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Not all Epimorphisms are Surjective in the Class of Diagonal Cylindric Algebras

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Abstract

Let α be an infinite ordinal. We show that not all epimorphisms are surjective in the class $\mathbf{Di}_\alpha$ of diagonal cylindric algebras (regarded as a concrete category). It follows that $\mathbf{Di}_\alpha$ does not have the strong amalgamation property. This answers a question of Pigozzi.

Keywords: *Algebraic logic, amalgmation property, cylindric algebras, diagonal algebras*

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1 Introduction

The class of diagonal cylindric algebras is studied by Henkin Monk and Tarski, cf. Theorem 2.6.50 in [2]. In [10] p. 327 this class is denoted by $\mathbf{Di}_\alpha$ where α is an infinite ordinal. We follow the notation of [10] which is in conformity with that adopted in [2]. One of the main results in [10] is that $\mathbf{Di}_\alpha$ has the amalgamation property AP, cf. Theorem 2.2.20 therein, and it was asked in [10] whether this class has the strong AP (SAP). Here we give a negative answer to this question. In fact, we prove a stronger result. We show that not all epimorphisms (i.e right cancellative maps) are surjective in $\mathbf{Di}_\alpha$. Independently of us Madárasz [7] proves that $\mathbf{Di}_\alpha$ does not have SAP even if the amalgam is sought in the bigger class of representable cylindric algebras. [1] In

[1]This result is only announced in [4] without proof.

our proof we use extensively the notation of [2] without reference or any kind of warning. All these are collected at the end of [2] under the title "index and symbols" p.489. In what follows α is an infinite ordinal.

2 The Main Result

Definition 2.1 *$A \in \mathbf{Di}_\alpha$ if all non-zero $x \in A$ and finite $\Gamma \subseteq \alpha$, there are distinct $k, l \notin \Gamma$ such that $x \cdot \mathsf{d}_{kl} \neq 0$.*

Theorem 2.2 *In $\mathbf{Di}_\alpha$ not all epimorphisms are surjective. In fact, there are $A, A_0 \in \mathbf{Di}_\alpha$ such that the inclusion map $A \subseteq A_0$ is not surjective and such that for all $A_1 \in \mathbf{Di}_\alpha$ and homomorphisms $m : A_0 \to A_1$ and $n : A_0 \to A_1$, if m and n coincide on A, then $m = n$. In particular, $\mathbf{Di}_\alpha$ does not have the strong amalgamation property.*

We shall need several lemmas before embarking on the proof:

Lemma 2.3 *Let $\alpha \in \Gamma \subseteq \beta$ and $i, j \in \beta \setminus \Gamma$. Let $A \in \mathbf{CA}_\beta$. Then ${}_\alpha\mathsf{s}(i,j)$ is a complete one to one endomorphism of $Cl_\Gamma A$. Furthermore, ${}_\alpha\mathsf{s}(0,1)$ is an automorphism if $|\Gamma| > 1$*

Proof We may assume that $i \neq j$ since ${}_\alpha\mathsf{s}(i,i)|Cl_\Gamma A = Id$ by [2] 1.5.13(iii) and 1.5.8(i). ${}_\alpha\mathsf{s}(i,j)$ is a complete boolean endomorphism of BlA by [2] 1.5.16. To prove that it is one to one on $Cl_\Gamma A$ it is enough to show that $x > 0 \to {}_\alpha\mathsf{s}(i,j)\mathsf{c}_\alpha x > 0$. By definition

$${}_\alpha\mathsf{s}(i,j)\mathsf{c}_\alpha x = \mathsf{s}_i^\alpha \mathsf{s}_j^i \mathsf{s}_\alpha^j \mathsf{c}_\alpha x = \mathsf{c}_\alpha(\mathsf{d}_{\alpha i}.\mathsf{c}_i(\mathsf{d}_{ij}.\mathsf{c}_j(\mathsf{d}_{j\alpha}.\mathsf{c}_\alpha x))).$$

By 1.3.8 [2], $0 < x \to 0 < \mathsf{d}_{kl}.\mathsf{c}_l x$ for every $k, l \in \beta$. The required follows. The rest of the statement follows from [2] 1.6.13 and 1.5.17.

Lemma 2.4 *Let $k, l, u \neq v$ all in α. Let $A \in \mathbf{CA}_\alpha$. Then the following hold for all $x \in A$:*

$${}_u\mathsf{s}(k,l)\mathsf{c}_u\mathsf{c}_v x = {}_u\mathsf{s}(l,k)\mathsf{c}_u\mathsf{c}_v x$$

$${}_u\mathsf{s}(k,l)_u\mathsf{s}(k,l)\mathsf{c}_u\mathsf{c}_v x = \mathsf{c}_u\mathsf{c}_v x.$$

Proof [2] 1.5.14, 1.5.17.

The equations in the above Lemma are sometimes called the merry go round indentities (MGR).

Lemma 2.5 *Let A_0 and $A_1 \in \mathbf{Di}_\alpha$. Then there exist B_0, $B_1 \in \mathbf{CA}_{\alpha+\omega}$ $i_0 : A_0 \to Nr_\alpha B_0$ and $i_1 : A_1 \to Nr_\alpha B_1$ such that for every homomorphism $f : A_0 \to A_1$ there exists a homomorphism $g : B_0 \to B_1$ such that $g \circ i_0 = i_1 \circ f$.*

Proof The argument we use is a typical step-by step construction. Let $A_0, A_1 \in \mathbf{Di}_\alpha$. We construct the desired algebras using ultraproducts. Let R be the set of all ordered quadruples $\langle \Gamma, n, k, l\rangle$ such that: $\Gamma \subseteq_\omega \omega$, $n \in \omega$, k, l are one to one (finite) sequences with

$$k, l \in {}^n(\omega \sim \Gamma), \quad Rng(k) \cap Rng(l) = \emptyset.$$

For $\Gamma \subseteq_\omega \omega$, and $n \in \omega$ put

$$X_{\Gamma,n} = \{\langle \Delta, m, k, l\rangle \in R : \ \Gamma \subseteq \Delta, \quad n \leq m\}.$$

It is straighforward to check that the set consisting of all the $X_{\Gamma,n}$'s is closed under finite intersections. Accordingly, we let M be the proper filter of $\wp(R)$ generated by the $X_{\Gamma,n}$'s. so that

$$M = \{Y \subseteq R : X_{\Gamma,n} \subseteq Y, \exists \Gamma \subseteq_\omega \omega, \ n \in \omega\}.$$

For each $\langle \Gamma, n, k, l\rangle \in R$, choose a bijection $\rho(\langle \Gamma, n, k, l\rangle)$ from $\alpha + \omega$ onto α such that

$$\rho(\langle \Gamma, n, k, l\rangle)|\Gamma \subseteq Id$$

and

$$\rho(\langle \Gamma, n, k, l\rangle)(\alpha + j) = k_j, \forall j < n.$$

Now fix $i \in \{0, 1\}$. Let

$$\mathbf{F}(A_i) = \prod_{\phi \in R} Rd^{\rho(\phi)} A_i/M$$

Here $Rd^{\rho(\phi)}A_i$ - the $\rho(\phi)$ reduct of A_i - is a $\mathbf{CA}_{\alpha+\omega}$, and so $\mathbf{F}(A_i)$ - an ultraproduct of these - is also a $\mathbf{CA}_{\alpha+\omega}$. Note too, that for each $\phi \in R$, the algebra $Rd^{\rho\phi}A_i$ has universe A_i. Now let j_i be the function from A_i into $\mathbf{F}(A_i)$ defined as follows

$$j_i x = \langle (\mathsf{s}^{k_0}_{l_0})^{A_i} \circ \ldots (\mathsf{s}^{k_{n-1}}_{l_{n-1}})^{A_i} x : \langle \Gamma, n, k, l\rangle \in R\rangle/M.$$

In [10] it is proved in Theorem 2.2.19 that $j_i \in Ism(A_i, Nr_\alpha \mathbf{F}(A_i))$. Let g be the function from $\mathbf{F}(A_0)$ into $\mathbf{F}(A_1)$ defined by:

$$g(\langle x_\phi : \phi \in R\rangle/M) = \langle f x_\phi : \phi \in R\rangle/M.$$

Then it is straightforward to check that g is the desired "lifting" function.

Proof of the Main theorem Let $\alpha \geq \omega$ and F is field of characteristic 0. Let

$$V = \{s \in {}^\alpha F : |\{i \in \alpha : s_i \neq 0\}| < \omega\}.$$

As is the custom in algebraic logic V a weak space is denoted by ${}^{\alpha}F^{(\mathbf{0})}$, where $\mathbf{0}$ is the constant 0 sequence. Note that V is a vector space over the field F. Let

$$C = \langle \wp(V), \cup, \cap, \sim, \emptyset, V, \mathsf{c}_i, \mathsf{d}_{ij} \rangle_{i,j\in\alpha}.$$

Let y denote the following α-ary relation:

$$y = \{s \in V : s_0 + 1 = \sum_{i>0} s_i\}.$$

and

$$w = \{s \in V : s_1 + 1 = \sum_{i\neq 1} s_i\}.$$

For each $s \in y$ we let y_s be the singleton containing s, i.e. $y_s = \{s\}$. Let

$$A = Sg^C(\{y, y_s : s \in y\})$$

$$A_0 = Sg^C(\{y, w, y_s : s \in y\}).$$

Here Sg is short for the subalgebra generated by. Clearly A and A_0 are in $\mathbf{Di}_\alpha$. We first show that $w \notin A$. We follow closely the argument in [8], where a similar construction using the field of rationals is used. Let

$$Pl = \{\{s \in {}^{\alpha}F^{(\mathbf{0})} : t + \sum (r_i s_i) = 0\} : \{t, r_i : i < \alpha\} \subseteq F\}.$$

$$Pl^< = \{p \in Pl : \exists i < \alpha, \mathsf{c}_i p = p\}.$$

Note that for $p \in Pl$, $p = \{s \in {}^{\alpha}F^{(\mathbf{0})} : t + \sum_i r_i s_i = 0\}$ say, then $\mathsf{c}_i p = p$ (i.e. p is parallel to the i-th axis) iff $r_i = 0$. Note too, that

$$\{y, w, \mathsf{d}_{ij} : i, j \in \alpha\} \subseteq Pl.$$

$$y, w \notin Pl^<, 1 \in Pl^<$$

and

$$\{\mathsf{d}_{ij} : i \neq j, i, j \in \alpha\} \subseteq Pl^< \leftrightarrow \alpha \geq 3.$$

Now let

$$G = \{y, -y,\ p, -p,\ \mathsf{c}_{(\Delta)}\{\mathbf{0}\}, -\mathsf{c}_{(\Delta)}\{\mathbf{0}\} : p \in Pl^< \cup \{\mathsf{d}_{01}\}\ \Delta \subseteq_\omega \alpha,\ 0 \in \Delta\}.$$

$$G^* = \{\bigcap_{i\in n} g_i : n \in \omega, g_i \in G\}.$$

and

$$G^{**} = \{\bigcup_{i\in n} g_i : n \in \omega, g_i \in G^*\}.$$

It is easy to see that $\{y, y_s : s \in y\} \subseteq G^{**}$, and G^{**} is a boolean field of sets. We prove that $w \notin G^{**}$ and that G^{**} is closed under cylindrifications. To this end, we set:

$$L = \{p \in Pl^{<} : \mathsf{c}_0 p \neq p\}, \quad P(0) = L \cup \{\mathsf{d}_{01}\}.$$

Next we define

$$G_1 = \{g \in G^* : g \subseteq y\}$$

and

$$G_2 = \{g \in G^* : g \not\subseteq y, \quad g \subseteq p, \ p \in P(0)\}.$$

We have $G_1 \cap G_2 = \emptyset$. Now let

$$G_3 = \{p_1 \cap p_2 \ldots \cap p_k : \ k \in \omega, \ \{p_1, p_2, \ldots, p_k\} \subseteq G \sim (\{y\} \cup P(0))\}.$$

It is easy to see that $G^* = G_1 \cup G_2 \cup G_3$. To prove that $w \notin G^{**}$ we need: If $g \in G_3$ and $0 \neq g$, then $g \not\subset w$. But this follows from the following. Assume that $g = p_1 \cap p_2 \ldots \cap p_k$ say, with $p_i \in G$ and $p_i \notin (\{y\} \cup P(0))$ for $1 \leq i \leq k$, and let $z \in g$. Let $[]$ be the function from Pl into F defined as follows:

$$[p] = \{1/r_0(-t - \sum r_i z_i)\}, \ p = -\{s \in {}^{\alpha}F^{(\mathbf{0})} : t + \sum r_i s_i = 0\}, r_0 \neq 0,$$

and else

$$[p] = 0.$$

Let

$$r \in F \sim ((\bigcup_{1 \leq i \leq k} [p_i]) \cup [-w])$$

be arbitrary, and let

$$z_r^0 = z \sim \{(0, z_0)\} \cup \{(0, r)\}.$$

Then

$$z_r^0 \in g \sim w, \quad g \not\subseteq w.$$

(Here we are using that when $\mathsf{c}_{(\Delta)}\{\mathbf{0}\} \in G$, then $0 \in \Delta$.) We now proceed to show that $w \notin G^{**}$. Assume that

$$x = \bigcup\{g_i^1 : i < n_1\} \cup \bigcup\{g_i^2 : i < n_2\} \cup \bigcup\{g_i^3 : i < n_3\}$$

where

$$\{g_i^j : i < n_j\} \subseteq G_j, \ g_i^j \subseteq w, \forall j \in \{1, 2, 3\}.$$

We show that $x \neq w$. By the above, we have $x \subseteq \bigcup_{i<n} p_i$ for some $\{p_i : i < n\} \subseteq P(0)$. Note that if $\alpha > 2$ then $P(0) = L$ and $P(0) = L \cup \{\mathsf{d}_{01}\}$ otherwise. If $\alpha = 2$ then $w \subseteq -\mathsf{d}_{01}$ otherwise $P(0) = L$. Now it is enough to show that w is not contained in $\bigcup E$ for any finite $E \subseteq L$. But it can be seen by implementing easy linear algebraic arguments that, for every $n \in \omega$, and for every system

$$t_0 + \sum(r_{0i}x_i) = 0$$
$$.$$
$$.$$
$$t_n + \sum(r_{ni}x_i) = 0,$$

of equations, such that for all $j \leq n$, there exists $i < \alpha$, such that

$$r_{ji} = 0 \quad r_{j0} \neq 0,$$

the equation

$$\sum_{i<\alpha} x_i = 2x_1 + 1$$

has a solution s in the weak space ${}^{\alpha}F^{(\mathbf{0})}$, such that s is not a solution of

$$t_j + \sum_{i<\alpha}(r_{ji}x_i) = 0,$$

for every $j \leq n$. We have proved that $w \notin G^{**}$. To show that $w \notin A$, we will show that G^{**} is closed under the cylindric operations (i.e it is the universe of a $\mathbf{CA}_\alpha$. It is enough to show that (since the c_i's are additive), that for $j \in \alpha$ and $g \in G^*$ arbitrary, we have $\mathsf{c}_j g \in G^{**}$. For this purpose, put for every $p \in Pl$

$$p(j|0) = \mathsf{c}_j\{s \in p : s_j = 0\}, \quad (-p)(j|0) = -p(j|0).$$

Then it is not hard to see that

$$p(j|0) = \{s \in {}^{\alpha}F^{(\mathbf{0})} : t + \sum_{i\neq j}(r_i s_i) = 0\},$$

if

$$p = \{s \in {}^{\alpha}F^{(\mathbf{0})} : t + \sum_{i<\alpha}(r_i s_i) = 0\},$$

and so

$$p(j|0) \in Pl^{<} \forall p \in Pl.$$

Let j and g be as indicated above. We can assume that

$$g = e \cap p_1 \cap \ldots \cap p_n \cap -P_1 \ldots \cap -P_m \cap z$$

$$\cap - \mathsf{c}_{(\Delta_1)}\{\mathbf{0}\} \ldots \cap -\mathsf{c}_{(\Delta_N)}\{\mathbf{0}\},$$

where

$$e \in \{y, -y, 1\}$$

$$n, m, N \in \omega \sim \{0\}, p_i, P_i \in Pl^{<} \cup \{\mathsf{d}_{01}\},$$

$$\mathsf{c}_j p_i \neq p_i, \ \mathsf{c}_j P_i \neq P_i,$$

$$z \in \{\mathsf{c}_{(\Delta)}\{\mathbf{0}\}, 1 : \Delta \in \wp_\omega \alpha, \ 0 \in \Delta, \ j \notin \Delta\},$$

and

$$\{\Delta_1, \ldots, \Delta_n\} \subseteq \{x \in \wp_\omega \alpha : \ j \notin x, 0 \in x\}.$$

We distinguish between 2 cases:

Case 1.

$$z = \mathsf{c}_{(\Delta)}\{\mathbf{0}\}, \ j \notin \Delta.$$

Then

$$\mathsf{c}_j(e \cap p_1 \ldots \cap p_n \cap -P_1 \ldots \cap -P_m$$

$$\cap \mathsf{c}_{(\Delta)}\{\mathbf{0}\} \cap -\mathsf{c}_{(\Delta_1)}\{\mathbf{0}\} \ldots \cap -\mathsf{c}_{(\Delta_N)}\{\mathbf{0}\})$$

$$p_1(j|0) \cap \ldots p_n(j|0) \cap -P_1(j|0) \ldots \cap -P_m(j|0)$$

$$\cap \mathsf{c}_j \mathsf{c}_{(\Delta)}\{\mathbf{0}\} \cap -\mathsf{c}_j \mathsf{c}_{(\Delta_1)}\{\mathbf{0}\} \cap \cap - \mathsf{c}_j \mathsf{c}_{(\Delta_N)}\{\mathbf{0}\}.$$

Case 2.

$$z = 1$$

Then

$$\mathsf{c}_j(e.p_1 \cap \ldots \cap p_n \cap -P_1 \ldots \cap -P_m$$

$$\cap - \mathsf{c}_{(\Delta_1)}\{\mathbf{0}\} \ldots \cap -\mathsf{c}_{(\Delta_N)}\{\mathbf{0}\})$$

$$= f(e) \cap_{k \leq n} ((\cap_{i \leq n} \mathsf{c}_j(p_k \cap p_i) \cap \cap_{i \leq m} \mathsf{c}_j(p_k - P_i)$$

$$\cap_{i \leq N} \mathsf{c}_j(p_k - \mathsf{c}_{(\Delta_i)}\{\mathbf{0}\})).$$

where

$$f(y) = ((\cap_{i \leq n} \mathsf{c}_j(y \cap p_i) \cap \cap_{i \leq m} \mathsf{c}_j(y - P_i)$$

$$\cap_{i \leq N} \mathsf{c}_j(y - \mathsf{c}_{(\Delta_i)}\{\mathbf{0}\})).$$

$$f(-y) = \cap_{k \leq n} \mathsf{c}_j(p_k - y)$$

$$f(1) = 1.$$

Now for every $p, q \in Pl$, there are p', q', p'' and $q'' \in Pl^{<}$ such that

$$\mathsf{c}_j(p \cap q) = p' \cap q',$$

$$\mathsf{c}_j(p \sim q) = p'' \sim q''$$

and if $j \in \Delta p \sim \Gamma$, then

$$\mathsf{c}_j(p \setminus \mathsf{c}_{(\Gamma)}\{\mathbf{0}\}) = {}^{\alpha}F^{(\mathbf{0})} \sim p(j|0) \cup (p(j|0) \sim \mathsf{c}_j \mathsf{c}_{(\Gamma)}\{\mathbf{0}\}).$$

We have proved that $w \notin A$. Now let $A_1 \in \mathbf{Di}_\alpha$ and assume that h and k are given homomorphisms from A_0 to A_1 that agree on A. It clearly suffices to show that $k(w) = h(w)$. By the above Lemma, B_0,,B_1 be ω-extensions of A_0 and A_1 via i_0 and i_1, respectively, that is $i_0 : A_0 \to Nr_\alpha B_0$ and $i_1 : A_1 \to Nr_\alpha B_1$. Let $k^* : B_0 \to B_1$ be a homomorphism such that

$$k^* \circ i_0 = i_1 \circ k,$$

and let $h^* : B_0 \to B_1$ be a homomorphism such that

$$h^* \circ i_0 = i_1 \circ h.$$

We define

$$\tau_\alpha(x) = {}_\alpha\mathsf{s}(0,1)x.$$

We will show that (*)

$$\tau_\alpha^{B_0}(i_0 y) = i_0 w.$$

By (*) we will be done because of the following:

$$k^* \circ i_0(w) = k^*(\tau_\alpha^{B_0}(i_0 y)) = \tau_\alpha^{B_1}(k^* \circ i_0(y)).$$

But since h and k agree on A and $y \in A$, we have

$$k^* \circ i_0(y) = i_1 \circ k(y) = i_1 \circ h(y) = h^* \circ i_0(y).$$

From which we get that

$$k^* \circ i_0(w) = \tau_\alpha^{B_1}(k^* \circ i_0(y)) = \tau_\alpha^{B_1}(h^* \circ i_0(y))$$

$$= h^*(\tau_\alpha^{B_0}(i_0(y)) = h^*(i_0 w) = h^* \circ i_0(w).$$

We have shown that

$$k^* \circ i_0(w) = h^* \circ i_0(w).$$

Thus

$$i_1 \circ k(w) = i_1 \circ h(w).$$

But since i_1 is one to one, it readily follows thus that

$$k(w) = h(w).$$

We are done modulu (*). We now prove (*). We write i instead of i_0. Now $t_\alpha^{B_0} x = {}_\alpha\mathsf{s}(0,1)^{B_0} x$ is always evaluated in B_0, hence for better readability we omit the superscript B_0. Let $\tau(x)$ be the following $\mathbf{CA}_2$ term:

$$\tau(x) = \mathsf{s}_1^0 \mathsf{c}_1 x . \mathsf{s}_0^1 \mathsf{c}_0 x.$$

Let

$$X = \{y_s : s \in y\}, \quad a \in X.$$

We show that

$$(1) \quad i(\tau(a)) \leq \tau_\alpha(i(y)).$$

We start by showing that

$$(+) \quad i(\tau(a)) = \tau_\alpha(i(a)).$$

Note first that $a = \mathsf{c}_1 a.\mathsf{c}_0 a$. Now we have

$$_\alpha s(0,1)i(a) = {_\alpha\mathsf{s}}(0,1)(\mathsf{c}_1 i(a).\mathsf{c}_0 i(a))$$

$$= {_\alpha\mathsf{s}}(0,1)\mathsf{c}_1 i(a).{_\alpha\mathsf{s}}(1,0)\mathsf{c}_0 i(a)$$

Here we use that ${_\alpha\mathsf{s}}(0,1)$ is an endomorphism and the MGR, namely that

$${_\alpha\mathsf{s}}(0,1)\mathsf{c}_0 i(a) = {_\alpha\mathsf{s}}(0,1)\mathsf{c}_\alpha\mathsf{c}_{\alpha+1}\mathsf{c}_0 i(a)$$

$$= {_\alpha\mathsf{s}}(1,0)\mathsf{c}_\alpha\mathsf{c}_{\alpha+1}\mathsf{c}_0 i(a) = {_\alpha\mathsf{s}}(1,0)\mathsf{c}_0 i(a).$$

We compute

$$\begin{aligned} {_\alpha\mathsf{s}}(0,1)\mathsf{c}_1 i(a) &= \mathsf{s}_0^\alpha\mathsf{s}_1^0\mathsf{s}_\alpha^1\mathsf{c}_1(i(a)) = \mathsf{s}_0^\alpha\mathsf{s}_1^0\mathsf{c}_i(i(a)) \\ &= \mathsf{s}_0^\alpha\mathsf{s}_1^0\mathsf{c}_\alpha\mathsf{c}_1(a) = \mathsf{s}_0^\alpha\mathsf{c}_\alpha\mathsf{s}_1^0\mathsf{c}_i(i(a)) = \mathsf{s}_1^0\mathsf{c}_1(i(a)). \end{aligned}$$

Similarly

$${_\alpha\mathsf{s}}(1,0)\mathsf{c}_0 i(a) = \mathsf{s}_0^1\mathsf{c}_0 i(a).$$

From this we get (+). (1) follows from (+) by noting that $\tau_\alpha(i(a)) \leq \tau_\alpha(i(y))$.

Let $X" = \{\tau(y_s) : s \in y\}$. Then clearly $w = \bigcup X"$. Now A is atomic. Indeed, A contains all singletons. To see this, let $s \in {^\alpha F^{(\mathbf{0})}}$ be arbitrary. Then

$$\langle s_0, s_0 + 1 - \sum_{i>1} s_i, s_i\rangle_{i>1}$$

and

$$\langle \sum_{0<i<\alpha} s_i - 1, s_i\rangle_{i\geq 1}$$

are elements in y. Since

$$\{s\} = \mathsf{c}_1\{\langle s_0, s_0 + 1 - \sum_{i>1} s_i, s_i\rangle_{i>1}\} \cap \mathsf{c}_0\{\langle \sum_{0\neq i<\alpha} s_i - 1, s_i\rangle_{i\geq 1}\},$$

it follows that $\{s\} \in A$. Let $At(A)$ denote the set of all atoms of A, i.e. the singletons. We can assume that $B_0 = Sg^{B_0} i(A_0)$. Upon noting that A contains all singletons, we obtain the following density condition.

$$(2)(\forall d)(d \in Nr_\alpha B_0 \wedge d \neq 0 \rightarrow \exists a \in At(A) \wedge i(a) \leq d).$$

From (1), (2) we get the desired conclusion i.e that $i(w) = \tau_\alpha(i(y))$, because, roughly, any atom in A below w is of the form $\tau(a)$ for singleton a below y. In more detail, we we shall show that

$$i(w) \leq \tau_\alpha(i(y)), \quad \tau_\alpha(i(y)) \leq i(w).$$

Let us start with the first inclusion. Assume seeking a contradiction that it does not hold. This means that

$$i(w) - \tau_\alpha(i(y)) \neq 0.$$

But then applying (2), we get an atom $z \in A$, such that $i(z) \leq i(w)$ and

$$i(z) \leq -\tau_\alpha(i(y)).$$

But $z = \tau(a)$ for some $a \in X$, thus

$$i(z) = \tau(i(a)) \leq \tau_\alpha(i(y)).$$

But this means that $i(z) \leq -\tau_\alpha(i(y)).\tau_\alpha(i(y)) = 0$. This is impossible since z is an atom and i is one to one. Now we want to establish the other inclusion, namely that

$$\tau_\alpha(i(y)) \leq i(w).$$

Now assume again, seeking a contradiction, that it is not the case that

$$\tau_\alpha(i(y)) \leq i(w).$$

Thus we have

$$\tau_\alpha(i(y)) \cdot -i(w) \neq 0.$$

By (2) there exists an atom $z \in A$ such that

$$i(z) \leq \tau_\alpha(i(y)), \quad i(z) \leq -i(w).$$

From the first inclusion we get

$$i(z)._\alpha\mathsf{s}(0,1)i(y) \neq 0,$$

hence

$$(++) \quad i(z) \leq {}_\alpha\mathsf{s}(0,1)i(y),$$

since z is a singleton. Let

$$a = \mathsf{s}_0^1\mathsf{c}_0 z.\mathsf{s}_1^0\mathsf{c}_1 z = {}_\alpha\mathsf{s}(0,1)z.$$

Applying ${}_\alpha\mathsf{s}(0,1)$ to both sides of (++) we get

$$i(a) \leq {}_\alpha\mathsf{s}(0,1)_\alpha\mathsf{s}(0,1)i(y) = i(y).$$

The latter equality follows from the MGR, indeed

$${}_\alpha\mathsf{s}(0,1)_\alpha\mathsf{s}(0,1)i(y) = {}_\alpha\mathsf{s}(0,1)_\alpha\mathsf{s}(0,1)\mathsf{c}_\alpha\mathsf{c}_{\alpha+1}i(y) = i(y).$$

Then $a \leq y$ and so $z = \tau(a) \leq w$. From this we get $i(z) \leq i(w)$ which is a contradiction since z is an atom and i is one to one and $i(z) \leq -i(w)$. By this (*) is proved and so is our main Theorem.

3 Conclusion

This paper solves a long outstanding problem in algebraic logic posed by Pigozzi in his landmark paper [10] published in algebra universalis in 1971. The proof is an adaptation of techniques of Nemeti used in [8], to solve a problem on neat reducts for cylindric algebras. The notion of neat reducts is strongly related to the amalgamation property, see [1]. More on that and related problems can be found in [4].

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