CARLESON EMBEDDINGS

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ABSTRACT. In this paper we discuss several operator ideal properties for so called Carleson embeddings of tent spaces into specific $L^q(\mu)$-spaces, where μ is a Carleson measure on the complex unit disc. Characterizing absolutely q-summing, absolutely continuous and q-integral Carleson embeddings in terms of the underlying measure is our main topic. The presented results extend and integrate results especially known for composition operators on Hardy spaces as well as embedding theorems for function spaces of similar kind.

1. Introduction and Main Results

Carleson measures proved to be an effective tool to discuss composition operators on classical Hardy spaces $H^q(\mathbf{D})$ on the complex unit disc $\mathbf{D}$ (see Hunziker, Jarchow [6], or Zhu [9, Chapter 8]). Central idea in these discussions is to translate the given problem into an embedding problem for Hardy spaces into specific $L^q(\mu)$-spaces. We enlarge Hardy spaces to so called *tent spaces* $T^q(\mathbf{D})$, which are spaces of continuous functions on $\mathbf{T}$ still possessing the same boundary behavior as functions in classical Hardy spaces. Embedding these bigger spaces into $L^q(\mu)$-spaces allows us to apply new techniques in this field. We call the corresponding embeddings $I_\mu\colon T^q(\mathbf{D}) \to L^{\beta q}(\mu)$, and their restrictions to subspaces of $T^q(\mathbf{D})$, *Carleson embeddings.* Especially discussing cases when they are *absolutely q-summing* or do have related properties profits from this approach. These operator ideal properties correspond intimately to geometric and distributional properties of the corresponding measure. Before going into details let us state the main results. Precise definitions are given below.

Throughout this paper assume that μ is a positive regular Borel measure on the closed complex unit disc $\overline{\mathbf{D}}$. To show what we have in mind let us recall a well-known result in this context ([6, 2.4]).

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Theorem A. *Suppose that $\beta \geq 1$. Then μ is a vanishing β-Carleson measure if and only if the formal identity $H^q(\mathbf{D}) \to L^{\beta q}(\mu)$ is compact for some, and hence all, $1 \leq q < \infty$.*

We will see that enlarging the domain of the Carleson embedding breaks up this equivalence. Clearly, as restrictions of compact operators are still compact, a compact Carleson embedding is induced by a vanishing Carleson measure. But the converse does not need to hold any longer.

Theorem 1. *Assume that $\beta \geq 1$ and $q \geq \beta^{-1}$ is finite. If μ is a vanishing β-Carleson measure the Carleson embedding $I_\mu \colon T^q(\mathbf{D}) \to L^{\beta q}(\mu)$ can be approximated in operator norm by (βq)-integral operators,*

$$\begin{aligned}\operatorname{dist}(I_\mu, \{\, u \colon T^q(\mathbf{D}) \to L^{\beta q}(\mu) \,:\, i_{\beta q}(u) \leq h^{-\frac{1}{q}} \,\})^{\beta q} \\ \leq \big\| \mu | \mathbf{D} \setminus A_h \big\|_{M^\beta} \big\| \mu | A_h \big\|_M\end{aligned}$$

with $0 < h < 1$ and $A_h := \big\{\, z \in \overline{\mathbf{D}} \,:\, |B(z)| < h \,\big\}$.

In particular,

Corollary 1. *If μ is a vanishing β-Carleson measure then, for all $q \geq \beta^{-1}$ the induced $I_\mu \colon T^q(\mathbf{D}) \to L^{\beta q}(\mu)$ is absolutely continuous.*

Remark. In general, an arbitrary vanishing β-Carleson measure does not induce a compact Carleson embedding. Let, for example, μ be the area measure restricted to the disc $\frac{1}{2}\overline{\mathbf{D}}$. Of course, this is a vanishing β-Carleson measure for all $\beta > 0$. But the Carleson embedding $I_\mu \colon T^q(\mathbf{D}) \to L^{\beta q}(\mu)$ is not compact because the restriction operator $I_\mu \circ J \colon C(\overline{\mathbf{D}}) \to L^{\beta q}(\mu)$, $f \mapsto f\big|_{\frac{1}{2}\overline{\mathbf{D}}}$ lacks this property. Here $J \colon C(\overline{\mathbf{D}}) \to T^q(\mathbf{D})$ is the canonical embedding. On the other hand, by a *normal families* argument, the restriction of this particular I_μ to the Hardy space $H^q(\mathbf{D})$ is compact.

Thus the ideal of compact operators must be replaced by another operator ideal. In most cases the ideal of absolutely continuous operators is the appropriate one.

Theorem 2. *Suppose that either $\beta = q = 1$ or $q > 1$ and $\beta \geq 1$. Then μ is a vanishing β-Carleson measure if and only if the induced Carleson embedding $I_\mu \colon T^q(\mathbf{D}) \to L^{\beta q}(\mu)$ is absolutely continuous.*

Finally, we characterize cases when Carleson embeddings are absolutely (βq)-summing or when they are (βq)-integral. In fact, we show that this properties are equivalent and there is a precise condition on the distribution of the corresponding measure guaranteeing them, and vice versa.

Theorem 3. *Let $\beta > 0$ and assume that $\mu(\mathbf{T}) = 0$. Then the following statements are equivalent:*

1. *The map $b \colon \mathbf{D} \to \mathbf{C}$, $z \mapsto (1 - |z|^2)^{-1}$ is in $L^\beta(\mu)$;*
2. *for some, and then all, finite $q \geq \beta^{-1}$ there is a $g \in L^{\beta q}(\mu)$ such that for all f in the unit ball of $T^q(\mathbf{D})$ we have $|f| \leq g$ μ-almost everywhere;*

3. *$I_\mu \colon T^q(\mathbf{D}) \to L^{\beta q}(\mu)$ is (βq)-integral for some, and then all, finite $q \ge \beta^{-1}$;*
4. *$I_\mu \colon T^q(\mathbf{D}) \to L^{\beta q}(\mu)$ is absolutely (βq)-summing for some, and then all, finite $q \ge \beta^{-1}$.*

2. Preliminaries

Before going into details, we recall necessary definitions and notations: The sets of all complex numbers with absolute value less than, equal to, or not bigger than 1 are denoted by $\mathbf{D}$, $\mathbf{T}$, and $\overline{\mathbf{D}}$, respectively. The normalized arc length on $\mathbf{T}$ is $d\zeta$, and $|E|$ stands for the normalized "length" of a measurable $E \subset \mathbf{T}$. As usual, L^q is short for $L^q(\mathbf{T}, d\zeta)$, (quasi-)normed by $\| \; \|_q$ $(0 < q \le \infty)$.

Let $f \colon \mathbf{D} \to \mathbf{C}$ be measurable. If $0 < r < 1$ then $f_r \colon \overline{\mathbf{D}} \to \mathbf{C}$ is the map which assigns $f(rz)$ to each $z \in \overline{\mathbf{D}}$. Moreover, $f_r\big|\mathbf{T}$ is $d\zeta$-measurable and so it makes sense to form the *q-th mean*

$$M_q(r, f) := \left\| f_r \big| \mathbf{T} \right\|_q \in [\,0, \infty].$$

The Hardy space $h^q(\mathbf{D})$ ($H^q(\mathbf{D})$) consists of all harmonic (analytic) functions $f \colon \mathbf{D} \to \mathbf{C}$ such that

$$\|f\|_{H^q} := \sup_{0<r<1} M_q(r, f)$$

is finite.

Burkholder, Gundy and Silverstein characterized Hardy spaces in terms of *nontangential suprema* (see Koosis [7, p. 246f]). In order to fit this characterization into our presentation, some more preparations are necessary. We assign to each $z \in \mathbf{D}$ a set $B(z) \subset \mathbf{T}$, which is the whole unit circle when $z = 0$, and which is, otherwise, the arc of length $1 - |z|^2$ centered at $z/\,|z|$. The *Stoltz domain* for a point $\zeta \in \mathbf{T}$ is $\Gamma(\zeta) := \{\, z \in \mathbf{D} \,:\, \zeta \in B(z) \,\}$, and the *tent* over an open $\Omega \subset \mathbf{T}$ is $\Theta(\Omega) := \{\, z \in \mathbf{D} \,:\, B(z) \subset \Omega \,\} \cup \Omega$.

Given $f \colon \mathbf{D} \to \mathbf{C}$ and $\zeta \in \mathbf{T}$, we define the *nontangential supremum of f at ζ* to be

$$\mathcal{N}f(\zeta) := \sup_{z \in \Gamma(\zeta)} |f(z)|\,.$$

Clearly, $\mathcal{N}f \colon \mathbf{T} \to [\,0, \infty]$ is lower semicontinuous and therefore measurable. Thus it makes sense to define

$$\|f\|_{T^q} := \|\mathcal{N}f\|_q \in [\,0, \infty].$$

In this way we get an extended (q-)norm on the space of all complex valued functions on $\mathbf{D}$ (see Heiming [4, pp. 36ff.] for details). Similar (q-)norms for functions on halfspaces were introduced by Coifman, Meyer, and Stein [1]. One easily checks that, for every $f \colon \mathbf{D} \to \mathbf{C}$,

$$|f(z)| \le |B(z)|^{-1/q}\, \|f\|_{T^q} \quad (z \in \mathbf{D}) \tag{1}$$

It is a standard procedure (see [4, 2.4]) to conclude from (1) that

$$\mathcal{T}^q(\mathbf{D}) := \{\, f\colon \mathbf{D} \to \mathbf{C} \,:\, \|f\|_{T^q} < \infty \,\}$$

is a (q-)Banach space. By (1), the space $C(\overline{\mathbf{D}})$ embeds injectively and contractively into $\mathcal{T}^q(\mathbf{D})$. We denote the closure of $C(\overline{\mathbf{D}})$ in $\mathcal{T}^q(\mathbf{D})$ by

$$(T^q(\mathbf{D}), \|\ \|_{T^q}).$$

These spaces will be called *tent spaces* (The name "tent space" for similar function spaces goes back at least to [1]). Using above notation, the Burkholder-Gundy-Silverstein Theorem can be reformulated as

$$h^q(\mathbf{D}) = \{\, f \in T^q(\mathbf{D}) \,:\, f\colon \mathbf{D} \to \mathbf{C} \text{ harmonic} \,\} \qquad (1 < q < \infty)$$

and

$$H^q(\mathbf{D}) = \{\, f \in T^q(\mathbf{D}) \,:\, f\colon \mathbf{D} \to \mathbf{C} \text{ analytic} \,\} \qquad (0 < q < \infty)$$

with equivalent (q-)norms (see [4, 2.25]).

Let $\beta > 0$. A regular Borel measure $\mu \in M(\overline{\mathbf{D}}) = C(\overline{\mathbf{D}})^*$ is called a *β-Carleson measure* if there is a constant C such that, for all open $\Omega \subset \mathbf{T}$,

$$|\mu|\,(\Theta(\Omega)) \le C\,|\Omega|^\beta\,.$$

The least such constant C is $\|\mu\|_{M^\beta}$ and

$$M^\beta(\overline{\mathbf{D}}) := \left\{\, \mu \in M(\overline{\mathbf{D}}) \,:\, \|\mu\|_{M^\beta} < \infty \,\right\}$$

is the space of all β-Carleson measures. Clearly, $(M^\beta(\overline{\mathbf{D}}), \|\ \|_{M^\beta})$ is a Banach space. A measure $\mu \in M^\beta(\overline{\mathbf{D}})$ with $|\mu|\,(\Theta(\Omega)) = \mathrm{o}(|\Omega|^\beta)$ as $|\Omega| \to 0$ is called a *vanishing β-Carleson measure*.

The following inequality is crucial for relating tent spaces and spaces of Carleson measures,

$$\|f\|_{L^{\beta q}(\overline{\mathbf{D}},|\mu|)} \le \|\mu\|_{M^\beta}^{1/\beta q}\,\|f\|_{T^q}\,, \tag{2}$$

where $f \in T^q(\mathbf{D})$ and $\mu \in M^\beta(\overline{\mathbf{D}})$. In addition,

$$\|\mu\|_{M^\beta}^{1/\beta q} = \sup\left\{\, \|f\|_{L^{\beta q}(\overline{\mathbf{D}},|\mu|)} \,:\, \|f\|_{T^q} \le 1 \,\right\}.$$

A proof for (2) in a version including Lorentz-type spaces is given in [4].

The classical Riesz Representation Theorem for measures in conjunction with (2) imply that $M^\beta(\overline{\mathbf{D}})$ is isometric to the dual space of $T^{1/\beta}(\mathbf{D})$. Another consequence of (2) — the main subject of this paper — is that the so called *Carleson embedding*, which is the formal identity

$$I_\mu\colon T^q(\mathbf{D}) \to L^{\beta q}(\mu), f \mapsto f,$$

is continuous exactly if μ is a β-Carleson measure. In this case $\|I_\mu\|^{\beta p}$ is equivalent to $\|\mu\|_{M^\beta}$.

Maybe the most prominent examples of Carleson measures are *composition measures*, which are defined as follows. For each analytic map $\phi\colon \mathbf{D} \to \mathbf{D}$ the *radial limit* $\lim_{r\nearrow 1} f(r\zeta)$ exists for almost all $\zeta \in \mathbf{T}$. Thus

$$m_\phi(A) := |\{\, \zeta \in \mathbf{T} : \lim_{r\nearrow 1} \phi(r\zeta) \in A \,\}| \quad (A \subset \overline{\mathbf{D}} \text{ measurable}) \tag{3}$$

defines a probability measure on $\overline{\mathbf{D}}$, which can be shown to be a Carleson measure. The composition operator $C_\phi\colon H^q(\mathbf{D}) \to H^{\beta q}(\mathbf{D})$, $f \mapsto f \circ \phi$ corresponds to

$$H^q(\mathbf{D}) \subset T^q(\mathbf{D}) \xrightarrow{I_{m_\phi}} L^{\beta q}(m_\phi)$$

(see [6, 9] and the references given there).

We assume the reader to be familiar with the notion and fundamental properties of specific *operator ideals*, namely, weakly compact, completely continuous, absolutely p-summing, p-integral, and, finally, absolutely continuous operators. An elaborate exposition of these is presented by Diestel, Jarchow, and Tonge [2].

3. Proof of Theorem 1

Let us first assume that our measure takes its support in the open disc. Then the corresponding Carleson embedding is (βq)-integral. Precisely, we have

Lemma 1. *Suppose that the positive, regular Borel measure μ is supported in $\mathbf{D}$. Then $I_\mu\colon T^q(\mathbf{D}) \to L^{\beta q}(\mu)$ is (βq)-integral for all $\beta q \geq 1$ and*

$$i_{\beta q}(I_\mu) \leq \sup\left\{\, |B(z)|^{-1/q} : z \in \operatorname{supp}(\mu) \,\right\}.$$

Proof. Let $z_0 \in \operatorname{supp}(\mu)$ be such that

$$B_0 := |B(z_0)| = \min\{\, |B(z)| : z \in \operatorname{supp}(\mu) \,\},$$

put $r_0 := |z_0|$, and set $D_0 := r_0\overline{\mathbf{D}}$. Then $\operatorname{supp}(\mu) \subset D_0$, and, denoting by ρ the restriction map $f \mapsto f\big|D_0$, we have the factorization

$$\begin{array}{ccc} T^q(\mathbf{D}) & \xrightarrow{I_\mu} & L^{\beta q}(\overline{\mathbf{D}}, \mu) \\ {\scriptstyle\rho}\big\downarrow & & \big\uparrow{\scriptstyle f \mapsto f} \\ C(D_0) & \xrightarrow[f \mapsto f]{} & L^{\beta q}(D_0, \mu). \end{array}$$

Due to (1), $\|\rho\| = B_0^{-1/q}$ settles our claim. ∎

Now we are in position to prove Theorem 1. Fix $\epsilon > 0$. As we assume that μ is a vanishing β-Carleson measure, there is an $0 < R < 1$ such that for all $R < r < 1$ and all open $\Omega \subset \mathbf{T}$ with $|\Omega| < r$ we have $\mu(\Theta(\Omega)) \leq \epsilon\, |\Omega|^\beta$. Thus, if we denote by μ_r the restricted measure $\mu\big|A_r$ we get $\|\mu_r\|_{M^\beta} \leq \epsilon$. On the other hand we can apply Lemma 1 to $\mu_r' := \mu - \mu_r$, which takes its support in $\mathbf{D} \setminus A_r$, and get $i_{\beta q}(I_{\mu_r'}) \leq r^{-1/q}$. This shows that I_μ can be approximated by (βq)-integral operators, and, after reordering the estimates given above, we get our hands on the quality of this approximation.

4. Proof of Theorem 2

The proof of Theorem 2 splits into two parts, depending on the range of the Carleson embedding under consideration. The next lemma makes use of the Dunford-Pettis characterization of weakly compact sets in L^1-spaces.

Lemma 2. *Let $0 < q < \infty$. If $I_\mu\colon T^q(\mathbf{D}) \to L^1(\mu)$ exists as a weakly compact operator then μ is a vanishing β-Carleson measure for $\beta = q^{-1}$.*

Proof. The tent space $T^q(\mathbf{D})$ is weakly separated and $L^1(\mu)$ is a Banach space. Hence the second adjoint $I_\mu^{**}\colon T^q(\mathbf{D})^{**} \to L^1(\mu)^{**}$ is weakly compact and takes its values in $L^1(\mu)$. It is an extension of I_μ to $T^q(\mathbf{D})^{**}$. As $T^q(\mathbf{D})^*$ is isomorphic to $M^\beta(\overline{\mathbf{D}})$ (cf. (2)) every nonvoid open subset Ω of $\mathbf{T}$ gives rise to a normalized linear functional ϕ_Ω on $T^q(\mathbf{D})^*$ via

$$\phi_\Omega(\nu) := |\Omega|^{-\beta}\,\nu(\Theta(\Omega)) \quad (\nu \in M^\beta(\overline{\mathbf{D}})).$$

Therefore,

$$U := I_\mu^{**}(\{\, \phi_\Omega \,:\, \emptyset \neq \Omega \subset \mathbf{T}\,\})$$

is a relatively weakly compact subset of $L^1(\mu)$. By the Dunford-Pettis Theorem (Dunford, Schwartz [3, IV.8.11]), U is uniformly absolutely continuous with respect to μ,

$$\lim_{\mu(E)\to 0} \sup_{f\in U} \left| \int_E f\, d\mu \right| = 0.$$

If $\Omega_n \subset \mathbf{T}$ are such that $\lim |\Omega_n| = 0$, then $\lim \mu(\Theta(\Omega_n)) = 0$, since $\mu \in M^\beta(\overline{\mathbf{D}})$. The conclusion is that

$$|\Omega|^{-\beta}\,\mu(\Theta(\Omega_n)) = \int_{\Omega_n} \phi_{\Omega_n}\, d\mu \le \sup_{f\in U} \left| \int_{\Omega_n} f\, d\mu \right| \to 0 \quad (n \to \infty).$$

So, μ is a vanishing β-Carleson measure. ∎

Lemma 3. *Assume that $q > 1$, $\beta \ge 1$ and $I_\mu\colon T^q(\mathbf{D}) \to L^{\beta q}(\mu)$ is completely continuous. Then μ is a vanishing β-Carleson measure.*

Proof. As $q > 1$ the Hardy space $H^q(\mathbf{D})$ is reflexive. Therefore the restriction of I_μ to $H^q(\mathbf{D})$ is compact. By Theorem A, μ must be a vanishing β-Carleson measure. ∎

The proof of Theorem 2 is now easy: As already stated in Corollary 1 every vanishing β-Carleson measure induces an absolutely continuous Carleson embedding. On the other hand, if I_μ is absolutely continuous, Lemma 2 and Lemma 3 imply that the measure under consideration must be a vanishing β-Carleson measure, because every absolutely continuous operator is weakly compact and completely continuous.

5. Proof of Theorem 3

The implication "**3.** $\Rightarrow$ **4.**" follows from the very definition of the considered operator properties.

"1. $\Rightarrow$ 3." Fix $\beta^{-1} \le q < \infty$. The linear operator $v\colon T^q(\mathbf{D}) \to C(\overline{\mathbf{D}})$, $f \mapsto b^{-\frac{1}{q}} f$ is continuous with $\|v\| = 1$. Since $b \in L^\beta(\mu)$, the formal identity $J\colon C(\overline{\mathbf{D}}) \to L^{\beta q}(b^\beta \mu)$ is bounded with $\|J\| = \|b\|_\beta^{\frac{1}{q}}$. Moreover, $w\colon L^{\beta q}(b^\beta \mu) \to L^{\beta q}(\mu)$, $f \mapsto b^{\frac{1}{q}} f$ is isometric, and we have the following factorization of $I_\mu\colon T^q(\mathbf{D}) \to L^{\beta q}(\mu)$,

$$\begin{array}{ccc} T^q(\mathbf{D}) & \xrightarrow{I_\mu} & L^{\beta q}(\mu) \\ {\scriptstyle v}\big\downarrow & & \big\uparrow{\scriptstyle w} \\ C(\overline{\mathbf{D}}) & \xrightarrow{J} & L^{\beta q}(b^\beta \mu). \end{array}$$

Consequently, I_μ is βq-integral with $i_{\beta q}(I_\mu) \le \|b\|_\beta^{\frac{1}{q}}$.

"1. $\Rightarrow$ 2." Fix $\beta^{-1} \le q < \infty$. First of all we show that μ is a β-Carleson measure. Given an open $\Omega \subset \mathbf{T}$, we have $|B(z)| \le |\Omega|$ for all $z \in \Theta(\Omega)$, and so

$$\frac{\mu(\Theta(\Omega))}{|\Omega|^\beta} \le \int_{\Theta(\Omega)} |B(z)|^{-\beta}\, d\mu \le \|b\|_\beta \,.$$

This proves μ to be a β-Carleson measure. For each f in the unit ball of $T^q(\mathbf{D})$ and every $z \in \mathbf{D}$ we have $|f(z)| \le |B(z)|^{-1/q} = b(z)^{-1/q}$. Our assumption $b \in L^\beta(\mu)$ implies $b^{1/q} \in L^{\beta q}(\mu)$ and so we can take $g := b^{1/q}$.

"2. $\Rightarrow$ 1." For $z \in \mathbf{D}$ we have

$$b(z)^{1/q} = \|\delta_z\|_{T^q}^* = \sup\{\, |f(z)| \,:\, f \in T^q(\mathbf{D}),\ \|f\|_{T^q} \le 1 \,\} \le g(z).$$

This clearly implies $b \in L^\beta(\mu)$.

"4. $\Rightarrow$ 2." Assume that $I_\mu\colon T^q(\mathbf{D}) \to L^{\beta q}(\mu)$ is absolutely (βq)-summing. Let K be the unit ball of $T^q(\mathbf{D})^*$ endowed with the weak* topology. Then Pietsch's Domination Theorem yields a probability measure λ on K such that

$$\|f\|_{\beta q} \le \pi_{\beta q}(I_\mu) \left(\int_K |\langle f \,,\, \omega\rangle|^{\beta q}\, d\lambda(\omega) \right)^{\frac{1}{\beta q}}.$$

This implies that for all positive $f \in T^q(\mathbf{D})$,

$$\|f\|_{\beta q} \le \pi_{\beta q}(I_\mu) \left(\int_K \langle f \,,\, |\omega|\rangle^{\beta q}\, d\lambda(\omega) \right)^{\frac{1}{\beta q}}. \tag{4}$$

We define a map $\sigma\colon T^q(\mathbf{D}) \to C(K)$ by

$$\sigma f(\omega) := \langle f \,,\, |\omega|\rangle \quad (f \in T^q(\mathbf{D}),\ \omega \in K).$$

σ is continuous, linear, positive and $|\sigma f(\omega)| \leq 1_K(\omega)$ for each $\omega \in K$ and all f in the unit ball of $T^q(\mathbf{D})$. Let Y be the image of $\sigma(T^q(\mathbf{D}))$ in $L^{\beta q}(K, \lambda)$ under the canonical embedding of $C(K)$ into the latter space. Then, by (4),

$$\rho\colon Y \to L^{\beta q}(\mu), \sigma f \mapsto f$$

is a well-defined, continuous operator. Since K contains all normalized point evaluations $\delta_z/\|\delta_z\|_{T^q}^*$ $(z \in \mathbf{D})$, pointwise ordering on K is stronger than pointwise ordering on $\mathbf{D}$. Thus ρ is also positive. Put

$$F := \left\{ g \in L^{\beta q}(K, \lambda) \ : \ \exists f \in Y : |g| \leq f \right\}.$$

This is a sublattice of $L^{\beta q}(K, \lambda)$ containing Y. As $L^{\beta q}(\mu)$ is a Dedekind complete Riesz space, the Kantorovich Extension Theorem (see Meyer-Nieberg [8, 1.5.9]) provides a positive, continuous extension of ρ to F and, by continuity, even to the closure of F,

$$\overline{\rho}\colon \overline{F} \to L^{\beta q}(\mu), \quad \overline{\rho}\big|Y = \rho.$$

Since $T^q(\mathbf{D})$ is separable, there is countable dense subset

$$\{ f_n \in T^q(\mathbf{D}) \ : \ n \in \mathbf{N} \}$$

of the unit ball of $T^q(\mathbf{D})$. Set $g_n := \max_{k \leq n} \sigma(|f_k|)$, then (g_n) is an increasing, positive sequence in the unit ball of $\sigma(T^q(\mathbf{D}))$. Therefore it is also an increasing sequence in $F \subset L^{\beta q}(K, \lambda)$ and it is dominated by 1_K. Hence an appeal to Lebesgue's Dominated Convergence Theorem yields $g \in \overline{F}$ such that

$$\lim_{n \to \infty} \|g_n - g\|_{\beta q} = 0 \quad \text{and} \quad g_n \leq g \quad (n \in \mathbf{N}).$$

The construction of g reveals that $\sigma(|f_n|) \leq g$ and so

$$|f_n| = \overline{\rho}(\sigma(|f_n|) \leq \overline{\rho}(g) \quad \mu\text{-a.e.}$$

for all $n \in \mathbf{N}$. Now fix f in the unit ball of $T^q(\mathbf{D})$. There is a sequence $(f_{n_k})_k$ such that $\lim_{k \to \infty} \|f - f_{n_k}\|_{T^q} = 0$ and thus $\lim_{k \to \infty} \||f| - |f_{n_k}|\|_{\beta q} = 0$. This implies, up to selecting once more a subsequence,

$$\overline{\rho}(g) - |f| = \lim_{k \to \infty} (\overline{\rho}(g) - |f_{n_k}|) \geq 0$$

μ-almost everywhere. Hence $\overline{\rho}(g)$ is the function we are seeking. ∎

6. Concluding Remarks

Composition operators. As pointed out in (3) every analytic $\phi\colon \mathbf{D} \to \mathbf{D}$ gives rise to a Carleson measure m_ϕ. By 'change of variables', for $q \geq 1$, the associated Carleson embedding corresponds to

$$C_\phi\colon T^q(\mathbf{D}) \to L^q(\,d\zeta), \ f \mapsto \left[\zeta \mapsto \lim_{r \nearrow 1} f(\phi(r\zeta))\right].$$

Under this assumptions condition 1. of Theorem 3 is equivalent to the finiteness of $\sum_n \|\phi^n\|_{H^1}$ (see Hunziker [5, Satz 6.3] for Hardy spaces).

Embedding Hardy spaces into weighted Bergman spaces. It is easy to verify that, for $\beta > 1$,

$$dA_\beta(z) := (\beta - 1)(1 - |z|^2)^{\beta-2}\, dz$$

defines a probability measure on $\mathbf{D}$, which additionally is a β-Carleson measure with $\| dA_\beta \|_{M^\beta} = 1$. These measures appear in the theory of weighted Bergman spaces (cf. [9, 6.4.1]),

$$L^q_a(\mathbf{D},\, dA_\beta) := \{\, f \in L^q(\mathbf{D},\, dA_\beta) \;:\; f \text{ is analytic} \,\}\,.$$

Clearly, this implies that $H^q(\mathbf{D})$ embeds continuously into $L^{\beta q}_a(\mathbf{D},\, dA_\beta)$. Due to the fact that every β-Carleson measure is a vanishing α-Carleson measure ($\alpha < \beta$), for all $p < \beta q$, the embedding $H^q(\mathbf{D}) \subset L^p_a(\mathbf{D},\, dA_\beta)$ is absolutely continuous. If $q > 1$ it is even compact, because its domain is reflexive. Moreover, for $\beta > 2$, as $(1 - |z|^2)^{-1} \in L^1(\mathbf{D},\, dA_\beta)$, we get that the embedding of $H^q(\mathbf{D})$ into $L^q_a(\mathbf{D},\, dA_\beta)$ is 1-integral.

Main open question. In Theorem 2 it is left as an open question, whether the absolute continuity of an Carleson embedding $I_\mu \colon T^1(\mathbf{D}) \to L^\beta(\mu)$ ($\beta > 1$) is sufficient for μ to be a vanishing β-Carleson measure.

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