*

Proof: Since $\alpha_n \to \alpha$ and $\beta_n \to \beta$, for any positive number $\epsilon > 0$, there exists a positive integer n_0 such that
$\inf \|\alpha_n\# - \alpha\| < \frac{\epsilon}{2} \ \ \forall\ n \geq n_0$ and $\inf \|\beta_n\# - \beta\| < \frac{\epsilon}{2} \ \ \forall\ n \geq n_0$.
Let $x \in \alpha_n\# - \alpha$ and $y \in \beta_n\# - \beta$
$\Rightarrow \alpha_n \in x\#\alpha$ and $\beta_n \in y\#\beta$, (by the last axiom of definition 2.2)
$\Rightarrow \alpha \in \alpha_n\# - x$ and $\beta \in \beta_n\# - y$
$\Rightarrow \alpha\#\beta \subseteq \alpha_n\# - x\#\beta_n\# - y$
Let γ be any element of $\alpha\#\beta$.
Therefore $\gamma \in \alpha_n\#\beta_n\# - x\# - y$
Therefore for every n, there exists $t_n \in \alpha_n\#\beta_n$ and $z_n \in -x\# - y$ such that
$\gamma \in t_n\#z_n \Rightarrow z_n \in \gamma\# - t_n$
$\Rightarrow \inf \|\gamma\# - t_n\| \leq \|z_n\| \leq \sup \| - x\# - y\| \leq \| - x\| + \| - y\| = \|x\| + \|y\|$
$\Rightarrow \inf \|t_n\# - \gamma\| \leq \|x\| + \|y\|$.
Since $t_n\# - \gamma \subseteq (\alpha_n\#\beta_n)\# - \gamma$, $\ \inf \|(\alpha_n\#\beta_n)\# - \gamma\| \leq \inf \|t_n\# - \gamma\|$.
So, $\inf \|(\alpha_n\#\beta_n)\# - \gamma\| \leq \|x\| + \|y\|$, this is true for all $x \in \alpha_n\# - \alpha$ and for all $y \in \beta_n\# - \beta$. Therefore,
$\inf \|(\alpha_n\#\beta_n)\# - \gamma\| \leq \inf\{\|x\| : \ x \in \alpha_n\# - \alpha\} + \inf\{\|y\| : \ y \in \beta_n\# - \beta\}$.
$\Rightarrow \inf \|(\alpha_n\#\beta_n)\# - \gamma\| \leq \inf \|\alpha_n\# - \alpha\| + \inf \|\beta_n\# - \beta\|$.
Therefore for every n, there exists $\gamma_n \in \alpha_n\#\beta_n$ such that
$\inf \|\gamma_n\# - \gamma\| \leq \inf \|\alpha_n\# - \alpha\| + \inf \|\beta_n\# - \beta\|$ for all positive integer n. Thus $\inf \|\gamma_n\# - \gamma\| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$ whenever $n \geq n_0$.
Therefore there exists a sequence $\{\gamma_n\}$ which converges to γ.
This completes the proof.

Theorem 4.12 *Let the sequence $\{\alpha_n\}$ converges to α in V and the sequence $\{\lambda_n\}$ converges to λ in $\mathbf{R}$. Then for every $\beta \in \lambda * \alpha$, there exists a sequence $\{\beta_n\}$, where $\beta_n \in \lambda_n * \alpha_n \ \forall\ n$ such that $\beta_n \to \beta$ in V.*

Proof: Since $\{\alpha_n\}$ and $\{\lambda_n\}$ are convergent sequences in V and $\mathbf{R}$ respectively, so by theorems 4.9 and 4.10, $\{\alpha_n\}$ and $\{\lambda_n\}$ are bounded sequences. Therefore there exist positive integers M and N such that
$\|\alpha_n\| \leq M$ and $|\lambda_n| \leq N$, for all positive integer n.
Again, since $\alpha_n \to \alpha$ and $\lambda_n \to \lambda$, for any positive number $\epsilon > 0$, there exists a positive integer n_0 such that
$\inf \|\alpha_n\# - \alpha\| < \frac{\epsilon}{M+N}$, $\forall\ n \geq n_0$ and $\inf |\lambda_n \oplus -\lambda| < \frac{\epsilon}{M+N}$, $\forall\ n \geq n_0$.

Let $x \in \alpha_n \# -\alpha$ and $a \in \lambda_n \oplus -\lambda$
$\Rightarrow \alpha \in \alpha_n \# -x$ and $\lambda \in \lambda_n \oplus -a$
$\Rightarrow \lambda * \alpha \subseteq (\lambda_n \oplus -a) * (\alpha_n \# -x) = \{b * \beta :\ b \in \lambda_n \oplus -a,\ \beta \in \alpha_n \# -x\}$.
Let β be any element of $\lambda * \alpha$.
Therefore $\beta \in (\lambda_n \oplus -a) * (\alpha_n \# -x) \subseteq \{\lambda_n * (\alpha_n \# -x)\} \# \{(-a) * (\alpha_n \# -x)\}$
$\Rightarrow \beta \in \lambda_n * \alpha_n \# \lambda_n * (-x) \# (-a) * \alpha_n \# (-a) * (-x)$
$\Rightarrow \beta \in t_n \# y_n$, for some $t_n \in \lambda_n * \alpha_n$ and $y_n \in \lambda_n * (-x) \# (-a) * \alpha_n \# (-a) * (-x)$
$\Rightarrow y_n \in \beta \# -t_n$
$\Rightarrow \inf \|\beta \# -t_n\| \leq \|y_n\| \leq \sup \|\lambda_n * (-x) \# (-a) * \alpha_n \# (-a) * (-x)\|$
$\Rightarrow \inf \|\beta \# -t_n\| \leq \sup \|\lambda_n * (-x)\| + \sup \|(-a) * \alpha_n\| + \sup \|(-a) * (-x)\|$, by proposition 3.7
$\Rightarrow \inf \|\beta \# -t_n\| \leq |\lambda_n| \|-x\| + |-a| \|\alpha_n\| + |-a| \|-x\|$
$\Rightarrow \inf \|t_n \# -\beta\| \leq |\lambda_n| \|x\| + |a| \|\alpha_n\| + |a| \|x\|$, as $\inf \|\beta \# -t_n\| = \inf \|t_n \# -\beta\|$
$\Rightarrow \inf \|t_n \# -\beta\| \leq N\|x\| + |a|M + |a|\|x\|$,
Since $t_n \# -\beta \subseteq (\lambda_n * \alpha_n) \# -\beta,\ \inf \|(\lambda_n * \alpha_n) \# -\beta\| \leq \inf \|t_n \# -\beta\|$.
Therefore, $\inf \|(\lambda_n * \alpha_n) \# -\beta\| \leq N\|x\| + |a|M + |a|\|x\|$, this is true for every $x \in \alpha_n \# -\alpha$ and for every $a \in \lambda_n \oplus -\lambda$, which implies
$\inf \|(\lambda_n * \alpha_n) \# -\beta\| \leq N \inf\{\|x\| : x \in \alpha_n \# -\alpha\} + M \inf\{|a| : a \in \lambda_n \oplus -\lambda\} + \inf\{|a| : a \in \lambda_n \oplus -\lambda\}. \inf\{\|x\| : x \in \alpha_n \# -\alpha\}$
$\Rightarrow \inf \|(\lambda_n * \alpha_n) \# -\beta\| \leq N \inf \|\alpha_n \# -\alpha\| + M \inf |\lambda_n \oplus -\lambda| +$
$\inf |\lambda_n \oplus -\lambda|. \inf \|\alpha_n \# -\alpha\|$, this is true for each n.
Therefore for each n, there exists $\beta_n \in \lambda_n * \alpha_n$ such that
$\inf \|\beta_n \# -\beta\| \leq N \inf \|\alpha_n \# -\alpha\| + M \inf |\lambda_n \oplus -\lambda| + \inf |\lambda_n \oplus -\lambda|. \inf \|\alpha_n \# -\alpha\|$
$\Rightarrow \inf \|\beta_n \# -\beta\| \leq N \frac{\epsilon}{M+N} + M \frac{\epsilon}{M+N} + \frac{\epsilon}{M+N} . \frac{\epsilon}{M+N}$
$\Rightarrow \inf \|\beta_n \# -\beta\| \leq \{1 + \frac{\epsilon}{(M+N)^2}\}\epsilon < 2\epsilon$.
Therefore there exists a sequence $\{\beta_n\}$ which converges to β.
This completes the proof.

Theorem 4.13 *Let V be a normed hypervector space and $\alpha, \beta \in V$, then* $|\ \|\alpha\| - \|\beta\|\ | \leq \inf \|\alpha \# -\beta\|$.

Proof: Let $x \in \alpha \# -\beta \Rightarrow \alpha \in x \# \beta$
$\Rightarrow \|\alpha\| \leq \sup \|x \# \beta\| \leq \|x\| + \|\beta\|$
$\Rightarrow \|\alpha\| - \|\beta\| \leq \|x\|$, this is true for every $x \in \alpha \# -\beta$
$\Rightarrow \|\alpha\| - \|\beta\| \leq \inf\{\|x\| : x \in \alpha \# -\beta\}$
$\Rightarrow \|\alpha\| - \|\beta\| \leq \inf \|\alpha \# -\beta\|$. $\cdots$ (1)
Again let $x \in \alpha \# -\beta \Rightarrow -\beta \in x \# -\alpha$
$\Rightarrow \|-\beta\| \leq \sup \|x \# -\alpha\| \leq \|x\| + \|-\alpha\|$
$\Rightarrow \|\beta\| \leq \|x\| + \|\alpha\|$
$\Rightarrow \|\beta\| - \|\alpha\| \leq \|x\|$, this is true for every $x \in \alpha \# -\beta$
$\Rightarrow \|\beta\| - \|\alpha\| \leq \inf\{\|x\| : x \in \alpha \# -\beta\}$
$\Rightarrow \|\beta\| - \|\alpha\| \leq \inf \|\alpha \# -\beta\|$. $\cdots$ (2)
Therefore from (1) and (2), we get $|\ \|\alpha\| - \|\beta\|\ | \leq \inf \|\alpha \# -\beta\|$.

Corollary 4.14 *Let $\{\alpha_n\}$ be a convergent sequence in a normed hypervector space V converging to α. Then the sequence $\{\|\alpha_n\|\}$ converges to $\|\alpha\|$ in* R.

Proof: Follows from the theorem 4.13.

Theorem 4.15 *In a normed hypervector space V, every convergent sequence is a cauchy sequence.*

Proof: Let the sequence $\{\alpha_n\}$ be converge to α in V. Therefore for any $\epsilon > 0$, there exists a positive integer n_0 such that $\inf \|\alpha_n \# - \alpha\| < \frac{\epsilon}{2}, \forall\, n \geq n_0$.
Now by the theorem 3.4 we have
$\inf \|\alpha_n \# - \alpha_m\| \leq \inf \|\alpha_n \# - \alpha\| + \inf \|\alpha \# - \alpha_m\|$
$\Rightarrow \inf \|\alpha_n \# - \alpha_m\| \leq \inf \|\alpha_n \# - \alpha\| + \inf \|\alpha_m \# - \alpha\|$, by result 3.3.
$\Rightarrow \inf \|\alpha_n \# - \alpha_m\| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$ whenever $n, m \geq n_0$.
Therefore the sequence $\{\alpha_n\}$ in V is a Cauchy sequence.
This completes the proof.

Theorem 4.16 *If a sequence $\{\alpha_n\}$ converges to α in a normed hypervector space V, then every subsequence of $\{\alpha_n\}$ also converges to α in V.*

Proof: Let $\{\alpha_{n_k}\}$ be a subsequence of $\{\alpha_n\}$. Since $\{\alpha_n\}$ is a convergent sequence, by the theorem 4.15, $\{\alpha_n\}$ is a Cauchy sequence in V. Therefore for any $\epsilon > 0$, there exists a positive integer n_0 such that
$\inf \|\alpha_n \# - \alpha\| < \frac{\epsilon}{2}\ \forall\, n \geq n_0$ and $\inf \|\alpha_n \# - \alpha_m\| < \frac{\epsilon}{2}\ \forall\, n, m \geq n_0$.
Now by the theorem 3.4 we have
$\inf \|\alpha_{n_k} \# - \alpha\| \leq \inf \|\alpha_{n_k} \# - \alpha_n\| + \inf \|\alpha_n \# - \alpha\|$
$\Rightarrow \inf \|\alpha_{n_k} \# - \alpha\| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$ whenever $n_k, n \geq n_0$
$\Rightarrow \inf \|\alpha_{n_k} \# - \alpha\| < \epsilon$ whenever $n_k \geq n_0$. Therefore the subsequence $\{\alpha_{n_k}\}$ converges to α. This completes the proof.

Theorem 4.17 *In a normed hypervector space V, every Cauchy sequence is convergent if and only if it has a convergent subsequence.*

Proof: If a Cauchy sequence is convergent, then obviously it has a convergent subsequence.
Conversely, let $\{\alpha_n\}$ be a Cauchy sequence in V and $\{\alpha_{n_k}\}$ be a subsequence of $\{\alpha_n\}$ that converges to α.
Therefore for any $\epsilon > 0$, there exists a positive integer n_0 such that
$\inf \|\alpha_n \# - \alpha_m\| < \frac{\epsilon}{2}\ \forall\, n, m \geq n_0$ and $\inf \|\alpha_{n_k} \# - \alpha\| < \frac{\epsilon}{2}\ \forall\, n_k \geq n_0$.
Now by theorem 3.4 we have
$\inf \|\alpha_n \# - \alpha\| \leq \inf \|\alpha_n \# - \alpha_{n_k}\| + \inf \|\alpha_{n_k} \# - \alpha\|$
$\Rightarrow \inf \|\alpha_n \# - \alpha\| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$ whenever $n_k, n \geq n_0$.
Therefore the sequence $\{\alpha_n\}$ converges to α in V. This completes the proof.

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