ON QUASILINEAR ELLIPTIC EQUATIONS IN $I\!R^N$

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ABSTRACT. In this note we give a result for the operator p-Laplacian complementing a theorem by Brézis and Kamin concerning a necessary and sufficient condition for the equation $-\Delta u = h(x)u^q$ *in* $I\!R^N$, where $0 < q < 1$, to have a bounded positive solution. While Brézis and Kamin use the method of sub and super solutions, we employ variational arguments for the existence of solutions.

1. Introduction

We are concerned with existence of solutions for the quasilinear elliptic problem

$$(*) \qquad \begin{cases} -\Delta_p u = h(x)u^q \ \ in \ \ I\!R^N, \\ u \geq 0, \ \ u \not\equiv 0, \ \ \int |\nabla u|^p < \infty, \end{cases}$$

where $\Delta_p u = div(|\nabla u|^{p-2}\nabla u)$, $1 < p < N$, $0 \leq q < p^* - 1$, $p^* = \frac{Np}{N-p}$, and $h : I\!R^N \to I\!R$ is a measurable function with $h \not\equiv 0$.
In [3] Brézis and Kamin studied $(*)$ in the case $p = 2$ and obtained necessary and sufficient conditions for it to have bounded solutions. More precisely it was shown in [3], for $p = 2$ and by using a priori estimates and the method of sub and super solutions, that $(*)$ has a bounded solution iff both

$$(H_1) \qquad h \in L^\infty_{loc} \text{ and } h \geq 0 \text{ a.e. in } \Omega$$

and the linear problem

$$-\Delta u = h(x) \ in \ I\!R^N$$

have a bounded solution.

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The study of problem $(*)$ in this case $(p = 2)$ is related to the study of the asymptotic behaviour of solutions $U(x,t)$ (as $t \to \infty$) of

$$h(x)U_t + \Delta U^{\frac{1}{q}} = 0 \ in \ I\!R^N \times (0,\infty).$$

Actually,

$$U(x,t) \equiv Av(x)\frac{1}{(t+B)^{\frac{q}{1-q}}},$$

where $B > 0$ is any constant and $A > 0$ is an appropriate number, solves the evolution equation above if $v(x) \geq 0$ is a solution of the equation $-\Delta v^{\frac{1}{q}} = h(x)v \ \ in \ \ I\!R^N$, i.e., $u \equiv v^{\frac{1}{q}}$ solves $(*)$. Eidus [7] treats a situation in which $h(x) \to 0$ at ∞.
We point out that $(*)$ does not always have a solution, see again [3] and also Gidas and Spruck [8] for an important non-existence result.
There is by now an extensive literature on this kind of problem. We refer the reader also to Brézis and Nirenberg [4], Noussair and Swanson [9], Tshinanga [13], Alves-Goncalves and Maia [1], Rabinowitz [11], Costa and Miyagaki [5] and their references.
Our aim in this work is to use variational methods to prove the following result.

Theorem 1. *Assume that*

$$(H_2) \qquad h \in L^\infty_{loc} \ and \ h(x) \leq 0 \ \ a.e. \ \ in \ \Omega^c$$

for some bounded domain $\Omega \subset I\!R^N$ *and*

$$(H_3) \qquad \inf_{\omega} h > 0$$

for some domain $\omega \subset \Omega$. *Then* $(*)$ *has a solution* $u \in \mathcal{D}^{1,p}$ *provided that either*

$$(i) \qquad 0 \leq q < p-1$$

or

$$(ii) \qquad p-1 < q < p^* - 1.$$

Moreover, $u \in W^{1,p}_{loc}$ *when* $1 < p < N$ *and* $u \in W^{p,s}_{loc}$ *for* $s > N$ *when* $p = 2$.

Our approach to prove Theorem 1 will involve the consideration of the family of problems in the ball of radius $R > 0$ centered at the origin $I\!R^N$, $B_R \equiv B_R(0)$, namely

$$(*)_R \qquad \begin{cases} -\Delta_p u = h(x)u^q \ \ in \ \ B_R, \\ u \in W_R \equiv W^{1,p}_o(B_R), \\ u \geq 0 \ in \ B_R, \ \ u \not\equiv 0, \end{cases}$$

and the finding of critical points of the associated energy functional $I_R : W_R \to I\!R$ given by

$$I_R(u) = \frac{1}{p}\int_{B_R} |\nabla u|^p - \frac{1}{q+1}\int_{B_R} hu_+^{q+1}.$$

Actually, under appropriate assumptions such as the ones we have stated above, $I_R \in \mathcal{C}^1(W_R, I\!R)$ and its derivative $I'_R(u)$ is given by

$$\langle I'_R(u), \phi\rangle = \int_{B_R} |\nabla u|^{p-2}\nabla u \nabla\phi - \int_{B_R} hu_+^q\phi,\ \phi \in W_R.$$

A distributional solution of $(*)$will be found by estimating and passing to the limit as $R \to \infty$.

One reason for working out this reduction procedure is that the Euler-Lagrange functional of $(*)$ is not defined over either $W^{1,p}(I\!R^N)$ or $D^{1,p}(I\!R^N)$ since the integral

$$\int hu_+^{q+1}$$

may not be defined.

2. Preliminaries

Let $W_n \equiv W_o^{1,p}(B_n)$. Under the conditions of Section 1 we have $I_n \in C^1(W_n, I\!R)$. We state below some technical lemmas and remarks which will be useful in the next section.

Lemma 1. *Assume* $(H_2), (H_3)$ *and* $0 \le q < p-1$. *Then*

(i) $$I_n(u) \to \infty \ \textit{as}\ \|u\|_{W_n} \to \infty.$$

(ii) $$I_n(t\phi) < 0,\ 0 < t < t_n \ \textit{for some}\ t_n > 0 \ \textit{and}\ \phi \in C_0^\infty.$$

We remark that by Lemma 1 there is some $u_n \in W_n$ such that

$$I_n(u_n) = \inf_{W_n} I_n < 0$$

and

$$\left\langle I_n^{'}(u_n), \phi\right\rangle = 0,\ \phi \in W_n$$

so that in particular u_n is a weak solution of $(*)_n$ for each $n > 1$.

Lemma 2. *There are constants* $\widehat{c} < 0$, *and* $M > 0$ *independent of* n *such that*

(i) $$I_n(u_n) \le \widehat{c},\ n > 1$$

(ii) $$\|u_n\|_{D^{1,p}} \le M,\ n > 1.$$

Lemma 3. *Assume* $p-1 < q < p^*-1$ *and* $(H_2), (H_3)$. *Then there are constants* $\rho, r > 0$ *independent of* n *and* $e \in C_o^\infty(\Omega)$, $\|e\|_{W_n} > \rho$ *such that*

(i) $$I_n(u) \ge r \ \textit{for}\ \|u\|_{W_n} = \rho$$

(ii) $$I_n(e) \le 0,\ n > 1.$$

Moreover, I_n *satisfies the (PS) condition.*

Remark 1. By the Ambrosetti-Rabinowitz Mountain Pass theorem we find a critical point $u_n \in W_n$ of I_n such that

$$I_n(u_n) = c_n \ge r > 0$$

where

$$c_n = \inf_{\gamma\in\Gamma} \max_{0\le t\le 1} I_n(\gamma(t))$$

and

$$\Gamma = \{\gamma \in C([0,1], W_n) \mid \gamma(0) = 0, \gamma(1) = e\}$$

and using the fact that $W_n \subset W_{n+1}$ we actually have

$$c_1 \geq c_2 \geq \ \dots\ c_n \geq \ \dots\ \geq r > 0$$

so that in particular

$$I_n(u_n) = c_n \to c \text{ for some } c \geq r > 0.$$

Lemma 4. *The sequence $u_n \in W_n \subset D^{1,p}$ given above by the Mountain Pass theorem is bounded in $D^{1,p}$.*

Lemma 5. *Under either conditions (i) or (ii) of Theorem 1 we have*

(i) $$u_n \rightharpoonup u \text{ in } D^{1,p},$$

(ii) $$u_n \to u \text{ a.e. in } \mathbb{R}^N,$$

(iii) $$\nabla u_n \to \nabla u \text{ a.e. in } \mathbb{R}^N$$

for some $u \in D^{1,p}$.

3. Proofs

We now give the proofs of Theorem 1 and Lemmas 1-5.

Proof of Lemma 1.
(i). Since $h \in L^\infty_{loc}$, we have $h \in L^\theta(B_n)$. Thus, by Hölder's inequality and Sobolev's embedding theorem,

$$\int_{B_n} hu_+^{q+1} \leq C|h|_{\theta,B_n}\|u\|^{q+1}_{W_n},$$

where $\theta \equiv \frac{p^*}{p^*-(q+1)}$.
Hence

$$I_n(u) \geq \frac{1}{p}\|u\|^p_{W_n} - \frac{C|h|_{\theta,B_n}}{q+1}\|u\|^{q+1}_{W_n},$$

which shows that I_n is coercive along W_n.
(ii). Letting $\phi \in C_o^\infty$ with $\phi \geq 0$, $\phi \not\equiv 0$ and $supt(\phi) \subset \omega$, we get

$$I_n(t\phi) < 0 \text{ for } 0 < t < t_n$$

for some $t_n > 0$. ■

Proof of Lemma 2. We have

$$\begin{array}{rcl} I_n(u_n) & = & \frac{1}{p}\int_{B_n}|\nabla u_n|^p - \frac{1}{q+1}\int_{B_n} hu_{n+}^{q+1} \\ & \geq & \frac{1}{p}\int_{B_n}|\nabla u_n|^p - \frac{1}{q+1}\int_{\Omega} hu_{n+}^{q+1} \\ & \geq & \frac{1}{p}\|u_n\|^p_{W_n} - \frac{|h|_{\theta,\Omega}}{q+1}|u_n|^{q+1}_{p^*,\Omega} \\ & \geq & \frac{1}{p}\|u_n\|^p_{W_n} - C\frac{|h|_{\theta,\Omega}}{q+1}\|u_n\|^{q+1}_{W_n}. \end{array}$$

Now, it is easy to see that

$$I_n(u_n) \leq I_n(t^*\phi) = t^{*p}\int_\omega |\nabla\phi|^p - \frac{(t^*)^{q+1}}{q+1}\int_\omega h\phi^{q+1} \equiv \widehat{c} < 0$$

for some $t^* > 0$. Thus, we get

$$\|u_n\|^p_{W_n} = \|u_n\|^p_{D^{1,p}} \leq M$$

for some $M > 0$. ∎

Proof of Lemma 3. We remark first that

$$\begin{array}{rcl} I_n(u) & \geq & \frac{1}{p}\int_{B_n}|\nabla u|^p - \frac{1}{q+1}\int_\Omega hu_+^{q+1} \\ & \geq & \frac{1}{p}\|u\|^p_{W_n} - \frac{|h|_{\theta,\Omega}}{q+1}|u|^{q+1}_{p^*,\Omega} \\ & \geq & \frac{1}{p}\|u\|^p_{W_n} - S^{-\frac{q+1}{2}}\frac{|h|_{\theta,\Omega}}{q+1}\|u\|^{q+1}_{W_n}, \end{array}$$

where S is the best constant for the embedding $D^{1,p} \to L^{p^*}$. So, there are $r,\ \rho > 0$ such that

$$I_n(u) \geq r \text{ for } \|u\| = \rho.$$

Note that by taking the extension by zero of $u \in W_n$ to $I\!R^N$ we have $\|u\|^p_{W_n} = \|u\|^p_{D^{1,p}}$. S̃o r, ρ are independent of n.
On the other hand, as above, there is some $t_o > 0$ such that

$$I_n(t_o\phi) = t_o^p\int_\omega |\nabla\phi|^p - \frac{t_o^{q+1}}{q+1}\int_\omega h\phi^{q+1} \leq 0$$

and, taking $e \equiv t_o\phi$, we have $I_n(e) \leq 0$.
Now, letting $I_n = I$, $W_n = W$ and $B_n = B$, assume $u_k \in W$ is a sequence such that

$$I'(u_k) \to 0 \text{ with } I(u_k) \text{ bounded.}$$

We have

$$I(u_k) - \frac{1}{q+1}\langle I'(u_k), u_k\rangle = \left(\frac{1}{p} - \frac{1}{q+1}\right)\|u_k\|^p_W,$$

which shows that

$$\|u_k\|_W \text{ is bounded,}$$

and thus,

$$u_k \rightharpoonup u \ in\ W.$$

In addition,

$$u_k \to u \ in\ L^r(B),\ p \leq r < p^* \text{ and } u_k \to u \text{ a.e. in } B.$$

Hence from

$$\langle I'(u_k), u_k\rangle = o_k(1)$$

we have

$$\|u_k\|^p_W \to \int_B hu_+^{q+1}.$$

On the other hand, choosing $\psi_j \to u$ *in* W with $\psi_j \in C_o^\infty$, we have

$$\int_B |\nabla u|^{p-2}\nabla u\nabla\psi j \to \int_B |\nabla u|^p.$$

Since

$$\int_B |\nabla u|^{p-2}\nabla u \nabla \psi j = \int_B h u_+^q \psi_j$$

and

$$\int_B h u_+^q \psi_j \to \int_B h u_+^{q+1}$$

we infer that

$$\|u_k\|_W \to \|u\|_W,$$

which shows that $u_k \to u \ in \ W$. ■

Proof of Lemma 4. We shall use that $W_n \subset D^{1,p}$. By Remark 1,

$$I_n(u_n) = c + o_n(1)$$

and since

$$\left\langle I_n^{'}(u_n), u_n \right\rangle = 0, \text{ for } n > 1,$$

and recalling that

$$I_n(u_n) - \frac{1}{q+1}\left\langle I_n^{'}(u_n), u_n \right\rangle = c + o_n(1),$$

we get

$$\left(\frac{1}{p} - \frac{1}{q+1}\right)\int_{B_n} |\nabla u_n|^p = c + o_n(1).$$

Now, since

$$\int_{B_n} |\nabla u_n|^p = \int |\nabla u_n|^p$$

we find some $M > 0$ such that

$$\|u_n\|_{D^{1,p}} \le M \text{ for all } n > 1. \blacksquare$$

Next we present the proof of Theorem 1 and we leave the proof of Lemma 5 for a later step.

Proof of Theorem 1. At first we remark that from

$$0 = \left\langle I_n^{'}(u_n), u_{n-} \right\rangle = \int_{B_n} |\nabla u_{n-}|^p$$

it follows that $u_n \ge 0$ and we have already shown that $u_n \not\equiv 0$. On the other hand,

$$\int_{B_n} |\nabla u_n|^{p-2}\nabla u_n \nabla \psi - \int_{B_n} h u_n^q \psi = 0, \ \psi \in C_o^\infty.$$

Now, using Lemma 5 and passing to the limit we get

$$\int |\nabla u|^{p-2}\nabla u \nabla \psi - \int h u^q \psi = 0, \ \psi \in C_o^\infty,$$

which shows that u is a distributional solution of $(*)$, that is,

$$-\Delta_p u = h(x) u^q \ in \ D^{'}(I\!\!R^N).$$

Next, we show that $u \not\equiv 0$. Indeed, assuming that $p - 1 < q < p^* - 1$, we have

$$I_n(u_n) = c + o_n(1) \text{ and } \frac{1}{q+1}\left\langle I_n^{'}(u_n), u_n \right\rangle = 0.$$

Therefore,

$$c + o_n(1) = \left(\frac{1}{p} - \frac{1}{q+1}\right) \int h u_n^{q+1} \le \left(\frac{1}{p} - \frac{1}{q+1}\right) \int_\Omega h u_n^{q+1}.$$

Passing to the limit, we infer that

$$0 < r \le c \le \left(\frac{1}{p} - \frac{1}{q+1}\right) \int_\Omega h u^{q+1}$$

which shows that $u \not\equiv 0$. A similar argument works for the case $0 \le q < p-1$, recalling that $I_n(u_n) \le \widehat{c} < 0$.

Now, $u \in W^{1,p}_{loc}$ when $1 < p < N$, by Sobolev embeddings, and $u \in W^{p,s}_{loc}$ for $s > N$ when $p = 2$ by elliptic regularity theory (see DiBenedetto [6] for $p \neq 2$.) ▪

The proof of Lemma 5 is adapted from arguments by Noussair-Swanson and Jianfu [10] (see also Alves and Goncalves [2] and their references).

Proof of Lemma 5. We shall only show that

$$(iii) \qquad \nabla u_n \to \nabla u \text{ a.e. in } I\!\!R^N.$$

since $(i)(ii)$ are more standard.

So, let us consider the cut-off function

$$\eta \in C^\infty_o,\ 0 \le \eta \le 1,\ \eta = 1 \text{ on } B_1,\ \eta = 0 \text{ on } B_2^c.$$

Let $\rho > 0$ and $\eta_\rho(x) = \eta(\frac{x}{\rho})$ and let $n > 1$ such that $B_{2\rho} \subset B_n$. We have

$$\langle I_n'(u_n), \phi \rangle = 0,\ \phi \in W_n$$

and since $u_n - u \in D^{1,p}$, we get $(u_n - u)\eta_\rho \in W_n$, so that we also have

$$\int_{B_n} |\nabla u_n|^{p-2} \nabla u_n \nabla[(u_n - u)\eta_\rho] = \int_{B_n} h u_n^q (u_n - u)\eta_\rho.$$

Hence

$$\int_{B_n} |\nabla u_n|^{p-2} \nabla u_n (\nabla u_n - \nabla u)\eta_\rho =$$

$$\int_{B_n} h u_n^q (u_n - u)\eta_\rho - \int_{B_n} |\nabla u_n|^{p-2} (u_n - u) \nabla u_n \nabla \eta_\rho.$$

We claim that

$$(A) \qquad \int_{B_n} |\nabla u_n|^{p-2} (u_n - u) \nabla u_n \nabla \eta_\rho = o_n(1)$$

$$(B) \qquad \int_{B_n} h u_n^q (u_n - u)\eta_\rho = o_n(1)$$

Assume that (A) and (B) hold. Then

$$\int_{B_n} (|\nabla u_n|^{p-2} \nabla u_n - |\nabla u|^{p-2} \nabla u)(\nabla u_n - \nabla u)\eta_\rho =$$

$$- \int_{B_n} |\nabla u|^{p-2} \nabla u (\nabla u_n - \nabla u)\eta_\rho + o_n(1).$$

Recalling that

$$U_n \equiv (|\nabla u_n|^{p-2} \nabla u_n - |\nabla u|^{p-2} \nabla u)(\nabla u_n - \nabla u) \ge 0 \text{ a.e. in } I\!\!R^N,$$

we get

$$\int_{B_\rho} U_n \leq \int_{B_n} U_n \eta_\rho = -\int_{B_n} |\nabla u|^{p-2} \nabla u (\nabla u_n - \nabla u)\eta_\rho + o_n(1).$$

We claim that

$$(C) \qquad \int_{B_n} |\nabla u|^{p-2} \nabla u (\nabla u_n - \nabla u)\eta_\rho = o_n(1).$$

Let us assume that (C) holds. Then we have

$$\int_{B_\rho} (|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u)(\nabla u_n - \nabla u)\eta_\rho \leq o_n(1).$$

Now, from Tolksdorff [12, Lemma 1], we have

$$\int_{B_\rho} (1 + |\nabla u_n| + |\nabla u|)^{p-2} |\nabla u_n - \nabla u|^2 \leq o_n(1),\ 1 < p < 2$$

and

$$\int_{B_\rho} |\nabla u_n - \nabla u|^p \leq o_n(1),\ 2 \leq p < \infty$$

which shows that

$$\nabla u_n \to \nabla u \text{ a.e. in } B_\rho.$$

Taking a sequence $\rho_n \to \infty$ and using a diagonal argument, we infer that

$$\nabla u_n \to \nabla u \text{ a.e. in } \mathbb{R}^N.$$

Verification of (A). We recall that

$$D^{1,p} \to W^{1,p}(B_{2\rho}) \hookrightarrow L^r(B_{2\rho}),\ p \leq r < p^*$$

so that

$$u_n \to u \ in \ L^r(B_{2\rho})$$

and hence, using Hölder's inequality,

$$\begin{aligned} \textstyle\int_{B_n} ||\nabla u_n|^{p-2}(u_n - u)\nabla u_n \nabla \eta_\rho| &= \textstyle\int_{B_{2\rho}} ||\nabla u_n|^{p-2}(u_n - u)\nabla u_n \nabla \eta_\rho| \\ &\leq C_\rho \textstyle\int_{B_{2\rho}} |\nabla u_n|^{p-1}|u_n - u| \\ &\leq C_\rho ||\nabla u_n||^{p-1}_{L^p(B_{2\rho})} |u_n - u|_{L^p(B_{2\rho})}. \end{aligned}$$

Verification of (B). We have

$$\begin{aligned} \textstyle\int_{B_n} |h u_n^q (u_n - u)\eta_\rho| &= \textstyle\int_{B_{2\rho}} |h u_n^q||u_n - u|\eta_\rho \\ &\leq C_{h,\rho} \textstyle\int_{B_{2\rho}} |u_n^q||u_n - u| \\ &\leq C_{h,\rho} |u_n|^q_{L^{q+1}(B_{2\rho})} |u_n - u|_{L^{q+1}(B_{2\rho})} \end{aligned}$$

Verification of (C). Letting

$$\langle F, w \rangle \equiv \int_{B_{2\rho}} |\nabla u|^{p-2} \nabla u \nabla w \eta_\rho,\ w \in W^{1,p}(B_{2\rho})$$

and using Hölder's inequality we infer that

$$F \in (W^{1,p}(B_{2\rho}))'$$

and, consequently,

$$\langle F, u_n - u\rangle = \int_{B_{2\rho}} |\nabla u|^{p-2}\nabla u(\nabla u_n - \nabla u)\eta_\rho \to 0$$

This proves lemma 5. ∎

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