NONEXISTENCE RESULTS OF SOLUTIONS TO SYSTEMS OF SEMILINEAR DIFFERENTIAL INEQUALITIES ON THE HEISENBERG GROUP

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Received 25 December 2002

We establish nonexistence results to systems of differential inequalities on the $(2N+1)$-Heisenberg group. The systems considered here are of the type (ES_m). These nonexistence results hold for N less than critical exponents which depend on p_i and γ_i, $1 \le i \le m$. Our results improve the known estimates of the critical exponent.

1. Introduction

For the reader's convenience, we recall some background facts used here. The Heisenberg group $\mathbb{H}^N$, whose points will be denoted by $\eta = (x, y, \tau)$, is the Lie group $(\mathbb{R}^{2N+1}, \circ)$ with the group operation $\circ$ defined by

$$\eta \circ \tilde{\eta} = \left(x + \tilde{x}, y + \tilde{y}, \tau + \tilde{\tau} + 2\left(\langle x, \tilde{y}\rangle - \langle \tilde{x}, y\rangle\right)\right), \tag{1.1}$$

where $\langle \cdot, \cdot \rangle$ is the usual inner product in $\mathbb{R}^N$. The Laplacian $\Delta_{\mathbb{H}}$ over $\mathbb{H}^N$ is obtained, from the vector fields $X_i = \partial_{x_i} + 2y_i\partial_\tau$ and $Y_i = \partial_{y_i} - 2x_i\partial_\tau$, by

$$\Delta_{\mathbb{H}} = \sum_{i=1}^{N} \left(X_i^2 + Y_i^2\right). \tag{1.2}$$

Observe that the vector field $T = \partial_\tau$ does not appear in (1.2). This fact makes us presume a "loss of derivative" in the variable τ. The compensation comes from the relation

$$\left[X_i, Y_j\right] = -4T, \quad j,k \in \{1,2,\dots,N\}. \tag{1.3}$$

The relation (1.3) proves that $\mathbb{H}^N$ is a nilpotent Lie group of order 2. Incidently, (1.3) constitutes an abstract version of the canonical relations of commutation of Heisenberg between momentums and positions. Explicit computation gives the expression

$$\Delta_{\mathbb{H}} = \sum_{i=1}^{N} \left(\frac{\partial^2}{\partial x_i^2} + \frac{\partial^2}{\partial y_i^2} + 4y_i \frac{\partial^2}{\partial x_i \partial \tau} - 4x_i \frac{\partial^2}{\partial y_i \partial \tau} + 4\left(x_i^2 + y_i^2\right) \frac{\partial^2}{\partial \tau^2} \right). \tag{1.4}$$

Abstract and Applied Analysis 2004:2 (2004) 155–164
2000 Mathematics Subject Classification: 35J60, 35K60, 35L60
URL: http://dx.doi.org/10.1155/S108533750430802X

A natural group of dilatations on $\mathbb{H}^N$ is given by

$$\delta_\lambda(\eta) = (\lambda x, \lambda y, \lambda^2 \tau), \quad \lambda > 0, \tag{1.5}$$

whose Jacobian determinant is λ^Q, where

$$Q = 2N + 2 \tag{1.6}$$

is the homogeneous dimension of $\mathbb{H}^N$.

The operator $\Delta_{\mathbb{H}}$ is a degenerate elliptic operator. It is invariant with respect to the left translation of $\mathbb{H}^N$ and homogeneous with respect to the dilatations δ_λ. More precisely, we have

$$\begin{gathered} \Delta_{\mathbb{H}}(u(\eta \circ \tilde{\eta})) = (\Delta_{\mathbb{H}} u)(\eta \circ \tilde{\eta}), \\ \Delta_{\mathbb{H}}(u \circ \delta_\lambda) = \lambda^2 (\Delta_{\mathbb{H}} u) \circ \delta_\lambda \quad \forall (\eta, \tilde{\eta}) \in \mathbb{H}^N \times \mathbb{H}^N. \end{gathered} \tag{1.7}$$

It is natural to define a distance from η to the origin by

$$|\eta|_{\mathbb{H}} = \left(\tau^2 + \sum_{i=1}^N (x_i^2 + y_i^2)^2 \right)^{1/4}. \tag{1.8}$$

In [7], Pohozaev and Véron gave another proof of the result of Birindelli et al. [1] concerning the nonexistence of weak solutions of the differential inequality

$$\Delta_{\mathbb{H}}(au) + |\eta|_{\mathbb{H}}^{\gamma} |v|^p \le 0 \quad \text{in } \mathbb{H}^N \tag{1.9}$$

for $\gamma > -2$, $1 < p \le (Q+\gamma)/(Q-2)$, and $a \in L^\infty(\mathbb{H}^N)$.

They then addressed the question of nonexistence of weak solutions of the system (ES_2):

$$-\Delta_{\mathbb{H}}(a_1 u) \ge |\eta|_{\mathbb{H}}^{\gamma_1} |v|^{p_1}, \qquad -\Delta_{\mathbb{H}}(a_2 v) \ge |\eta|_{\mathbb{H}}^{\gamma_2} |u|^{p_2}, \tag{1.10}$$

where a_i, $i \in \{1,2\}$, are measurable and bounded functions defined on $\mathbb{H}^N$, and $p_i > 1$ and γ_i, $i = 1,2$, are real numbers. They showed that this system admits no solution defined in $\mathbb{H}^N$ whenever $\gamma_i > -2$ and $1 < p_i \le (Q+\gamma_i)/(Q-2)$, $i = 1,2$. The estimates on p_i, $i = 1,2$, are obtained using Young's inequality and are not optimal. Using the Hölder inequality, we obtain better estimates on p_i, $1 \le i \le m$. The same strategy is suitable to study the systems (PS_m) and (HS_m).

We also studied the following systems:

(PS_m) $\partial u_i/\partial t - \Delta_{\mathbb{H}}(a_i u_i) \ge |\eta|_{\mathbb{H}}^{\gamma_{i+1}} |u_{i+1}|^{p_{i+1}}$, $\eta \in \mathbb{H}^N$, $1 \le i \le m$, $u_{m+1} = u_1$,

(HS_m) $\partial^2 u_i/\partial t^2 - \Delta_{\mathbb{H}}(a_i u_i) \ge |\eta|_{\mathbb{H}}^{\gamma_{i+1}} |u_{i+1}|^{p_{i+1}}$, $\eta \in \mathbb{H}^N$, $1 \le i \le m$, $u_{m+1} = u_1$,

and showed the following results.

Theorem 1.1. *Assume that the initial data* $u_i^{(0)} \in L^1(\mathbb{R}^{2N+1})$ *and* $\int u_i^{(0)}(\eta)d\eta \geq 0$, $1 \leq i \leq m$. *If*

$$Q \leq \max\{X_1, X_2, \dots, X_m\}, \tag{1.11}$$

where the vector $(X_1, X_2, \dots, X_m)^T$ *is the solution of (3.1), then there is no nontrivial global weak solution* $(u_1, \dots, u_m)$ *of the system* (PS_m).

Theorem 1.2. *Assume that initial data (for the first derivatives of* u_i, $1 \leq i \leq m$*)* $u_i^{(1)} \in L^1(\mathbb{R}^{2N+1})$ *and* $\int u_i^{(1)}(\eta)d\eta \geq 0$, $1 \leq i \leq m$. *If*

$$Q \leq 1 + \max\{X_1, X_2, \dots, X_m\}, \tag{1.12}$$

where the vector $(X_1, X_2, \dots, X_m)^T$ *is the solution of (3.1), then there is no nontrivial global weak solution* $(u_1, \dots, u_m)$ *of the system* (HS_m).

In [2], the first author and Obeid presented results for systems of evolution type with higher-order time derivatives. Their results are the generalized versions of our previous results (Theorems 1.1 and 1.2) on (PS_m) and (HS_m).

For interesting results on elliptic equations and systems, we refer to the recent papers of Kartsatos and Kurta [3], Kurta [4, 5], and Mitidieri and Pohozaev [6].

To render the presentation very clear, we start with the case of systems of two inequalities.

2. Systems of two inequalities

In this section, we treat the case $m = 2$ and consider the system (ES_2).

We identify points in $\mathbb{H}^N$ with points in $\mathbb{R}^{2N+1}$. We also recall that the Haar measure on $\mathbb{H}^N$ is identical to the Lebesgue measure $d\eta = dx\,dy\,d\tau$ on $\mathbb{R}^{2N+1} = \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}$. In the sequel, the integral $\int_{\mathbb{R}^{2N+1}}$ will be simply denoted by $\int$; however, the measure of integration will be specified.

Definition 2.1. Let a_1 and a_2 be two bounded measurable functions on $\mathbb{R}^{2N+1}$. A weak solution (u, v) of the system (ES_2) on $\mathbb{R}^{2N+1}$ is a pair of locally integrable functions (u, v) such that

$$u \in L^{p_2}_{\mathrm{loc}}(\mathbb{R}^{2N+1}, |\eta|_{\mathbb{H}}^{\gamma_2} d\eta), \qquad v \in L^{p_1}_{\mathrm{loc}}(\mathbb{R}^{2N+1}, |\eta|_{\mathbb{H}}^{\gamma_1} d\eta), \tag{2.1}$$

satisfying

$$\begin{aligned} &\int_{\mathbb{R}^{2N+1}} \left(a_1 u \Delta_{\mathbb{H}} \varphi + |\eta|_{\mathbb{H}}^{\gamma_1} |v|^{p_1} \varphi\right) d\eta \leq 0, \\ &\int_{\mathbb{R}^{2N+1}} \left(a_2 v \Delta_{\mathbb{H}} \varphi + |\eta|_{\mathbb{H}}^{\gamma_2} |u|^{p_2} \varphi\right) d\eta \leq 0 \end{aligned} \tag{2.2}$$

for any nonnegative test function $\varphi \in C_c^2(\mathbb{R}^{2N+1})$.

Theorem 2.2. *Assume that*

$$Q \leq Q_e^* = 2 + \frac{1}{p_1 p_2 - 1} \max\{(\gamma_1 + 2) + p_1(\gamma_2 + 2); p_2(\gamma_1 + 2) + (\gamma_2 + 2)\}. \tag{2.3}$$

Then there is no nontrivial weak solution (u,v) of the system (ES_2).

Proof. Let $\varphi_R \in \mathfrak{D}(\mathbb{H}^N)$ be a nonnegative function such that

$$\varphi_R(\eta) = \Phi^\lambda\left(\frac{\tau^2 + |x|^4 + |y|^4}{R^4}\right), \tag{2.4}$$

where $\lambda \gg 1$, $R > 0$, and $\Phi \in \mathfrak{D}([0,+\infty[)$ is the "standard cutoff function"

$$\Phi(r) = \begin{cases} 1, & \text{if } 0 \leq r \leq 1, \\ 0, & \text{if } r \geq 2. \end{cases} \quad 0 \leq \Phi(r) \leq 1, \tag{2.5}$$

Note that $\operatorname{supp}(\varphi_R)$ is a subset of

$$\Omega_R = \{\eta \equiv (x,y,\tau) \in \mathbb{H}^N;\ 0 \leq \tau^2 + |x|^4 + |y|^4 \leq 2R^4\} \tag{2.6}$$

and $\operatorname{supp}(\Delta_{\mathbb{H}}\varphi_R)$ is included in

$$\mathscr{C}_R = \{\eta \equiv (x,y,\tau) \in \mathbb{H}^N;\ R^4 \leq \tau^2 + |x|^4 + |y|^4 \leq 2R^4\}. \tag{2.7}$$

Let

$$\rho = \frac{\tau^2 + |x|^4 + |y|^4}{R^4}, \tag{2.8}$$

then

$$\begin{aligned} \Delta_{\mathbb{H}}\varphi_R(\eta) = {} & \frac{4(N+4)\Phi'(\rho)}{R^4}\lambda\Phi^{\lambda-1}(\rho)(|x|^2 + |y|^2) \\ & + \frac{16\Phi''(\rho)}{R^8}\lambda\Phi^{\lambda-1}(\rho) \\ & \times\left(\left(|x|^6 + |y|^6\right) + \tau^2\left(|x|^2 + |y|^2\right) + 2\tau\langle x,y\rangle\left(|x|^2 - |y|^2\right)\right) \\ & + \frac{16\Phi'^2(\rho)}{R^8}\lambda(\lambda-1)\Phi^{\lambda-2}(\rho) \\ & \times\left(\left(|x|^6 + |y|^6\right) + \frac{\tau^2}{4}\left(|x|^2 + |y|^2\right) + 2\tau\langle x,y\rangle\left(|x|^2 - |y|^2\right)\right). \end{aligned} \tag{2.9}$$

It follows that there is a positive constant $C > 0$, independent of R, such that

$$|\Delta_{\mathbb{H}}\varphi_R(\eta)| \leq \frac{C}{R^2} \quad \forall \eta \in \Omega_R. \tag{2.10}$$

Let (u,v) be a nontrivial weak solution of (ES_2). Using (2.2) with $\varphi = \varphi_R$, one has

$$\begin{aligned}\int |\eta|_{\mathbb{H}}^{\gamma_1}|v|^{p_1}\varphi_R\,d\eta &\le -\int a_1 u\Delta_{\mathbb{H}}\varphi_R\,d\eta\\ &\le \|a_1\|_{L^\infty}\int |u|\,|\Delta_{\mathbb{H}}\varphi_R|\,d\eta \\ &\le \|a_1\|_{L^\infty}\left(\int |\eta|_{\mathbb{H}}^{\gamma_2}|u|^{p_2}\varphi_R\right)^{1/p_2}\left(\int |\Delta_{\mathbb{H}}\varphi_R|^{p_2'}\left(\varphi_R|\eta|_{\mathbb{H}}^{\gamma_2}\right)^{1-p_2'}\right)^{1/p_2'},\end{aligned}\tag{2.11}$$

$$\begin{aligned}\int |\eta|_{\mathbb{H}}^{\gamma_2}|u|^{p_2}\varphi_R\,d\eta &\le -\int a_2 v\Delta_{\mathbb{H}}\varphi_R\,d\eta\\ &\le \|a_2\|_{L^\infty}\left(\int |\eta|_{\mathbb{H}}^{\gamma_1}|v|^{p_1}\varphi_R\right)^{1/p_1}\left(\int |\Delta_{\mathbb{H}}\varphi_R|^{p_1'}\left(\varphi_R|\eta|_{\mathbb{H}}^{\gamma_1}\right)^{1-p_1'}\right)^{1/p_1'},\end{aligned}\tag{2.12}$$

thanks to the Hölder inequality. Setting

$$I(R) = \int |\eta|_{\mathbb{H}}^{\gamma_2}|u|^{p_2}\varphi_R\,d\eta, \qquad J(R) = \int |\eta|_{\mathbb{H}}^{\gamma_1}|v|^{p_1}\varphi_R\,d\eta, \tag{2.13}$$

we have

$$J(R) \le C_1 I(R)^{1/p_2}\mathcal{A}_{p_2,\gamma_2}(R)^{1/p_2'}, \tag{2.14}$$

where

$$\mathcal{A}_{p_2,\gamma_2}(R) = \int |\Delta_{\mathbb{H}}\varphi_R|^{p_2'}\left(\varphi_R|\eta|_{\mathbb{H}}^{\gamma_2}\right)^{1-p_2'}d\eta \tag{2.15}$$

and C_1 is a positive constant independent of R. Similarly, we have

$$I(R) \le C_2 J(R)^{1/p_1}\mathcal{A}_{p_1,\gamma_1}(R)^{1/p_1'}, \tag{2.16}$$

where

$$\mathcal{A}_{p_1,\gamma_1}(R) = \int |\Delta_{\mathbb{H}}\varphi_R|^{p_1'}\left(\varphi_R|\eta|_{\mathbb{H}}^{\gamma_1}\right)^{1-p_1'}d\eta \tag{2.17}$$

and C_2 is a positive constant independent of R.

Note that for λ sufficiently large, the integrals $\mathcal{A}_{p_i,\gamma_i}(R)$, $i \in \{1,2\}$, are convergent. Indeed, in the expression $\mathcal{A}_{p_i,\gamma_i}(R)$, $i \in \{1,2\}$, we have $|\eta|_{\mathbb{H}} \ge R^4$, and the exponent of φ_R is positive for λ large enough.

In order to estimate the integrals $\mathcal{A}_{p_i,\gamma_i}(R)$, $i \in \{1,2\}$, we introduce the scaled variables

$$\tilde{\tau} = R^{-2}\tau, \qquad \tilde{x} = R^{-1}x, \qquad \tilde{y} = R^{-1}y. \tag{2.18}$$

Using the fact that $\operatorname{supp}\varphi_R \subset \Omega_R$, we conclude that

$$\mathcal{A}_{p_i,\gamma_i}(R) \le CR^{2N+2-2p_i'+\gamma_i(1-p_i')}, \quad i \in \{1,2\}. \tag{2.19}$$

Using (2.16) and (2.19) in (2.14), we obtain

$$J(R)^{1-1/p_1p_2} \le C\mathcal{A}_{p_1,\gamma_1}(R)^{1/p_1'p_2}\mathcal{A}_{p_2,\gamma_2}(R)^{1/p_2'} \le CR^{\sigma_J}, \tag{2.20}$$

where

$$\begin{aligned}\sigma_J &= \frac{1}{p_2'}\left(2N+2-2p_2+\gamma_2\left(1-p_2'\right)\right)+\frac{1}{p_1'p_2}\left(2N+2-2p_1+\gamma_1\left(1-p_1'\right)\right)\\ &= Q\left(1-\frac{1}{p_1p_2}\right)-\frac{\left(2p_2+2+\gamma_2\right)p_1+\gamma_1}{p_1p_2}.\end{aligned} \tag{2.21}$$

Similarly, we have

$$I(R)^{1-1/p_1p_2} \le C\mathcal{A}_{p_1,\gamma_1}(R)^{1/p_1'}\mathcal{A}_{p_2,\gamma_2}(R)^{1/p_1p_2'} \le CR^{\sigma_I}, \tag{2.22}$$

where

$$\sigma_I = Q\left(1-\frac{1}{p_1p_2}\right)-\frac{\left(2p_1+2+\gamma_1\right)p_2+\gamma_2}{p_1p_2}. \tag{2.23}$$

Now, we require that $\sigma_I \le 0$ or $\sigma_J \le 0$, which is equivalent to

$$\begin{aligned}Q \le Q_e^* &= \frac{1}{p_1p_2-1}\max\left\{p_1\left(2\left(p_2+1\right)+\gamma_2\right)+\gamma_1;p_2\left(2\left(p_1+1\right)+\gamma_1\right)+\gamma_2\right\}\\ &= 2+\frac{1}{p_1p_2-1}\max\left\{\left(\gamma_1+2\right)+p_1\left(\gamma_2+2\right);p_2\left(\gamma_1+2\right)+\left(\gamma_2+2\right)\right\}.\end{aligned} \tag{2.24}$$

In this case, the integrals $I(R)$ and $J(R)$, increasing in R, are bounded uniformly with respect to R. Using the monotone convergence theorem, we deduce that $|\eta|_{\mathbb{H}}^{\gamma_1}|v|^{p_1}$ and $|\eta|_{\mathbb{H}}^{\gamma_2}|u|^{p_2}$ are in $L^1(\mathbb{R}^{2N+1})$. Note that instead of (2.11) we have, more precisely,

$$\begin{aligned}\int |\eta|_{\mathbb{H}}^{\gamma_1}|v|^{p_1}\varphi_R\,d\eta &\le \|a_1\|_{L^\infty}\left(\int_{\mathcal{C}_R}|\eta|_{\mathbb{H}}^{\gamma_2}|u|^{p_2}\varphi_R\,d\eta\right)^{1/p_2}\mathcal{A}_{p_2,\gamma_2}(R)^{1/p_2'}\\ &\le C\int_{\mathcal{C}_R}|\eta|_{\mathbb{H}}^{\gamma_2}|u|^{p_2}\varphi_R\,d\eta.\end{aligned} \tag{2.25}$$

Finally, using the dominated convergence theorem, we obtain that

$$\lim_{R\to+\infty}\int_{\mathcal{C}_R}|\eta|_{\mathbb{H}}^{\gamma_2}|u|^{p_2}\varphi_R d\eta = 0. \tag{2.26}$$

Hence,

$$\int |\eta|_{\mathbb{H}}^{\gamma_1}|v|^{p_1}d\eta = 0, \tag{2.27}$$

which implies that $v \equiv 0$ and $u \equiv 0$ via (2.12). This contradicts the fact that (u,v) is a nontrivial weak solution of (ES_2), which achieves the proof. □

Remark 2.3. The critical exponent Q_e^* can be written as

$$Q_e^* = 2+\max\left\{X_1,X_2\right\}, \tag{2.28}$$

where the vector $(X_1,X_2)^T$ is the solution of the linear system

$$\begin{pmatrix} -1 & p_1 \\ p_2 & -1 \end{pmatrix}\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} \gamma_1+2 \\ \gamma_2+2 \end{pmatrix}. \tag{2.29}$$

Comment 2.4. In their paper, Pohozaev and Véron [7] showed that if

$$1 < p_j \le \frac{Q+\gamma_j}{Q-2}, \quad j \in \{1,2\}, \tag{2.30}$$

then the system (ES_2) has no nontrivial weak solution. The condition (2.30) is equivalent to

$$Q \le 2 + \min\left\{\frac{\gamma_1+2}{p_1-1}; \frac{\gamma_2+2}{p_2-1}\right\}. \tag{2.31}$$

Theorem 2.2 gives a better estimate of the exponent. Indeed,

$$\frac{(\gamma_1+2)+p_1(\gamma_2+2)}{p_1p_2-1} - \frac{\gamma_2+2}{p_2-1} = -\frac{p_2(\gamma_1+2)+(\gamma_2+2)}{p_1p_2-1} + \frac{\gamma_1+2}{p_1-1}, \tag{2.32}$$

which implies that

$$\max\left\{\frac{(\gamma_1+2)+p_1(\gamma_2+2)}{p_1p_2-1}; \frac{p_2(\gamma_1+2)+(\gamma_2+2)}{p_1p_2-1}\right\} \ge \min\left\{\frac{\gamma_1+2}{p_1-1}; \frac{\gamma_2+2}{p_2-1}\right\}. \tag{2.33}$$

3. Systems of m semilinear inequalities

In this section, we give generalizations of the last results to systems with m inequalities, $m \in \mathbb{N}^*$.

Let $(X_1,X_2,\dots,X_m)$ be the solution of the linear system

$$\begin{pmatrix} 1 & -p_1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & -p_2 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & -p_{m-1} \\ -p_m & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_{m-1} \\ X_m \end{pmatrix} = \begin{pmatrix} -\gamma_1-2 \\ -\gamma_2-2 \\ \vdots \\ -\gamma_{m-1}-2 \\ -\gamma_m-2 \end{pmatrix}, \tag{3.1}$$

where $p_i > 1$ and γ_i are given real numbers, $i \in \{1,2,\dots,m\}$.

Consider the system (ES_m):

$$-\Delta_{\mathbb{H}}(a_iu_i) \ge |\eta|_{\mathbb{H}}^{\gamma_{i+1}} |u_{i+1}|^{p_{i+1}}, \quad \eta \in \mathbb{H}^N,\ 1 \le i \le m,\ u_{m+1} = u_1, \tag{3.2}$$

where $p_{m+1} = p_1$, $\gamma_{m+1} = \gamma_1$.

Definition 3.1. Let a_i, $i \in \{1,2,\dots,m\}$, be m bounded measurable functions on $\mathbb{R}^{2N+1}$. A weak solution $(u_1,\dots,u_m)$ of the system (ES_m) on $\mathbb{R}^{2N+1}$ is a vector of locally integrable functions $(u_1,\dots,u_m)$ such that

$$u_i \in L^{p_i}_{\mathrm{loc}}\left(\mathbb{R}^{2N+1}, |\eta|_{\mathbb{H}}^{\gamma_i} d\eta\right), \quad i \in \{1,2,\dots,m\}, \tag{3.3}$$

satisfying

$$\begin{gathered}\int_{\mathbb{R}^{2N+1}} \left(a_i u \Delta_{\mathbb{H}}\varphi + |\eta|_{\mathbb{H}}^{\gamma_{i+1}} \left| u_{i+1}\right|^{p_{i+1}} \varphi\right) d\eta \le 0, \quad i \in \{1,2,\dots,m-1\}, \\ \int_{\mathbb{R}^{2N+1}} \left(a_m u_m \Delta_{\mathbb{H}}\varphi + |\eta|_{\mathbb{H}}^{\gamma_1} |u|^{p_1} \varphi\right) d\eta \le 0\end{gathered} \tag{3.4}$$

for any nonnegative test function $\varphi \in C_c^2(\mathbb{R}^{2N+1})$.

Theorem 3.2. *If $Q \le 2+\max\{X_1,X_2,\dots,X_m\}$, then system ($\mathrm{ES}_m$) has no nontrivial solution.*

Proof. In order to simplify the proof, we treat only the case $m = 3$; the general case can be established in the same manner.

Let (u_1,u_2,u_3) be a nontrivial weak solution of (ES_m). The inequalities (3.4), with $\varphi = \varphi_R$ defined by (2.4), imply that

$$\begin{aligned}
&\int |\eta|_{\mathbb{H}}^{\gamma_1} \left| u_1\right|^{p_1} \varphi_R \, d\eta \\
&\quad \le \|a_3\|_{L^\infty} \left(\int |\eta|_{\mathbb{H}}^{\gamma_3} \left| u_3\right|^{p_3} \varphi_R \right)^{1/p_3} \left(\int \left|\Delta_{\mathbb{H}}\varphi_R\right|^{p_3'} \left(\varphi_R |\eta|_{\mathbb{H}}^{\gamma_3}\right)^{1-p_3'} \right)^{1/p_3'}, \\
&\int |\eta|_{\mathbb{H}}^{\gamma_2} \left| u_2\right|^{p_2} \varphi_R \, d\eta \\
&\quad \le \|a_1\|_{L^\infty} \left(\int |\eta|_{\mathbb{H}}^{\gamma_1} \left| u_1\right|^{p_1} \varphi_R \right)^{1/p_1} \left(\int \left|\Delta_{\mathbb{H}}\varphi_R\right|^{p_1'} \left(\varphi_R |\eta|_{\mathbb{H}}^{\gamma_1}\right)^{1-p_1'} \right)^{1/p_1'}, \\
&\int |\eta|_{\mathbb{H}}^{\gamma_3} \left| u_3\right|^{p_3} \varphi_R \, d\eta \\
&\quad \le \|a_2\|_{L^\infty} \left(\int |\eta|_{\mathbb{H}}^{\gamma_2} \left| u_2\right|^{p_2} \varphi_R \right)^{1/p_2} \left(\int \left|\Delta_{\mathbb{H}}\varphi_R\right|^{p_2'} \left(\varphi_R |\eta|_{\mathbb{H}}^{\gamma_2}\right)^{1-p_2'} \right)^{1/p_2'}.
\end{aligned} \tag{3.5}$$

Let

$$\begin{aligned}
I_i(R) &= \int |\eta|_{\mathbb{H}}^{\gamma_i} \left| u_i\right|^{p_i} \varphi_R \, d\eta, \quad 1 \le i \le 3, \\
\mathcal{A}_i(R) &= \int \left|\Delta_{\mathbb{H}}\varphi_R\right|^{p_i'} \left(\varphi_R |\eta|_{\mathbb{H}}^{\gamma_i}\right)^{1-p_i'}, \quad 1 \le i \le 3,
\end{aligned} \tag{3.6}$$

then there is a positive constant C such that

$$I_1 \le C I_3^{1/p_3} \mathcal{A}_3^{1/p_3'}, \qquad I_2 \le C I_1^{1/p_1} \mathcal{A}_1^{1/p_1'}, \qquad I_3 \le C I_2^{1/p_2} \mathcal{A}_2^{1/p_2'}. \tag{3.7}$$

Hence, the estimates

$$I_1^{1-1/p_1p_2p_3} \le C\mathcal{A}_1^{1/p_1'p_2p_3}\mathcal{A}_2^{1/p_2'p_3}\mathcal{A}_3^{1/p_3'},$$
$$I_2^{1-1/p_1p_2p_3} \le C\mathcal{A}_1^{1/p_1'}\mathcal{A}_2^{1/p_1p_2'p_3}\mathcal{A}_3^{1/p_1p_3'}, \tag{3.8}$$
$$I_3^{1-1/p_1p_2p_3} \le C\mathcal{A}_1^{1/p_1'p_2}\mathcal{A}_2^{1/p_2'}\mathcal{A}_3^{1/p_1p_2p_3'}$$

hold true.

In order to estimate the expressions I_i, $1 \le i \le 3$, we use the scaled variables (2.18) and obtain

$$I_i^{1-1/p_1p_2p_3} \le CR^{\sigma_i}, \quad 1 \le i \le 3, \tag{3.9}$$

where

$$\begin{aligned}
\sigma_1 &= \left(1 - \frac{1}{p_1p_2p_3}\right)\left(Q - 2 - \frac{(\gamma_1+2) + p_1(\gamma_2+2) + p_1p_2(\gamma_3+2)}{p_1p_2p_3 - 1}\right),\\
\sigma_2 &= \left(1 - \frac{1}{p_1p_2p_3}\right)\left(Q - 2 - \frac{p_2p_3(\gamma_1+2) + (\gamma_2+2) + p_2(\gamma_3+2)}{p_1p_2p_3 - 1}\right),\\
\sigma_3 &= \left(1 - \frac{1}{p_1p_2p_3}\right)\left(Q - 2 - \frac{p_3(\gamma_1+2) + p_1p_3(\gamma_2+2) + (\gamma_3+2)}{p_1p_2p_3 - 1}\right).
\end{aligned} \tag{3.10}$$

Now, we require that, at least, one of σ_i, $1 \le i \le 3$, is less than zero, which is equivalent to $Q \le 2 + \max\{X_1, X_2, X_3\}$, where the vector $(X_1, X_2, X_3)^T$ is the solution of

$$\begin{pmatrix} 1 & -p_1 & 0 \\ 0 & 1 & -p_2 \\ -p_3 & 0 & 1 \end{pmatrix}\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \begin{pmatrix} -\gamma_1 - 2 \\ -\gamma_2 - 2 \\ -\gamma_3 - 2 \end{pmatrix}. \tag{3.11}$$

Following the arguments used in the proof of Theorem 2.2, we conclude that $(u_1, u_2, u_3) \equiv (0,0,0)$. This ends the proof by contradiction. □

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