ON THE EXISTENCE OF POSITIVE SOLUTIONS FOR PERIODIC PARABOLIC SUBLINEAR PROBLEMS

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We give necessary and sufficient conditions for the existence of positive solutions for sublinear Dirichlet periodic parabolic problems $Lu = g(x,t,u)$ in $\Omega \times \mathbb{R}$ (where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain) for a wide class of Carathéodory functions $g : \Omega \times \mathbb{R} \times [0,\infty) \to \mathbb{R}$ satisfying some integrability and positivity conditions.

1. Introduction

Let Ω be a smooth bounded domain in $\mathbb{R}^N$, $N \geq 2$. For $T > 0$, $1 \leq p \leq \infty$, and $1 \leq q \leq \infty$, let $L^p(L^q)$ be the Banach space of T-periodic functions f on $\Omega \times \mathbb{R}$ (i.e., satisfying $f(x,t) = f(x,t+T)$ a.e. $(x,t) \in \Omega \times \mathbb{R}$) such that

$$\|f\|_{L^p(L^q)} := \left\| \left\| f(\cdot,t) \right\|_{L^q(\Omega)} \right\|_{L^p(0,T)} < \infty. \tag{1.1}$$

Similarly, let L^p_T be the Banach space of T-periodic functions f such that $f|_{\Omega \times (0,T)} \in L^p(\Omega \times (0,T))$, equipped with the norm $\|f\|_{L^p_T} := \|f|_{\Omega \times (0,T)}\|_{L^p(\Omega \times (0,T))}$. Finally, let C_T be the space of continuous and T-periodic functions on $\overline{\Omega} \times \mathbb{R}$ provided with the L^∞-norm.

For the whole paper, we fix $\nu, s \in (1,\infty]$ such that $N/2\nu + 1/s < 1$, $s > 2$. Let $\{a_{ij}\}$ and $\{b_j\}$, $1 \leq i,\ j \leq N$, be two families of functions satisfying $a_{ij}, b_j \in L^\infty_T$ and $a_{ij} = a_{j,i}$. Assume that $\sum a_{ij}(x,t)\xi_i\xi_j \geq \alpha_0 |\xi|^2$ for some $\alpha_0 > 0$ and all $(x,t) \in \Omega \times \mathbb{R}$, $\xi \in \mathbb{R}^N$. Let A be the $N \times N$ matrix whose i,j entry is $a_{i,j}$, let $b = (b_1,\dots,b_N)$, let $0 \leq c_0 \in L^s(L^\nu)$, and let L be the parabolic operator given by

$$Lu = u_t - \operatorname{div}(A\nabla u) + \langle b, \nabla u\rangle + c_0 u. \tag{1.2}$$

Let $W = \{u \in L^2((0,T), H^1_0(\Omega)) : u_t \in L^2((0,T), H^{-1}(\Omega))\}$. Given $f \in L^1_{T,\mathrm{loc}}(\Omega \times \mathbb{R})$, we say that u is a (weak) solution of the Dirichlet periodic problem $Lu = f$

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in $\Omega\times\mathbb{R}$, $u=0$ on $\partial\Omega\times\mathbb{R}$, if u is T-periodic, $u|_{\Omega\times(0,T)}\in W$, and

$$\int_{\Omega\times(0,T)}\left[-u\frac{\partial h}{\partial t}+\langle A\nabla u,\nabla h\rangle+\langle b,\nabla u\rangle h+c_0uh\right]=\int_{\Omega\times(0,T)}fh \tag{1.3}$$

for all $h\in C_c^\infty(\Omega\times\mathbb{R})$ (and so for all $h\in L_T^\infty$ such that $h|_{\Omega\times(0,T)}\in V_0$, where $V_0:=L^2((0,T),H_0^1(\Omega)))$. For $u\in W$, the inequality $Lu\geq f$ (resp., $\leq$) will be understood in the same sense.

Let $\widetilde{W}=\{u\in L^2((0,T),H^1(\Omega)):u_t\in L^2((0,T),H^{-1}(\Omega))\}$. Following [6], we say that v is a supersolution of the above problem if $v|_{\Omega\times(0,T)}\in\widetilde{W}$, $v_t\in L^2((0,T),H^{-1}(\Omega))+L^{1+\eta}(\Omega\times(0,T))$ for $\eta>0$ small enough, $v|_{\partial\Omega\times(0,T)}\geq 0$, $v(\cdot,0)\geq v(\cdot,T)$ a.e. in Ω, and

$$\int_{\Omega\times(0,T)}\left[-v\frac{\partial h}{\partial t}+\langle A\nabla v,\nabla h\rangle+\langle b,\nabla v\rangle h+c_0vh\right]\geq\int_{\Omega\times(0,T)}fh \tag{1.4}$$

for all $0\leq h\in C_c^\infty(\Omega\times(0,T))$ (and so for all $h\in L_T^\infty$ such that $h|_{\Omega\times(0,T)}\in V_0$ with V_0 as above). A subsolution is similarly defined by reversing the above inequalities.

Let $m\in L^s(L^v)$ and let

$$P(m):=\int_0^T\operatorname{ess\,sup}_{x\in\Omega}m(x,t)dt \tag{1.5}$$

(with the value "$+\infty$" allowed). For such m (cf. [8, Theorem 3.6]), $P(m)>0$ is necessary and sufficient for the existence of a positive principal eigenvalue for the periodic parabolic Dirichlet problem with weight function m (i.e., an eigenvalue with a positive T-periodic eigenfunction associated to the problem $Lu=\lambda mu$ in $\Omega\times\mathbb{R}$, $u=0$ on $\partial\Omega\times\mathbb{R}$). Moreover, this positive principal eigenvalue denoted by $\lambda_1(L,m)$ (or $\lambda_1(m)$), if exists, is unique.

We are interested in the existence of positive solutions for the semilinear periodic parabolic problem

$$\begin{gathered}Lu=g(x,t,u)\quad\text{in }\Omega\times\mathbb{R},\\ u=0\quad\text{on }\partial\Omega\times\mathbb{R},\\ uT\text{-periodic,}\end{gathered} \tag{1.6}$$

where g is a given function on $\Omega\times\mathbb{R}\times[0,\infty)$.

In [9, Theorem 3.7], it is proved that

$$\lambda_1\left(\sup_{\xi>0}\frac{g(\cdot,\xi)}{\xi}\right)<1<\lambda_1\left(\inf_{\xi>0}\frac{g(\cdot,\xi)}{\xi}\right) \tag{1.7}$$

is a necessary and sufficient condition for the existence of positive solutions in C_T for (1.6) provided that g satisfies $\xi\to g(x,t,\xi)\in C^1[0,\infty)$, $\xi\to g(x,t,\xi)/\xi$

nonincreasing in $(0,\infty)$, and some integrability and positivity conditions. In [10, Theorem 3.1], with the same monotonicity and regularity assumptions, and assuming also some integrability conditions, it is proved that if either $\inf_{\xi>0}(g(\cdot,\xi)/\xi) \in L^s(L^\nu)$ and $P(\inf_{\xi>0}(g(\cdot,\xi)/\xi)) \le 0$ or $\inf_{\xi>0}(g(\cdot,\xi)/\xi) \le 0$, then

$$\lambda_1\left(\sup_{\xi>0}\frac{g(\cdot,\xi)}{\xi}\right) < 1 \tag{1.8}$$

is necessary and sufficient for the existence of a positive solution $u \in C_T$ of (1.6).

Our aim in this paper is to prove, following a different approach, similar results without monotonicity and C^1-regularity assumptions on g (see Theorems 3.1, 3.2, 3.3, and 3.4). Moreover, we will also cover some cases where $\lim_{\xi\to 0^+}(g(\cdot,\xi)/\xi) = \infty$. These theorems will be obtained using the well-known sub- and supersolutions method combined with some facts concerning linear problems with weight.

In order to relate our results to others in the literature, we mention that, for the case $\xi \to g(\cdot,\xi)/\xi$ nonincreasing, similar results to Theorem 3.1 for elliptic problems have been obtained, for example, in [4, 5, 13], assuming more regularity in the function g. In the periodic parabolic case, there are also well-known results if $\xi \to g(\cdot,\xi)/\xi$ is concave and Hölder-continuous, and $g(\cdot,0) = 0$ (see [2, 3, 12] and the references therein).

On the other side, necessary and sufficient conditions for the existence of positive solutions for equations of type $Lu = a(x)u - b(x)u^p$, $p > 1$, $b \ge 0$ (logistic equation), are also known (see, e.g., [11, 12]). More general equations of the form $Lu = a(x)u - b(x)f(x,u)$, with $b \ge 0$ and f superlinear, were studied, for example, in [7] for $f \in C^{\mu,1+\mu}(\overline{\Omega}\times[0,\infty))$, f strictly increasing, and $b > 0$, and, for the Laplacian, the case $f = f(u)$ is treated in [1] assuming $f \in C([0,\infty))$. Theorem 3.2 generalizes the aforementioned results, while Theorems 3.3 and 3.4 also extend some well-known results, see, for example, [2, 3, 11, 12].

Some examples are also given at the end of the paper.

2. Preliminaries and auxiliary results

As usual, for $\xi \in [0,\infty)$ and $u : \Omega\times\mathbb{R} \to [0,\infty)$, we write $g(\xi)$ and $g(u)$ for the functions $(x,t) \to g(x,t,\xi)$ and $(x,t) \to g(x,t,u(x,t))$, $(x,t) \in \Omega\times\mathbb{R}$. We assume, from now on, that $g : \Omega\times\mathbb{R}\times[0,\infty) \to \mathbb{R}$ is a Carathéodory function (i.e., $(x,t) \to g(x,t,\xi)$ is measurable for all $\xi \in [0,\infty)$, and $\xi \to g(x,t,\xi)$ is continuous in $[0,\infty)$ a.e. $(x,t) \in \Omega\times\mathbb{R}$) such that $\sup_{\sigma\ge\xi}(g(\sigma)/\sigma)$ and $\inf_{0<\sigma\le\xi}(g(\sigma)/\sigma)$ are measurable functions for all $\xi > 0$, and $\inf_{\xi>0}(g(\xi)/\xi) \ne \sup_{\xi>0}(g(\xi)/\xi)$, that is, (1.6) is not a linear problem.

We start recalling some facts about periodic parabolic problems with weight.

Remark 2.1. (a) Let $D = \{m \in L^s(L^\nu) : P(m) > 0\}$. Then D is open in $L^s(L^\nu)$ and the map $m \to \lambda_1(m)$ is continuous from D into $\mathbb{R}$ (cf. [8, Theorem 3.9]). Also, the following comparison principle holds: if $m_1, m_2 \in L^s(L^\nu)$ and $m_1 \leq m_2$ in $\Omega \times \mathbb{R}$, then $\lambda_1(m_1) \geq \lambda_1(m_2)$; and if, in addition, $m_1 < m_2$ in a set of positive measure, then $\lambda_1(m_1) > \lambda_1(m_2)$ (cf. [8, Remark 3.7]).

(b) For $\lambda \in \mathbb{R}$ and $m \in L^s(L^\nu)$, let $\mu_m(\lambda)$ be defined as the unique $\mu \in \mathbb{R}$ such that the Dirichlet periodic problem $Lu = \lambda m u + \mu_m(\lambda) u$ in $\Omega \times \mathbb{R}$ has a positive solution u. We recall that $\mu_m(\lambda)$ is well defined and that the map $(\lambda, m) \to \mu_m(\lambda)$ is continuous from $\mathbb{R} \times L^s(L^\nu)$ into $\mathbb{R}$ (cf. [9, Proposition 2.7]). Moreover, $\mu_m(0) > 0$, μ_m is concave and continuous, and a given $\lambda \in \mathbb{R}$ is a principal eigenvalue associated to the weight m if and only if $\mu_m(\lambda) = 0$ (cf. [8, Lemma 3.2]). Also, if $\lambda_1(m)$ exists, then for $\lambda > 0$, $\mu_m(\lambda) > 0$ if and only if $\lambda < \lambda_1(m)$, and if $\lambda_1(m)$ does not exist, $\mu_m(\lambda) > 0$ for all $\lambda > 0$.

(c) Let $m \in L^s(L^\nu)$ such that $P(m) > 0$ and let m_j be a sequence such that m_j converges to m in $L^s(L^\nu)$. Then it follows from [9, Remark 2.5] that $P(m_j) > 0$ for j large enough.

Remark 2.2. If $u \in L_T^\infty$ is a positive solution of (1.6) and

$$\begin{aligned} &\inf_{0<\xi\leq M}\left(\frac{g(\xi)}{\xi}\right) \in L^s(L^\nu), \\ &\sup_{0<\xi\leq M}\left(\frac{g(\xi)}{\xi}\right) \in L^s(L^\nu), \end{aligned} \tag{2.1}$$

for all $M > 0$, then $u \in C_T$ and $u(x,t) > 0$ for all $(x,t) \in \Omega \times \mathbb{R}$. Indeed, this follows from [9, Remark 2.2 and Corollary 2.12].

We introduce some additional notation. For $(x,t,\xi) \in \Omega \times \mathbb{R} \times (0,\infty)$, let

$$\begin{aligned} \overline{g}(x,t,\xi) &= \xi \sup_{0<\xi\leq\sigma}\left(\frac{g(x,t,\sigma)}{\sigma}\right), \\ \underline{g}(x,t,\xi) &= \xi \inf_{0<\sigma\leq\xi}\left(\frac{g(x,t,\sigma)}{\sigma}\right) \end{aligned} \tag{2.2}$$

(with the values "$\pm\infty$" allowed). It is easy to check that if $\underline{g}(\xi)$ is finite for $\xi \leq \xi_0$, then $\xi \to \underline{g}(\xi)$ is continuous in $(0, \xi_0)$ a.e. in $\Omega \times \mathbb{R}$, and that if $\overline{g}(\xi)$ is finite for $\xi_0 \leq \xi$, then $\xi \to \overline{g}(\xi)$ is continuous in (ξ_0, ∞) a.e. in $\Omega \times \mathbb{R}$. We also set

$$\begin{aligned} &\underline{m}_\infty(x,t) = \inf_{\xi>0}\left(\frac{g(x,t,\xi)}{\xi}\right), \qquad \overline{m}_0(x,t) = \sup_{\xi>0}\left(\frac{g(x,t,\xi)}{\xi}\right), \\ &\underline{m}_0(x,t) = \liminf_{\xi\to0^+}\left(\frac{g(x,t,\xi)}{\xi}\right), \qquad \overline{m}_\infty(x,t) = \limsup_{\xi\to\infty}\left(\frac{g(x,t,\xi)}{\xi}\right). \end{aligned} \tag{2.3}$$

Note that

$$\begin{aligned} \underline{m}_\infty &= \lim_{\xi\to\infty}\left(\frac{\underline{g}(\xi)}{\xi}\right), & \overline{m}_0 &= \lim_{\xi\to 0^+}\left(\frac{\overline{g}(\xi)}{\xi}\right), \\ \underline{m}_0 &= \lim_{\xi\to 0^+}\left(\frac{\underline{g}(\xi)}{\xi}\right), & \overline{m}_\infty &= \lim_{\xi\to\infty}\left(\frac{\overline{g}(\xi)}{\xi}\right). \end{aligned} \tag{2.4}$$

Lemma 2.3. *Let $\xi_0 > 0$. Assume that $\overline{g}(\xi) \in L^s(L^\nu)$ for all $\xi \geq \xi_0$ and that either $\overline{m}_\infty \in L^s(L^\nu)$ with $\lambda_1(\overline{m}_\infty) > 1$ (if $\lambda_1(\overline{m}_\infty)$ exists) or $\overline{m}_\infty \leq 0$. Then, for all $c > 0$, there exists a supersolution $w \in C_T$ of (1.6) such that $w \geq c$.*

Proof. We first study the case $\overline{m}_\infty \in L^s(L^\nu)$. Let $c > 0$. We claim that there exists $\xi \geq c$ such that $\mu_{\overline{g}(\xi)/\xi}(1) > 0$. Indeed, for $\xi \geq \xi_0$, we have $\overline{m}_\infty \leq \overline{g}(\xi)/\xi \leq \overline{g}(\xi_0)/\xi_0$ and also $\lim_{\xi\to\infty}(\overline{g}(\xi)/\xi) = \overline{m}_\infty$ with convergence a.e. Thus, by dominated convergence, $\lim_{\xi\to\infty}(\overline{g}(\xi)/\xi) = \overline{m}_\infty$ in $L^s(L^\nu)$ and then Remark 2.1(b) implies $\lim_{\xi\to\infty}\mu_{\overline{g}(\xi)/\xi}(\lambda) = \mu_{\overline{m}_\infty}(\lambda)$ for all λ. Moreover, either if $P(\overline{m}_\infty) > 0$ and $\lambda_1(\overline{m}_\infty) > 1$ or if $P(\overline{m}_\infty) \leq 0$, the last statement in Remark 2.1(b) also gives $\mu_{\overline{m}_\infty}(1) > 0$. Thus, it follows that $\mu_{\overline{g}(\xi)/\xi}(1) > 0$ for ξ large enough.

We fix $\xi^* \geq \max(\xi_0, c)$ such that $\mu_{\overline{g}(\xi^*)/\xi^*}(1) > 0$. Let k be a function defined by $k(x,t) = \sup_{\xi\geq\xi^*}|\overline{g}(\xi)/\xi|$. Since $\overline{m}_\infty \leq k \leq \overline{g}(\xi^*)/\xi^*$, we get $k \in L^s(L^\nu)$. For $\xi \in [0,\infty)$, let $g^*(x,t,\xi) = \overline{g}(x,t,\xi) + k(x,t)\xi$. Then $g^*(x,t,\xi) \geq 0$ and $g^*(\xi)/\xi \in L^s(L^\nu)$ for $\xi \geq \xi^*$. Also, $\mu_{L+\lambda k, g^*(\xi^*)/\xi^*}(\lambda) = \mu_{L,\overline{g}(\xi^*)/\xi^*}(\lambda)$ for all λ. In particular, $\mu_{L+k,g^*(\xi)/\xi^*}(1) = \mu_{L,\overline{g}(\xi^*)/\xi^*}(1) > 0$. Thus, Lemma 2.9 in [9] says that the Dirichlet periodic problem $(L + k - g^*(\xi^*)/\xi^*)\Phi = g^*(\xi^*)$ in $\Omega\times\mathbb{R}$ has a solution $\Phi \in C_T$ satisfying $\Phi(x,t) > 0$ a.e. $(x,t) \in \Omega\times\mathbb{R}$. Now,

$$\begin{aligned} g(\xi^* + \Phi) &\leq \overline{g}(\xi^* + \Phi) \\ &\leq \frac{\overline{g}(\xi^*)}{\xi^*}(\xi^* + \Phi) \\ &\leq \overline{g}(\xi^*) + k\xi^* + \frac{\overline{g}(\xi^*)}{\xi^*}\Phi \\ &= g^*(\xi^*) + \frac{g^*(\xi^*)}{\xi^*}\Phi - k\Phi \\ &= L\Phi \leq L(\xi^* + \Phi), \end{aligned} \tag{2.5}$$

and therefore $\xi^* + \Phi$ is a supersolution for (1.6).

Consider now the case $\overline{m}_\infty \leq 0$. In this case, we have $\lim_{\xi\to\infty}(\overline{g}^+(\xi)/\xi) = 0$ a.e. in $\Omega\times\mathbb{R}$, where, as usual, we write $f = f^+ - f^-$. Also, $0 \leq \overline{g}^+(\xi)/\xi \leq \overline{g}^+(\xi_0)/\xi_0$ for all $\xi \geq \xi_0$, and thus $\lim_{\xi\to\infty}(\overline{g}^+(\xi)/\xi) = 0$ in $L^s(L^\nu)$. So, $\lim_{\xi\to\infty}\mu_{\overline{g}^+(\xi)/\xi}(\lambda) = \lambda_1$ for all λ, where λ_1 is the (positive) principal eigenvalue for L associated to the weight 1 (because for $m \equiv 1$, $\mu_m \equiv \lambda_1$). Thus, we can choose $\xi^* \geq \max(\xi_0, c)$ such that $\mu_{\overline{g}^+(\xi^*)/\xi^*} > 0$, and then, as above, the Dirichlet periodic problem $(L - \overline{g}^+(\xi^*)/\xi^*)\Phi = \overline{g}^+(\xi^*)$ in $\Omega\times\mathbb{R}$ has a solution $\Phi \in C_T$ satisfying $\Phi(x,t) > 0$ a.e.

(x,t) in $\Omega\times\mathbb{R}$. Also,

$$\begin{aligned} g(\xi^*+\Phi) &\le \overline{g}^+(\xi^*+\Phi)\\ &\le \frac{\overline{g}^+(\xi^*)}{\xi^*}(\xi^*+\Phi)\\ &= \overline{g}^+(\xi^*)+\frac{\overline{g}^+(\xi^*)}{\xi^*}\Phi\\ &= L\Phi \le L(\Phi+\xi^*), \end{aligned} \tag{2.6}$$

and this concludes the proof. □

Lemma 2.4. *Let $\xi_0>0$. Assume that $\underline{g}(\xi_0)\in L^s(L^\nu)$, $P(\underline{g}(\xi_0)/\xi_0)>0$, and $\lambda_1(\underline{g}(\xi_0)/\xi_0)\le 1$. Then there exists a subsolution $v\in C_T$ of (1.6) such that $v(x,t)>0$ for all $(x,t)\in\Omega\times\mathbb{R}$.*

Proof. Let Φ be the positive eigenfunction of

$$\begin{gathered} \left(L+\frac{\underline{g}^-(\xi_0)}{\xi_0}\right)\Phi=\lambda_1\left(\frac{\underline{g}^+(\xi_0)}{\xi_0}\right)\left(\frac{\underline{g}^+(\xi_0)}{\xi_0}\right)\Phi \quad \text{in } \Omega\times\mathbb{R},\\ \Phi=0 \quad \text{on } \partial\Omega\times\mathbb{R},\\ \Phi T\text{-periodic.} \end{gathered} \tag{2.7}$$

Then $\Phi\in C_T$ and $\Phi(x,t)>0$ for all $(x,t)\in\Omega\times\mathbb{R}$. Now, $\lambda_1(L,\underline{g}(\xi_0)/\xi_0)<1$ implies $\mu_{L,\underline{g}(\xi_0)/\xi_0}(1)\le 0$. Thus, since $\mu_{L,\underline{g}(\xi_0)/\xi_0}(1)=\mu_{L+\underline{g}^-(\xi_0)/\xi_0,\underline{g}^+(\xi_0)/\xi_0}(1)$, we get $\lambda_1(\underline{g}^+(\xi_0)/\xi_0)\le 1$.

Let $\varepsilon>0$ be such that $\varepsilon<\xi_0/\|\Phi\|_\infty$. Taking into account the above-mentioned facts and that $\xi\to\underline{g}(\xi)/\xi$ is nonincreasing, we have

$$\begin{aligned} L(\varepsilon\Phi)+\underline{g}^-(\varepsilon\Phi) &\le \left(L+\frac{\underline{g}^-(\varepsilon\|\Phi\|)}{\varepsilon\|\Phi\|}\right)\varepsilon\Phi\\ &\le \left(L+\frac{\underline{g}^-(\xi_0)}{\xi_0}\right)\varepsilon\Phi\\ &\le \left(\frac{\underline{g}^+(\xi_0)}{\xi_0}\right)\varepsilon\Phi\\ &\le \left(\frac{\underline{g}^+(\varepsilon\|\Phi\|)}{\varepsilon\|\Phi\|}\right)\varepsilon\Phi\\ &\le \underline{g}^+(\varepsilon\Phi), \end{aligned} \tag{2.8}$$

and the lemma follows. □

3. The main results

Theorem 3.1. (a) *Assume that*

(1) $\underline{m}_0, \overline{m}_\infty \in L^s(L^\nu)$, $P(\underline{m}_0) > 0$, *and* $P(\overline{m}_\infty) > 0$,
(2) $\overline{g}(\xi_0) \in L^s(L^\nu)$ *for some* $\xi_0 > 0$ *and* $\underline{g}(\xi_1) \in L^s(L^\nu)$ *for some* $\xi_1 > 0$.

Then, if $\lambda_1(\underline{m}_0) < 1 < \lambda_1(\overline{m}_\infty)$, *there exists a solution* $u \in L_T^\infty$ *of (1.6) satisfying* $u(x,t) > 0$ *for all* $(x,t) \in \Omega \times \mathbb{R}$.

(b) *Assume (1),* $\overline{m}_0 = \underline{m}_0$, $\overline{m}_\infty = \underline{m}_\infty$, *and that for all* $\xi > 0$,

$$\overline{m}_0 \neq \frac{\overline{g}(\xi)}{\xi}, \tag{3.1}$$

$$\underline{m}_\infty \neq \frac{\underline{g}(\xi)}{\xi}. \tag{3.2}$$

Then there exists a positive solution $u \in L_T^\infty$ *of (1.6) if and only if* $\lambda_1(\underline{m}_0) < 1 < \lambda_1(\overline{m}_\infty)$.

Proof. Suppose that $\lambda_1(\underline{m}_0) < 1 < \lambda_1(\overline{m}_\infty)$. Since, for $0 < \xi \le \xi_1$, we have $\underline{g}(\xi_1)/\xi_1 \le \underline{g}(\xi)/\xi \le \underline{m}_0$ and $\lim_{\xi \to 0^+} \underline{g}(\xi)/\xi = \underline{m}_0$ a.e. in $\Omega \times \mathbb{R}$, taking into account (1) and (2), we get $\underline{g}(\xi)/\xi \in L^s(L^\nu)$ for such ξ and so $\lim_{\xi \to 0^+} \underline{g}(\xi)/\xi = \underline{m}_0$ with convergence in $L^s(L^\nu)$. Then, by Remark 2.1(c), we have $\lim_{\xi \to 0^+} P(\underline{g}(\xi)/\xi) = P(\underline{m}_0) > 0$, and thus there exists $\lambda_1(\underline{g}(\xi)/\xi)$ for $\xi > 0$ small enough. Moreover, Remark 2.1(a) says that $\lim_{\xi \to 0^+} \lambda_1(\underline{g}(\xi)/\xi) = \lambda_1(\underline{m}_0) < 1$ and so $\lambda_1(\underline{g}(\xi)/\xi) < 1$ for such ξ. Hence, Lemma 2.4 can be applied to give a subsolution $v \in C_T$ of (1.6) with $v(x,t) > 0$ for all $(x,t) \in \Omega \times \mathbb{R}$.

On the other hand, for all $\xi \ge \xi_0$, we have $\overline{m}_\infty \le \overline{g}(\xi)/\xi \le \overline{g}(\xi_0)/\xi_0$, and so $\overline{g}(\xi)/\xi \in L^s(L^\nu)$. Therefore, taking $c = \|v\|_\infty$ in Lemma 2.3, we obtain a supersolution $w \in C_T$ of (1.6) with $w \ge c \ge v$. Now, [6, Theorem 1] gives a solution $u \in L_T^\infty$ such that $v \le u \le w$ and then $u(x,t) > 0$ for all $(x,t) \in \Omega \times \mathbb{R}$. Thus (a) is proved.

To prove (b), suppose that $u \in L_T^\infty$ is a positive solution of (1.6). By Remark 2.2, we have $u(x,t) > 0$ for all (x,t). Let $m_u : \Omega \times \mathbb{R} \to \mathbb{R}$ be defined by $m_u = g(u)/u$. Since m_u is measurable and $\underline{m}_\infty \le m_u \le \overline{m}_0$, it follows that $m_u \in L^s(L^\nu)$. Moreover, we have $Lu = m_u u$ and so $1 = \lambda_1(m_u)$. Now, the comparison principle in Remark 2.1(a) gives $1 = \lambda_1(m_u) \ge \lambda_1(\overline{m}_0) = \lambda_1(\underline{m}_0)$ and also $1 \le \lambda_1(\underline{m}_\infty) = \lambda_1(\overline{m}_\infty)$. Suppose $\lambda_1(\overline{m}_0) = 1$. Since $\lambda_1(m_u) = 1$ and $m_u \le \overline{m}_0$, we must have $m_u(x,t) = \overline{m}_0(x,t)$ a.e. $(x,t) \in \Omega \times \mathbb{R}$ (see Remark 2.1(a)), but $\sup_{0<\xi\le\|u\|_\infty}(g(\xi)/\xi) \ge g(u)/u = \overline{m}_0$ in $\Omega \times \mathbb{R}$ contradicting (3.1). Then $\lambda_1(\overline{m}_0) < 1$. Suppose now that $\lambda_1(\underline{m}_\infty) = 1$. Reasoning as above, we get $1 = \lambda_1(m_u) \le \lambda_1(\underline{m}_\infty) = 1$ and so $m_u = \underline{m}_\infty$. Thus, $\inf_{0<\xi\le\|u\|_\infty}(g(\xi)/\xi) \le g(u)/u = \inf_{\xi>0}(g(\xi)/\xi)$ a.e., which is again a contradiction. Then $\lambda_1(\underline{m}_\infty) > 1$. □

Theorem 3.2. (a) *Assume that*

(3) $\underline{m}_0 \in L^s(L^\nu)$, $P(\underline{m}_0) > 0$,

(4) $\overline{g}(\xi_0) \in L^s(L^\nu)$ *for some* $\xi_0 > 0$ *and* $\underline{g}(\xi) \in L^s(L^\nu)$ *for all* $\xi > 0$,

(5) *either* $\overline{m}_\infty \in L^s(L^\nu)$ *and* $P(\overline{m}_\infty) \le 0$ *or* $\overline{m}_\infty \le 0$.

Then, if $\lambda_1(\underline{m}_0) < 1$, *there exists a solution* $u \in L_T^\infty$ *of (1.6) satisfying* $u(x,t) > 0$ *for all* $(x,t) \in \Omega \times \mathbb{R}$.

(b) *Assume, in addition, (3.1) and* $\overline{m}_0 = \underline{m}_0$. *Then there exists a positive solution* $u \in L_T^\infty$ *of (1.6) if and only if* $\lambda_1(\underline{m}_0) < 1$.

Proof. As in the above theorem, we have $\underline{g}(\xi)/\xi \in L^s(L^\nu)$ and $\lambda_1(\underline{g}(\xi)/\xi) < 1$ for $\xi > 0$ small enough, and so Lemma 2.4 gives a subsolution $v \in C_T$ satisfying $v(x,t) > 0$ for all (x,t). On the other side, since $\underline{g}(\xi)/\xi \le \overline{g}(\xi)/\xi \le \overline{g}(\xi_0)/\xi_0$ for $\xi \ge \xi_0$, from (4), we have $\overline{g}(\xi)/\xi \in L^s(L^\nu)$ for such ξ. Therefore, (a) follows as in Theorem 3.1 taking $c = \|v\|_\infty$ in Lemma 2.3, and the proof of (b) follows similarly to part (b) of Theorem 3.1. □

Theorem 3.3. (a) *Assume (2) and that*

(6) $\overline{m}_\infty \in L^s(L^\nu)$ *and* $P(\overline{m}_\infty) > 0$,

(7) $P(\underline{g}(\xi)/\xi) > 0$ *for* $\xi > 0$ *small and* $\lim_{\xi \to 0^+} \lambda_1(\underline{g}(\xi)/\xi) = 0$.

Then, if $\lambda_1(\overline{m}_\infty) > 1$, *there exists a solution* $u \in L_T^\infty$ *of (1.6) satisfying* $u(x,t) > 0$ *for all* $(x,t) \in \Omega \times \mathbb{R}$.

(b) *Assume, in addition, (3.2) and* $\overline{m}_\infty = \underline{m}_\infty$. *Then there exists a positive solution* $u \in L_T^\infty$ *of (1.6) if and only if* $\lambda_1(\overline{m}_\infty) > 1$.

Proof. Reasoning as above, (a) follows from Lemmas 2.3, 2.4, and [6, Theorem 1]. Suppose now that $u \in L_T^\infty$ is a positive solution of (1.6). Let $\varepsilon > 0$ such that $\varepsilon < \|u\|_\infty$. Let $\underline{g}_\varepsilon$ be defined by $\underline{g}_\varepsilon(\xi) = \underline{g}(\xi)$ if $\xi \ge \varepsilon$ and $\underline{g}_\varepsilon(\xi) = \underline{g}(\varepsilon)$ if $\xi < \varepsilon$. We have $Lu = g(u) \ge \underline{g}(u) \ge \underline{g}_\varepsilon(u)$ and also $\underline{g}_\varepsilon(u)/u \in L^s(L^\nu)$. Thus, $1 \le \lambda_1(\underline{g}_\varepsilon(u)/u)$. Moreover, since $\underline{g}_\varepsilon(u)/u \ge \underline{m}_\infty$, the comparison principle in Remark 2.1(a) gives $1 \le \lambda_1(\underline{m}_\infty)$. Suppose $1 = \lambda_1(\underline{m}_\infty)$. Then $\underline{g}_\varepsilon(u)/u = \underline{m}_\infty$. But $\underline{g}_\varepsilon(u)/u \ge \underline{g}_\varepsilon(\|u\|)/\|u\| = \underline{g}(\|u\|)/\|u\|$, and therefore $\underline{m}_\infty = \underline{g}(\|u\|)/\|u\|$ in contradiction with (3.2). □

Theorem 3.4. *Assume (4), (5), and (7). Then (1.6) has a positive solution* $u \in L_T^\infty$ *satisfying* $u(x,t) > 0$ *for all* $(x,t) \in \Omega \times \mathbb{R}$.

Proof. The theorem follows again from Lemmas 2.3, 2.4, and [6, Theorem 1]. □

3.1. Examples. (a) Suppose there exist $\lim_{\xi \to 0^+}(g(\xi)/\xi)$ and $\lim_{\xi \to \infty}(g(\xi)/\xi)$ and assume $\inf_{\xi > 0}(g(\xi)/\xi), \sup_{\xi > 0}(g(\xi)/\xi) \in L^s(L^\nu)$, with $P(\inf_{\xi > 0}(g(\xi)/\xi)) > 0$. If $\lim_{\xi \to 0^+}(g(\xi)/\xi) = \sup_{\xi > 0}(g(\xi)/\xi)$ and $\lim_{\xi \to \infty}(g(\xi)/\xi) = \inf_{\xi > 0}(g(\xi)/\xi)$, from Theorem 3.1, we conclude that (1.6) has a positive solution $u \in L_T^\infty$ if and only if $\lambda_1(\lim_{\xi \to 0^+}(g(\xi)/\xi)) < 1 < \lambda_1(\lim_{\xi \to \infty}(g(\xi)/\xi))$.

(b) Consider the Dirichlet periodic problem $Lu = \sin u$ in $\Omega \times \mathbb{R}$. Theorem 3.2 says that this problem has a positive T-periodic solution if and only if $\lambda_1 < 1$, where λ_1 is the positive principal eigenvalue corresponding to the weight 1.

(c1) Consider the problem

$$
\begin{gathered}
Lu = a(x,t)u^{\gamma} - f(x,t,u)u \quad \text{in } \Omega \times \mathbb{R}, \\
u = 0 \quad \text{on } \partial\Omega \times \mathbb{R}, \\
u T\text{-periodic},
\end{gathered} \tag{3.3}
$$

where $0 < \gamma \leq 1$ and f is a Carathéodory function such that $f(\xi) \in L^s(L^\nu)$ for all $\xi > 0$ and $f(0) = 0$. Assume that $\gamma = 1$, $a \in L^s(L^\nu)$, $P(a) > 0$, $a \leq \lim_{\xi \to \infty} f(\xi) \leq \infty$, $\inf_{\xi_0 \leq \xi} f(\xi) \in L^s(L^\nu)$ for some $\xi_0 > 0$, and $\inf_{0<\xi\leq\xi_0} f(\xi) \in L^s(L^\nu)$ for all $\xi_0 > 0$. From Theorem 3.2, it follows that (3.3) has a positive solution $u \in L_T^\infty$ if and only if $\lambda_1(a) < 1$.

(c2) Consider now the case $0 < \gamma < 1$ and $a(x,t) \geq 0$ a.e. $(x,t) \in \Omega \times \mathbb{R}$. If $f(\xi) = -b$ with $b \in L^s(L^\nu)$ and $P(b) > 0$, then Theorem 3.3 says that (3.3) has a positive solution $u \in L_T^\infty$ if and only if $1 < \lambda_1(b)$. On the other hand, suppose $\lim_{\xi\to\infty} f(\xi) = \infty$, $\inf_{\xi_0 \leq \xi} f(\xi) \in L^s(L^\nu)$ for some $\xi_0 > 0$, and $\sup_{0<\xi\leq\xi_0} f(\xi) \in L^s(L^\nu)$ for all $\xi_0 > 0$. Then Theorem 3.4 gives a positive solution $u \in L_T^\infty$ for (3.3).

We note that in all the cases, the positive solution u satisfies $u(x,t) > 0$ for all (x,t). Moreover, recalling Remark 2.2, we also have that in (a), (b), and (c1) $u \in C_T$.

Remark 3.5. An inspection of the proofs shows that all the above results remain true for the corresponding elliptic problem, replacing $L^s(L^\nu)$ by $L^r(\Omega)$ with $r > N/2$, and $P(m)$ by $\operatorname{ess\,sup}_{x\in\Omega} m(x)$.

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