INVARIANT SETS FOR NONLINEAR EVOLUTION EQUATIONS, CAUCHY PROBLEMS AND PERIODIC PROBLEMS

NORIMICHI HIRANO AND NAOKI SHIOJI

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In the case of $K \neq \overline{D(A)}$, we study Cauchy problems and periodic problems for nonlinear evolution equation $u(t) \in K$, $u'(t) + Au(t) \ni f(t,u(t))$, $0 \leq t \leq T$, where A is a maximal monotone operator on a Hilbert space H, K is a closed, convex subset of H, V is a subspace of H, and $f : [0,T] \times (K \cap V) \to H$ is of Carathéodory type.

1. Introduction

Let E be a Banach space and let $A \subset E \times E$ be an m-accretive operator. Let K be a closed subset of E, let $T > 0$, and let $f : [0,T] \times K \to E$. In the case of $K = \overline{D(A)}$, many researchers have studied initial value problems or periodic problems for nonlinear evolution equation

$$u(t) \in K, \quad u'(t) + Au(t) \ni f\big(t,u(t)\big) \quad \text{for } 0 \leq t \leq T; \tag{1.1}$$

see [3, 5, 13, 14, 16, 17, 18, 21, 22, 23, 24, 25, 26, 27, 28]. Recently, in the case when $K \neq \overline{D(A)}$ and f is of Carathéodory type, Bothe [7] showed the existence of solutions of the initial value problem with $u(0) = x \in K \cap \overline{D(A)}$ for (1.1) under *a tangential condition*:

$$\varliminf_{s \to +0} \frac{1}{s} \inf_{z \in K \cap \overline{D(A)}} \|S_{f(t,x)}(s)x - z\| = 0 \quad \text{for every } (t,x) \in [0,T) \times \big(K \cap \overline{D(A)}\big), \tag{1.2}$$

where $S_{f(t,x)}(\cdot)x$ is the solution of $w(0) = x$ and $w'(s) + Aw(s) \ni f(t,x)$ for $s \geq 0$. Bothe [8] also showed the existence of T-periodic solutions for (1.1) under *a subtangential condition*: *K-invariance of resolvent operators* and *Nagumo-type condition*

$$\varliminf_{s \to +0} \frac{1}{s} \inf_{z \in K} \|x + sf(t,x) - z\| = 0 \quad \text{for a.e. } t \in (0,T) \text{ and for every } x \in K. \tag{1.3}$$

A typical case for $K \neq \overline{D(A)}$ is given by $K = \{v \in L^2(\Omega) : v \geq 0\}$; see [1, 15]. For a periodic problem, Bothe's result can be applied to a nonlinear parabolic boundary value problem

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of the form

$$\begin{gathered}
\frac{\partial u}{\partial t}(t,x)+Au(t,x)=g(t,x,u(t,x)) \quad \text{in } \mathbb{R}\times\Omega, \\
Bu(t,x)=0 \quad \text{on } \mathbb{R}\times\partial\Omega, \\
u(t,x)=u(t+T,x) \quad \text{in } \mathbb{R}\times\Omega, \\
0\le u(t,x)\le c \quad \text{in } \mathbb{R}\times\Omega,
\end{gathered} \tag{1.4}$$

where $\Omega\subset\mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$, $g:\mathbb{R}\times\Omega\times\mathbb{R}\to\mathbb{R}$ is a continuous mapping, A is a nonlinear elliptic operator, B is a boundary operator, and c is a real number.

For semilinear cases, Amann [2] considered initial value problems and periodic problems for (1.1) in the case when $K\neq\overline{D(A)}$ and f is *not* necessarily of Carathéodory type with respect to the topology of E. The results in [2] can be applied to derive the existence of T-periodic solutions of the problem

$$\begin{gathered}
\frac{\partial u}{\partial t}(t,x)+Lu(t,x)=g(t,x,u(t,x),\nabla u(t,x)) \quad \text{in } \mathbb{R}\times\Omega, \\
Bu(t,x)=0 \quad \text{on } \mathbb{R}\times\partial\Omega, \\
u(t,x)=u(t+T,x) \quad \text{in } \mathbb{R}\times\Omega, \\
u(t,x)\ge 0 \quad \text{in } \mathbb{R}\times\Omega,
\end{gathered} \tag{1.5}$$

where L is a second-order linear elliptic operator, B is a first-order boundary operator, and $g:\mathbb{R}\times\Omega\times\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}$ is a continuous function. We can see that in problem (1.5), function g cannot be of Carathéodory type in $L^2(\Omega)$. To deal with this kind of problems, it was assumed in [2] that $f(t,\cdot)$ is defined on a subspace V, which is endowed with a stronger topology than that of E and $f(t,\cdot):V\to E$ is continuous with respect to this topology. Under these conditions, the existence of solutions of the problems were established in [2] imposing *a subtangential condition*: *K-invariance of evolution operators* and *Nagumo type condition*.

Our purpose in this paper is to establish existence results which can cover problems of the form (1.5) with L replaced by *nonlinear elliptic operators*. That is, in the case of $K\neq\overline{D(A)}$, we give existence results for solutions of initial value problems and periodic problems for (1.1) under a tangential or subtangential condition in the case when H is a Hilbert space, V is a subspace of H, and $f:[0,T]\times V\to H$ is a mapping, which is *not* necessarily of Carathéodory type with respect to the topology of H.

The organization of this paper is the following. Section 2 is devoted to some preliminaries and notations. We state our main results in Section 3 and we prove them in Section 4. Finally, we study an example to which our results are applicable.

2. Preliminaries and notations

Throughout this paper, we denote by $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{R}_+$ the set of positive integers, the set of real numbers, and the set of nonnegative real numbers, respectively. For a subset X of a normed linear space, we denote by ∂X the boundary of X.

Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space. We denote by $|\cdot|$ the norm defined by $|x|^2 = \langle x,x\rangle$ for $x \in H$. We also denote by $B_H(x,r)$ the closed ball in H with center $x \in H$ and radius $r > 0$. Let K be a closed, convex subset of H and let P be *the metric projection* from H onto K, that is, for each $x \in H$, Px is the unique point in K with $|x - Px| = d_H(x,K)$, where $d_H(x,K) = \min_{y\in K}|x - y|$. We know that $\langle y - Px, x - Px\rangle \le 0$ for all $x \in H$ and $y \in K$. We define *a tangential cone* $T_K(x)$ for K at $x \in K$ by

$$T_K(x) = \left\{ y \in H : \lim_{s\to+0} \frac{1}{s} d_H(x+sy,K) = 0 \right\}. \tag{2.1}$$

Let A be a maximal monotone subset of $H \times H$. For each $\lambda > 0$, we define *a resolvent* and *a Yosida approximation* by $J_\lambda = (I + \lambda A)^{-1}$ and $A_\lambda = (I - J_\lambda)/\lambda$, respectively. We denote by $\{S(t) : t \ge 0\}$ *the semigroup generated by the negative of* A; see [4, 10, 19]. We say the semigroup $\{S(t)\}$ is *compact* if for each $t > 0$, $S(t) : \overline{D(A)} \to \overline{D(A)}$ is compact. Let $a, b \in \mathbb{R}$ with $a < b$, let $g \in L^1(a,b;H)$, and let $x \in \overline{D(A)}$. We say a function $u : [a,b] \to H$ is *an integral solution* of the initial value problem

$$u(a) = x, \quad u'(t) + Au(t) \ni g(t) \quad \text{for } a \le t \le b, \tag{2.2}$$

if u is continuous on $[a,b]$, $u(a) = x$, $u(t) \in \overline{D(A)}$ for every $a \le t \le b$, and

$$|u(t) - y|^2 \le |u(s) - y|^2 + 2\int_s^t \langle g(\tau) - z, u(\tau) - y\rangle \, d\tau \tag{2.3}$$

for every $(y,z) \in A$ and s, t with $a \le s \le t \le b$. It is known that the initial value problem (2.2) has a unique integral solution; see [4, 6]. We remark that for each $x \in \overline{D(A)}$, $S(\cdot)x$ is the integral solution of

$$u(0) = x, \quad u'(t) + Au(t) \ni 0 \quad \text{for } t \ge 0. \tag{2.4}$$

For each $x \in \overline{D(A)}$ and $z \in H$, we denote by $S_z(\cdot)x$ the integral solution of

$$u(0) = x, \quad u'(t) + Au(t) \ni z \quad \text{for } t \ge 0, \tag{2.5}$$

and we define T_K^A by

$$T_K^A(x) = \left\{ z \in H : \underline{\lim}_{s\to+0} \frac{1}{s} d_H(S_z(s)x,K) = 0 \right\} \quad \text{for each } x \in K \cap \overline{D(A)}. \tag{2.6}$$

We remark that in the case of $K \subset \overline{D(A)}$, T_K^A coincides with the one in [7].

Let $(V, \|\cdot\|)$ be a reflexive Banach space which is continuously imbedded into H. We identify V with a subspace of H. Let $\omega, \varepsilon \ge 0$, let $p > 1$, and let A be a maximal monotone subset of $H \times H$ such that $D(A) \subset V$ and $\langle y_1 - y_2, x_1 - x_2\rangle \ge \omega\|x_1 - x_2\|^p + \varepsilon|x_1 - x_2|^2$ for every $(x_1,y_1),(x_2,y_2) \in A$. In this case, if u, v are the integral solutions of (2.2) corresponding to $(x,g),(y,h) \in \overline{D(A)} \times L^1(a,b;H)$, respectively, then

$$|u(t) - v(t)| \le e^{-\varepsilon(t-s)}|u(s) - v(s)| + \int_s^t e^{-\varepsilon(t-\tau)}|g(\tau) - h(\tau)| \, d\tau \tag{2.7}$$

for $a \le s \le t \le b$ and

$$\begin{aligned} &|u(t)-v(t)|^2 - |u(s)-v(s)|^2 + 2\omega\int_s^t \|u(\tau)-v(\tau)\|^p d\tau + 2\varepsilon\int_s^t |u(\tau)-v(\tau)|^2 d\tau \\ &\quad \le 2\int_s^t \langle u(\tau)-v(\tau), g(\tau)-h(\tau)\rangle d\tau \end{aligned} \tag{2.8}$$

for $a \le s \le t \le b$.

To prove our results, we need the following propositions and theorems. The first one is a property of the Dini derivative. For a proof, see [9, Proposition 9.1].

Proposition 2.1. *Let g be a continuous function from $[a,b]$ into $\mathbb{R}$ with $a,b \in \mathbb{R}$ and $a < b$ such that*

$$\overline{\lim_{s\to+0}} \frac{g(t+s)-g(t)}{s} \le 0 \quad \text{for every } t \in (a,b). \tag{2.9}$$

Then g is decreasing on $[a,b]$.

The next one is a fixed-point theorem, which can be derived from the Leray-Schauder degree theory [11, 20].

Theorem 2.2. *Let X be a bounded, closed, convex subset of a normed linear space E with nonempty interior. Let H be a continuous mapping from $[0,1]\times X$ into a compact subset of E such that*

(i) *$H(1,X) \subset X$;*
(ii) *for every $\varepsilon \in [0,1]$, $H(\varepsilon,\cdot)$ has no fixed point on ∂X.*

Then $H(0,\cdot)$ has a fixed point in X.

The next proposition shows a sufficient condition that the negative of a maximal monotone operator generates a compact semigroup; see [18, Lemma 2].

Proposition 2.3. *Let $(V,\|\cdot\|)$ be a reflexive Banach space which is compactly imbedded into a Hilbert space $(H,\langle\cdot,\cdot\rangle)$ and let A be a maximal monotone subset of $H\times H$ which satisfies $D(A) \subset V$ and*

$$\langle y_1 - y_2, x_1 - x_2\rangle \ge \omega\|x_1 - x_2\|^p \quad \text{for every } (x_1,y_1),(x_2,y_2) \in A \tag{2.10}$$

with a constant $p > 1$. Then the negative of A generates a compact semigroup.

The following two theorems are concerning properties of integral solutions; see [4, Lemma III.2.1, Theorem III.2.2, Corollary III.2.1].

Theorem 2.4. *Let H be a Hilbert space and let A be a maximal monotone subset of $H\times H$. Let $u_0 \in \overline{D(A)}$, let $T > 0$, and let $g \in L^1(0,T;H)$. Let $\lambda > 0$ and let u_λ be the solution of the initial value problem*

$$u_\lambda(0) = u_0, \quad u_\lambda'(t) + A_\lambda u_\lambda(t) = g(t) \quad \text{for almost every } 0 < t < T. \tag{2.11}$$

Then $\{u_\lambda\}$ *converges to some* $u \in C(0,T;H)$ *as* $\lambda \to +0$ *with respect to the topology of* $C(0,T;H)$*, and the limit function* u *is the integral solution of the initial value problem*

$$u(0) = u_0, \quad u'(t) + Au(t) \ni g(t) \quad \text{for } 0 \le t \le T. \tag{2.12}$$

Theorem 2.5. *Let* H *and* A *be as those in Theorem 2.4. Let* $u_0 \in D(A)$*, let* $T > 0$*, and let* $g \in W^{1,1}(0,T;H)$*. Then the solution* $u(t)$ *of the initial value problem (2.12) is everywhere differentiable from the right on* $[0,T)$*,*

$$\frac{d^+}{dt}u(t) + \left(Au(t) - g(t)\right)^0 = 0 \quad \text{for every } 0 \le t < T, \tag{2.13}$$

and

$$\left|\frac{d^+}{dt}u(t)\right| \le \left|\left(Au(0) - g(0)\right)^0\right| + \int_0^t \left|\frac{dg}{ds}(s)\right| ds \quad \text{for every } 0 \le t < T, \tag{2.14}$$

where $(Au(t) - g(t))^0$ *is the unique element* z *of* $Au(t) - g(t)$ *satisfying* $|z| = \min\{|w| : w \in Au(t) - g(t)\}$.

The following compactness result is crucial in our argument; see [27, Theorem 2].

Theorem 2.6 (Vrabie). *Let* H *be a Hilbert space and let* A *be a maximal monotone subset of* $H \times H$*, whose negative generates a compact semigroup. Let* B *be a bounded subset of* $\overline{D(A)}$*, let* $T > 0$*, and let* G *be a uniformly integrable subset of* $L^1(0,T;H)$*. Let* $\mathcal{S}$ *be the set of all integral solutions of*

$$u(0) = x, \quad u'(t) + Au(t) \ni g(t), \quad 0 \le t \le T \tag{2.15}$$

for $x \in B$ *and* $g \in G$*. Then* $\{u(T) : u \in \mathcal{S}\}$ *is relatively compact in* H*. Furthermore, if* B *is relatively compact in* H*, then* $\mathcal{S}$ *is relatively compact in* $C(0,T;H)$*.*

3. Main results

We begin this section with hypotheses and notations which we will use in our results. The following are the hypotheses for our general framework:

(H1) $(V, \|\cdot\|)$ is a reflexive Banach space which is compactly imbedded into a Hilbert space $(H, \langle\cdot,\cdot\rangle)$ with the norm $|\cdot|$;

(H2) $A \subset H \times H$ is a maximal monotone subset such that $D(A) \subset V$ and

$$\langle y_1 - y_2, x_1 - x_2\rangle \ge \omega\|x_1 - x_2\|^p \tag{3.1}$$

for every $(x_1, y_1), (x_2, y_2) \in A$, where $1 < p < \infty$ and $\omega > 0$ are constants;

(H3) K is a closed, convex subset of H such that $K \cap V \ne \emptyset$ and $K \cap \overline{D(A)} \ne \emptyset$, where $\overline{D(A)}$ is the closure of $D(A)$ with respect to the topology of H, and the metric projection P from H onto K with respect to the metric in H satisfies

(i) $P(V) \subset V$ and $P : (V, \|\cdot\|) \to (V, \|\cdot\|)$ is continuous;

(ii) $\|Px\| \le c_1\|x\| + c_2$ for every $x \in V$, where c_1, c_2 are nonnegative constants;

(H4) $T > 0$ and f is a mapping from $[0,T] \times (K \cap V)$ into H such that
 (i) $f(\cdot,x)$ is strongly measurable for every $x \in K \cap V$;
 (ii) $f(t,\cdot)$ is continuous from $K \cap V$ with respect to the topology of V into H for almost every $t \in (0,T)$;
 (iii) there exist $\alpha \in [0,p)$, $a_1 \in L^{p/(p-\alpha)}(0,T;\mathbb{R}_+)$, and $a_2 \in L^1(0,T;\mathbb{R}_+)$ such that

$$|f(t,x)| \leq a_1(t)\|x\|^{\alpha} + a_2(t) \tag{3.2}$$

for almost every $t \in (0,T)$ and for every $x \in K \cap V$.

Each one of the following hypotheses guarantees the boundedness of solutions of (1.1). We remark that if $|f(t,x)|$ is bounded, (B2) is satisfied:

(B1) K is bounded in H;
(B2) $\alpha \in [0,p-1)$, $a_1 \in L^{pq/(p-q\alpha)}(0,T;\mathbb{R}_+)$, and $a_2 \in L^q(0,T;\mathbb{R}_+)$, where q is the constant with $1/p + 1/q = 1$;
(B3) there exist $\beta \in [0,p)$, $b_1 \in L^{p/(p-\beta)}(0,T;\mathbb{R}_+)$, $b_2 \in L^1(0,T;\mathbb{R}_+)$, and $\gamma \in [0,p/\alpha)$ such that

$$\langle Ax - f(t,x),x\rangle \geq \omega\|x\|^p - b_1(t)\|x\|^{\beta} - b_2(t)(|x|^{\gamma}+1) \tag{3.3}$$

for every $x \in D(A)$ and for almost every $t \in [0,T]$;
(B4) $0 \in D(A) \cap K$ and there exist $\beta \in [0,p)$, $b_1 \in L^{p/(p-\beta)}(0,T;\mathbb{R}_+)$, $b_2 \in L^1(0,T;\mathbb{R}_+)$, and $\gamma \in [0,(p/\alpha)\min\{1,p-1\})$ such that

$$\langle Ax - f(t,Px),x\rangle \geq \omega\|x\|^p - b_1(t)\|x\|^{\beta} - b_2(t)(|x|^{\gamma}+1) \tag{3.4}$$

for every $x \in D(A)$ and for almost every $t \in [0,T]$.

Each one of the following hypotheses is a tangential or subtangential condition which guarantees K-invariance of solutions for (1.1). In applications to elliptic-parabolic problems, K-invariance of the semigroup in (T2) corresponds to *the comparison principle for parabolic equations*, and K-invariance of the resolvents in (T3) corresponds to *the comparison principle for elliptic equations*; see examples in [2] and this paper:

(T1) $K \subset \overline{D(A)}$ and $f(t,x) \in T_K^A(x)$ for almost every $t \in [0,T]$ and for every $x \in K \cap V$;
(T2) $K \subset \overline{D(A)}$, $S(t)K \subset K$ for every $t \geq 0$, and $f(t,x) \in T_K(x)$ for almost every $t \in [0,T]$ and for every $x \in K \cap V$;
(T3) $J_\lambda K \subset K$ for every $\lambda > 0$, and $f(t,x) \in T_K(x)$ for almost every $t \in [0,T]$ and for every $x \in K \cap V$.

Now, we state our viability theorem.

Theorem 3.1. *Assume (H1), (H2), (H3), and (H4) and one of the conditions of (B1), (B2), and (B3). Assume also one of the conditions of (T1), (T2), and (T3). Then, for each* $x \in K \cap \overline{D(A)}$, *there exists an integral solution* u *of*

$$u(0) = x, \quad u'(t) + Au(t) \ni f(t,u(t)) \quad \text{for } 0 \leq t \leq T \tag{3.5}$$

which satisfies

$$u(t) \in K \quad \text{for every } t \in [0,T]. \tag{3.6}$$

Next, we state the existence of periodic solutions.

Theorem 3.2. *Assume (H1), (H2), (H3), and (H4) and one of the conditions of (B1), (B2), and (B4). Assume also one of the conditions of (T1), (T2), and (T3). Then there exists a T-periodic, integral solution u of*

$$u'(t) + Au(t) \ni f(t,u(t)) \quad \text{for } 0 \le t \le T \tag{3.7}$$

which satisfies (3.6).

In the case of $K = H$, we have the following corollaries as direct consequences of Theorems 3.1 and 3.2 with assumption (T3); see also [26].

Corollary 3.3 (Vrabie). *Assume (H1), (H2), $K = H$, (H4), and (B3). Then for each $x \in \overline{D(A)}$ and $h \in L^1(0,T;H)$, there exists an integral solution of*

$$u(0) = x, \quad u'(t) + Au(t) \ni f(t,u(t)) + h(t) \quad \text{for } 0 \le t \le T. \tag{3.8}$$

Corollary 3.4. *Assume (H1), (H2), $K = H$, (H4), and (B4) as P is identity. Assume also $(p/\alpha)\min\{1, p-1\} > 1$, in particular $p \ge 2$. Then for each $h \in L^1(0,T;H)$, there exists a T-periodic, integral solution of*

$$u'(t) + Au(t) \ni f(t,u(t)) + h(t) \quad \text{for } 0 \le t \le T. \tag{3.9}$$

In the case when f is t-independent, we can solve an elliptic problem as follows.

Corollary 3.5. *Assume that the hypotheses of Theorem 3.2 hold. Assume in addition that f is t-independent, a_1, a_2 are nonnegative constants, and b_1, b_2 are nonnegative constants in the case of (B4). Then there exists $x \in K \cap D(A)$ with $Ax \ni f(x)$.*

4. Proof of theorems

Throughout this section, we assume (H1), (H2), (H3), and (H4) and $|\cdot| \le \|\cdot\|$ without loss of generality. We consider that space $C(0,T;H) \cap L^p(0,T;V)$ is endowed with a norm $\|\cdot\|_{C(0,T;H)} + \|\cdot\|_{L^p(0,T;V)}$.

First, we give the proof of Theorem 3.2. The reason is that we want to give the proof of Theorem 3.2 precisely since its proof is more complicated than that of Theorem 3.1.

For each $\delta > 0$, we set $K_\delta = \{x \in H : d_H(x,K) \le \delta\}$. Since $K \cap \overline{D(A)} \ne \emptyset$, for each $\delta > 0$, there exists $(x_\delta, y_\delta) \in A$ with $B_H(x_\delta, \delta/2) \subset K_\delta$. In the case of (B4), we can set $(x_\delta, y_\delta) = (0, y_*)$ for all $\delta > 0$, where y_* is an element of $A0$. Within all lemmas below, we fix $\delta > 0$ and $(x_\delta, y_\delta) \in A$.

The following Lemmas 4.1, 4.2, 4.4, and 4.5 are obtained by similar arguments as those in [18].

Lemma 4.1. *For each $\varepsilon > 0$ and $g \in L^1(0,T;H)$, there exists a unique T-periodic, integral solution of*

$$u'(t) + \varepsilon\big(u(t) - x_\delta\big) + Au(t) \ni g(t) \quad \text{for } 0 \le t \le T. \tag{4.1}$$

Proof. Let $\varepsilon > 0$ and let $g \in L^1(0,T;H)$. We define a mapping $U : \overline{D(A)} \to \overline{D(A)}$ by $Ux = u(T)$ for $x \in \overline{D(A)}$, where u is the unique integral solution of the initial value problem (4.1) with $u(0) = x$. From (2.7), we have $|Ux - Uy| \le e^{-\varepsilon T}|x - y|$ for every $x, y \in \overline{D(A)}$. By the Banach contraction principle, U has the unique fixed point z. Then the integral solution u of (4.1) with $u(0) = z$ satisfies $u(0) = u(T)$. □

We define $Q_\delta : (0,\infty) \times L^1(0,T;H) \to C(0,T;H) \cap L^p(0,T;V)$ by $Q_\delta(\varepsilon,g) = u$ for each $(\varepsilon,g) \in (0,\infty) \times L^1(0,T;H)$, where $u \in C(0,T;H) \cap L^p(0,T;V)$ is the unique T-periodic, integral solution of (4.1).

Lemma 4.2. *There exist $k_1, k_2 \ge 0$ such that*

$$\sup_{\varepsilon>0} \|Q_\delta(\varepsilon,g)\|_{C(0,T;H)} \le k_1 \|g\|_{L^1(0,T;H)}^{\max\{1,1/(p-1)\}} + k_2 \quad \text{for every } g \in L^1(0,T;H). \tag{4.2}$$

In particular, for each bounded subset B of $L^1(0,T;H)$, $Q_\delta((0,\infty) \times B)$ is bounded in $C(0,T;H)$.

Proof. Let $(\varepsilon,g) \in (0,\infty) \times L^1(0,T;H)$ and let $u = Q_\delta(\varepsilon,g)$. We set $m = \min\{|u(t) - x_\delta| : 0 \le t \le T\}$ and $M = \max\{|u(t) - x_\delta| : 0 \le t \le T\}$. We also set $C = T|y_\delta| + \int_0^T |g(t)|\,dt$. From (2.7), we know $M \le m + C$. If $M \le 2C$, then the conclusion holds with arbitrary $k_1 \ge 2$ and $k_2 \ge 2T|y_\delta| + |x_\delta| + 1$. So we assume $M > 2C$. From (2.8), we have

$$\begin{aligned} \omega T\left(\frac{M}{2}\right)^p &\le \omega T m^p \le \omega \int_0^T |u(t) - x_\delta|^p\,dt \\ &\le \omega \int_0^T \|u(t) - x_\delta\|^p\,dt \le \int_0^T \langle g(t) - y_\delta, u(t) - x_\delta\rangle\,dt \le MC, \end{aligned} \tag{4.3}$$

and we obtain $\omega T M^{p-1} \le 2^p C$. Hence, it is easy to see that the conclusion holds. □

Remark 4.3. Using any $(z,w) \in A$ instead of (x_δ, y_δ), by the same proof, we can show that there exist $k_1, k_2 \ge 0$ such that

$$\|u\|_{C(0,T;H)} \le k_1 \|g\|_{L^1(0,T;H)}^{\max\{1,1/(p-1)\}} + k_2 \tag{4.4}$$

for every $g \in L^1(0,T;H)$ and T-periodic, integral solution u of

$$u'(t) + Au(t) \ni g(t) \quad \text{for } 0 \le t \le T. \tag{4.5}$$

Lemma 4.4. *$Q_\delta : (0,\infty) \times L^1(0,T;H) \to C(0,T;H) \cap L^p(0,T;V)$ is continuous.*

Proof. Fix $(\varepsilon_0, g_0) \in (0,\infty) \times L^1(0,T;H)$ and set $u_0 = Q_\delta(\varepsilon_0, g_0)$. By the previous lemma, there exists $C > 0$ such that $\|Q_\delta(\varepsilon,g)\|_{C(0,T;H)} \le C$ for every $(\varepsilon,g) \in (0,\infty) \times L^1(0,T;H)$

with $\int_0^T |g(\tau)-g_0(\tau)|d\tau \le 1$. Let $(\varepsilon,g)\in(0,\infty)\times L^1(0,T;H)$ with $\int_0^T |g(\tau)-g_0(\tau)|d\tau \le 1$ and set $u=Q_\delta(\varepsilon,g)$. By (2.7) and the periodicity of u and u_0, we have

$$|u(0)-u_0(0)| \le e^{-\varepsilon_0 T}|u(0)-u_0(0)| + \int_0^T |g(\tau)-g_0(\tau)-(\varepsilon-\varepsilon_0)(u(\tau)-x_\delta)|d\tau, \tag{4.6}$$

and hence we obtain

$$|u(t)-u_0(t)| \le \left(1+\frac{1}{1-e^{-\varepsilon_0 T}}\right)\left(\int_0^T |g(\tau)-g_0(\tau)|d\tau + |\varepsilon-\varepsilon_0|(C+|x_\delta|)\right) \tag{4.7}$$

for all $t\in[0,T]$. From (2.8), we also have

$$\omega\int_0^T \|u(\tau)-u_0(\tau)\|^p d\tau \le 2C\int_0^T |g(\tau)-g_0(\tau)|d\tau + 2C(C+|x_\delta|)|\varepsilon-\varepsilon_0|. \tag{4.8}$$

From these two inequalities, we know that Q_δ is continuous at (ε_0,g_0). □

Lemma 4.5. *For each $\varepsilon_0>0$ and uniformly integrable subset B of $L^1(0,T;H)$, $Q_\delta((0,\varepsilon_0]\times B)$ is relatively compact in $C(0,T;H)\cap L^p(0,T;V)$.*

Proof. Let $\varepsilon_0>0$ and let B be a uniformly integrable subset of $L^1(0,T;H)$. We know from Lemma 4.2 that $\{g-\varepsilon(Q_\delta(\varepsilon,g)-x_\delta):\varepsilon\in(0,\varepsilon_0],\ g\in B\}$ is uniformly integrable and $\{Q_\delta(\varepsilon,g)(0):\varepsilon\in(0,\varepsilon_0],\ g\in B\}$ is bounded in H. By Theorem 2.6, $\{Q_\delta(\varepsilon,g)(0):\varepsilon\in(0,\varepsilon_0],\ g\in B\}=\{Q_\delta(\varepsilon,g)(T):\varepsilon\in(0,\varepsilon_0],\ g\in B\}$ is relatively compact in H. Using Theorem 2.6 again, we know that $Q_\delta((0,\varepsilon_0]\times B)$ is relatively compact in $C(0,T;H)$. Next, we will show that $Q_\delta((0,\varepsilon_0]\times B)$ is relatively compact in $L^p(0,T;V)$ by the method employed in the proof of [26, Theorem 3.1]. Fix $\eta>0$. Then there exists $\{(\varepsilon_1,g_1),\dots,(\varepsilon_n,g_n)\}\subset(0,\varepsilon_0]\times B$ such that for each $(\varepsilon,g)\in(0,\varepsilon_0]\times B$, there exists i such that $\|Q_\delta(\varepsilon,g)-Q_\delta(\varepsilon_i,g_i)\|_{C(0,T;H)}\le\eta$. Let $C=\sup\{\|Q_\delta(\varepsilon,g)\|_{C(0,T;H)}:\varepsilon\in(0,\varepsilon_0],\ g\in B\}$. Since $\omega\int_0^T\|Q_\delta(\varepsilon,g)(t)-Q_\delta(\varepsilon_i,g_i)(t)\|^p dt\le\eta(\int_0^T|g(t)-g_i(t)|dt+2(C+|x_\delta|)\varepsilon_0)$ and B is bounded in $L^1(0,T;H)$, $Q_\delta((0,\varepsilon_0]\times B)$ is totally bounded in $L^p(0,T;V)$. Hence $Q_\delta((0,\varepsilon_0]\times B)$ is relatively compact in $C(0,T;H)\cap L^p(0,T;V)$. □

Remark 4.6. By a similar proof, we can show that for each uniformly integrable subset B of $L^1(0,T;H)$, the set of all T-periodic, integral solutions of (4.5) for $g\in B$ is relatively compact in $C(0,T;H)\cap L^p(0,T;V)$.

We show an a priori estimate for fixed points of the mapping $u\mapsto Q_\delta(\varepsilon,f(\cdot,Pu))$ with respect to norm $\|\cdot\|_{L^p(0,T;V)}$. We remark that after we obtain this estimate, Lemma 4.2 yields an a priori estimate with respect to norm $\|\cdot\|_{C(0,T;H)}$. We also remark that assumption $0\in D(A)\cap K$ in (B4) is used to show the following.

Lemma 4.7. *Assume one of the conditions of (B1), (B2), and (B4). Then there exists $R_1>0$ such that for every $\varepsilon>0$ and $u\in C(0,T;H)\cap L^p(0,T;V)$ satisfying $u=Q_\delta(\varepsilon,f(\cdot,Pu))$ and $u(t)\in K_\delta$ for all $t\in[0,T]$, there holds $\|u\|_{L^p(0,T;V)}<R_1$.*

Proof. Let $\varepsilon > 0$ and let $u \in C(0,T;H) \cap L^p(0,T;V)$ satisfying $u = Q_\delta(\varepsilon, f(\cdot, Pu))$ and $u(t) \in K_\delta$ for all $t \in [0,T]$. We know that u is a T-periodic, integral solution of $u'(t) + Au(t) + \varepsilon(u(t) - x_\delta) \ni f(t, Pu(t))$ for $0 \le t \le T$. First, we consider the case of (B1). Since we have

$$\begin{aligned}
\omega \int_0^T ||u(\tau) - x_\delta||^p d\tau &\le \int_0^T \langle f(\tau, Pu(\tau)) - y_\delta, u(\tau) - x_\delta \rangle d\tau \\
&\le \sup_{0 \le t \le T} |u(t) - x_\delta| \left(\int_0^T |f(\tau, Pu(\tau))| d\tau + T|y_\delta| \right) \\
&\le \sup_{z \in K_\delta} |z - x_\delta| \left[\left(\int_0^T |a_1(\tau)|^{p/(p-\alpha)} d\tau \right)^{(p-\alpha)/p} (c_1 R_1 + c_2 T^{1/p})^\alpha \right. \\
&\qquad \left. + \int_0^T |a_2(\tau)| d\tau + T|y_\delta| \right],
\end{aligned} \tag{4.9}$$

where c_1, c_2 are the constants in (H3), we can choose sufficiently large $R_1 > 0$ such that $\|u\|_{L^p(0,T;V)} < R_1$ for every $\varepsilon > 0$ and $u \in C(0,T;H) \cap L^p(0,T;V)$ satisfying $u = Q_\delta(\varepsilon, f(\cdot, Pu))$ and $u(t) \in K_\delta$ for all $t \in [0,T]$. Next, we consider the case of (B2). Since we have

$$\begin{aligned}
&\omega \int_0^T ||u(\tau) - x_\delta||^p d\tau \\
&\le \int_0^T \langle f(\tau, Pu(\tau)) - y_\delta, u(\tau) - x_\delta \rangle d\tau \\
&\le \left(\int_0^T ||u(\tau) - x_\delta||^p d\tau \right)^{1/p} \left[\left(\int_0^T |a_1(\tau)||Pu(\tau)||^\alpha + a_2(\tau)|^q d\tau \right)^{1/q} + T^{1/q}|y_\delta| \right] \\
&\le (\|u\|_{L^p(0,T;V)} + T^{1/p}||x_\delta||) \left[\left(\int_0^T |a_1(\tau)|^{pq/(p-q\alpha)} d\tau \right)^{(p-q\alpha)/pq} (c_1 \|u\|_{L^p(0,T;V)} + c_2 T^{1/p})^\alpha \right. \\
&\qquad \left. + \left(\int_0^T |a_2(\tau)|^q d\tau \right)^{1/q} + T^{1/q}|y_\delta| \right],
\end{aligned} \tag{4.10}$$

where q is the constant in (B2), we obtain the conclusion in this case. Finally, we consider the case of (B4). From (B4) and $x_\delta = 0$, we have

$$\begin{aligned}
0 &\ge \omega \int_0^T ||u(t)||^p dt - \int_0^T b_1(t)||u(t)||^\beta dt - \int_0^T b_2(t)(|Q_\delta(\varepsilon, f(t, Pu(t)))|^\gamma + 1) dt \\
&\ge \omega \|u\|^p_{L^p(0,T;V)} - \left(\int_0^T |b_1(t)|^{p/(p-\beta)} dt \right)^{(p-\beta)/p} \|u\|^\beta_{L^p(0,T;V)} \\
&\quad - \left(\int_0^T b_2(t) dt \right) \left(\left[k_1 \left\{ \left(\int_0^T |a_1(t)|^{p(p-\alpha)} dt \right)^{(p-\alpha)/p} (c_1 \|u\|_{L^p(0,T;V)} + c_2 T^{1/p})^\alpha \right. \right. \right. \\
&\qquad \left. \left. \left. + \int_0^T |a_2(t)| dt \right\}^{\max\{1, 1/(p-1)\}} + k_2 \right]^\gamma + 1 \right),
\end{aligned} \tag{4.11}$$

where k_1, k_2 are the constants in Lemma 4.2. Hence, we also obtain the conclusion in this case. □

Remark 4.8. Using any $(z,w) \in A$ instead of (x_δ, y_δ), by the same proof, we can show that if one of the conditions of (B1), (B2), and (B4) holds, then there exists $R_1 > 0$ such that for every T-periodic, integral solution u of $u'(t) + Au(t) \ni f(t, Pu(t))$ for $0 \le t \le T$ satisfying $u(t) \in K_1$ for all $t \in [0,T]$, there holds $\|u\|_{L^p(0,T;V)} < R_1$.

We fix R_1 as in the previous lemma, and we define a subset X_δ of $C(0,T;H) \cap L^p(0,T;V)$ by

$$X_\delta = \{u \in C(0,T;H) \cap L^p(0,T;V) : \|u\|_{L^p(0,T;V)} \le R_1, u(t) \in K_\delta \text{ for every } t \in [0,T]\} \tag{4.12}$$

in the case of (B1), and by

$$\begin{aligned} X_\delta = \{u \in C(0,T;H) \cap L^p(0,T;V) :& \\ \|u\|_{L^p(0,T;V)} \le R_1,\ u(t) \in K_\delta \cap B_H(x_\delta, R_2) &\text{ for every } t \in [0,T]\} \end{aligned} \tag{4.13}$$

in each case of (B2) or (B4), where R_2 is a positive constant satisfying

$$R_2 > \sup\{\|Q_\delta(\varepsilon, f(\cdot, Pv)) - x_\delta\|_{C(0,T;H)} : \varepsilon > 0,\ \|v\|_{L^p(0,T;V)} \le R_1\}. \tag{4.14}$$

We remark that we can choose such R_2 by Lemma 4.2.

Next, we will show $\{Q_\delta(\varepsilon, f(\cdot, Pv)) : v \in X_\delta\} \subset X_\delta$ for sufficiently large $\varepsilon > 0$. The following is needed to show this property.

Lemma 4.9. *Let $a \in L^1(0,T;\mathbb{R}_+)$. Then $\sup_{0\le t\le T} \int_0^t e^{-\varepsilon(t-\tau)} a(\tau)d\tau \to 0$ as $\varepsilon \to \infty$.*

Proof. Let $\eta > 0$. There exists $\rho > 0$ such that $\int_E a(\tau)d\tau \le \eta$ for each Lebesgue measurable subset E of $[0,T]$, whose Lebesgue measure is less than or equal to ρ. Choose a positive number ε with $e^{-\varepsilon\rho}\int_0^T a(\tau)d\tau \le \eta$. Let $t \in [0,T]$. If $t \le \rho$, we have $\int_0^t e^{-\varepsilon(t-\tau)} a(\tau)d\tau \le \eta$. If $t > \rho$, we have

$$\int_0^t e^{-\varepsilon(t-\tau)} a(\tau)d\tau \le e^{-\varepsilon\rho}\int_0^{t-\rho} e^{-\varepsilon(t-\rho-\tau)} a(\tau)d\tau + \int_{t-\rho}^t a(\tau)d\tau \le 2\eta. \tag{4.15}$$

Hence, we obtain the conclusion. □

Lemma 4.10. *There exists $\varepsilon_\delta \ge 1$ such that $\{Q_\delta(\varepsilon_\delta, f(\cdot, Pv)) : v \in X_\delta\} \subset X_\delta$.*

Proof. Fix $\eta \in (0, \delta/2)$ such that $\|u\|_{L^p(0,T;V)} \le R_1$ for every $u \in L^p(0,T;V)$ with

$$\begin{aligned} \omega &\int_0^T \|u(\tau) - x_\delta\|^p d\tau \\ &\le \eta\left[\left(\int_0^T |a_1(\tau)|^{p/(p-\alpha)} d\tau\right)^{(p-\alpha)/p} (c_1R_1 + c_2T^{1/p})^\alpha + \int_0^T |a_2(\tau)|d\tau + T|y_\delta|\right] \end{aligned} \tag{4.16}$$

and that $\eta \leq R_2$ in the case when X_δ is defined by (4.13). Let $\varepsilon > 0$ with $1/(1-e^{-\varepsilon T}) \leq 2$. Let v be any element of X_δ and set $u = Q_\delta(\varepsilon, f(\cdot, Pv))$. From (2.7), we have

$$|u(0) - x_\delta| \leq \frac{1}{1-e^{-\varepsilon T}} \int_0^T e^{-\varepsilon(T-\tau)} |f(\tau, Pv(\tau)) - y_\delta| \, d\tau, \tag{4.17}$$

and hence

$$\begin{aligned}
\|u - x_\delta\|_{C(0,T;H)} &\leq \sup_{0\leq t\leq T} \left(e^{-\varepsilon t} |u(0) - x_\delta| + \int_0^t e^{-\varepsilon(t-\tau)} |f(\tau, Pv(\tau)) - y_\delta| \, d\tau \right) \\
&\leq 3 \sup_{0\leq t\leq T} \left(\int_0^t e^{-\varepsilon(t-\tau)} \left(a_1(\tau)\|Pv(\tau)\|^\alpha + a_2(\tau) + |y_\delta|\right) d\tau \right) \\
&\leq 3 \sup_{0\leq t\leq T} \left[\left(\int_0^t e^{(p\varepsilon/(p-\alpha))(t-\tau)} |a_1(\tau)|^{p/(p-\alpha)} d\tau \right)^{(p-\alpha)/p} \left(c_1 R_1 + c_2 T^{1/p}\right)^\alpha \right. \\
&\qquad\qquad \left. + \int_0^t e^{-\varepsilon(t-\tau)} \left(a_2(\tau) + |y_\delta|\right) d\tau \right].
\end{aligned} \tag{4.18}$$

By the previous lemma, there exists $\varepsilon_\delta \geq 1$ such that $\sup_{v\in X_\delta} \|Q_\delta(\varepsilon_\delta, f(\cdot, Pv)) - x_\delta\|_{C(0,T;H)} \leq \eta$. Then we also have

$$\begin{aligned}
\omega &\int_0^T \|Q_\delta(\varepsilon_\delta, f(\tau, Pv(\tau))) - x_\delta\|^p d\tau \\
&\leq \int_0^T \langle f(\tau, Pv(\tau)) - y_\delta, Q_\delta(\varepsilon_\delta, f(\tau, Pv(\tau))) - x_\delta \rangle d\tau \\
&\leq \eta \left[\left(\int_0^T |a_1(\tau)|^{p/(p-\alpha)} d\tau \right)^{(p-\alpha)/p} \left(c_1 R_1 + c_2 T^{1/p}\right)^\alpha + \int_0^T |a_2(\tau)| \, d\tau + T|y_\delta| \right],
\end{aligned} \tag{4.19}$$

which implies $\sup_{v\in X_\delta} \|Q_\delta(\varepsilon_\delta, f(\cdot, Pv))\|_{L^p(0,T;V)} \leq R_1$. Hence, we obtain the conclusion. □

Next, we will show that the mapping $u \mapsto Q_\delta(\varepsilon, f(\cdot, Pu))$ has no fixed point on ∂X_δ for every $\varepsilon > 0$. The following play an important role to show this property.

Lemma 4.11. *The following hold:*

(i) $\langle A_\lambda x, x - Px \rangle \geq 0$ *for each* $\lambda > 0$ *and* $x \in H$*;*
(ii) $\langle z, x - Px \rangle \leq 0$ *for each* $x \in H$ *and* $z \in T_K(Px)$*;*
(iii) *if* $K \subset \overline{D(A)}$ *and* $S(t)K \subset K$ *for every* $t \geq 0$*, then* $\langle y, x - Px \rangle \geq 0$ *for each* $(x, y) \in A$*;*
(iv) *if* $K \subset \overline{D(A)}$*, then* $\langle y - z, x - Px \rangle \geq 0$ *for each* $(x, y) \in A$ *and* $z \in T_K^A(Px)$*.*

Proof. (i) Let $\lambda > 0$ and let $x \in H$. We know $|J_\lambda x - PJ_\lambda x| \leq |J_\lambda x - J_\lambda Px| \leq |x - Px|$. Hence, we have

$$\begin{aligned}
\langle x - J_\lambda x, x - Px \rangle &= |x - Px|^2 + \langle Px - PJ_\lambda x, x - Px \rangle + \langle PJ_\lambda x - J_\lambda x, x - Px \rangle \\
&\geq |x - Px|^2 - |PJ_\lambda x - J_\lambda x| \, |x - Px| \geq 0.
\end{aligned} \tag{4.20}$$

(ii) Let $x \in H$ and let $z \in T_K(Px)$. From

$$0 \le \left| x - P(x+sz) \right|^2 = \left| x + sz - P(x+sz) \right|^2 - 2s\langle z, x+sz-P(x+sz) \rangle + s^2 |z|^2 \quad (4.21)$$

for every $s > 0$, we have $\langle z, x - Px \rangle \le 0$.

(iii) Assume $K \subset \overline{D(A)}$ and $S(t)K \subset K$ for every $t \ge 0$. Let $(x,y) \in A$. Then we have

$$\int_0^t \langle y, x - S(\tau)Px \rangle d\tau \ge \frac{1}{2} \left| x - S(t)Px \right|^2 - \frac{1}{2} |x - Px|^2 \ge 0 \quad \text{for every } t > 0. \quad (4.22)$$

Hence, we have

$$\langle y, x - Px \rangle = \lim_{t \to +0} \frac{1}{t} \int_0^t \langle y, x - S(\tau)Px \rangle d\tau \ge 0. \quad (4.23)$$

(iv) Assume $K \subset \overline{D(A)}$. Let $(x,y) \in A$ and let $z \in T_K^A(Px)$. We set $w(\cdot) = S_z(\cdot)Px$. Then we have for every $t > 0$,

$$\begin{aligned} \int_0^t \langle y - z, x - w(\tau) \rangle d\tau &\ge \frac{1}{2} \left| x - w(t) \right|^2 - \frac{1}{2} |x - Px|^2 \\ &\ge \frac{1}{2} \left| Pw(t) - w(t) \right|^2 + \langle Pw(t) - w(t), x - Px \rangle. \end{aligned} \quad (4.24)$$

Hence, by $z \in T_K^A(Px)$, we obtain

$$\langle y - z, x - Px \rangle = \lim_{t \to +0} \frac{1}{t} \int_0^t \langle y - z, x - w(\tau) \rangle d\tau \ge 0. \quad (4.25)$$

□

The reason why we define an approximate equation by (4.1) can be found in the proof of the next lemma.

Lemma 4.12. *Assume one of the conditions of (B1), (B2), and (B4), and assume also one of the conditions of (T1), (T2), and (T3). Then for each $\varepsilon > 0$ and $u \in X_\delta$ with $u = Q_\delta(\varepsilon, f(\cdot, Pu))$, u is an interior point of X_δ.*

Proof. Let $\varepsilon > 0$ and $u \in X_\delta$ with $u = Q_\delta(\varepsilon, f(\cdot, Pu))$. We know that u is a T-periodic, integral solution of

$$u'(t) + \varepsilon\big(u(t) - x_\delta\big) + Au(t) \ni f\big(t, Pu(t)\big), \quad 0 \le t \le T. \quad (4.26)$$

Since one of the conditions of (B1), (B2), and (B4) is assumed, by Lemma 4.7 and the definition of X_δ, we have $\|u\|_{L^p(0,T;V)} < R_1$ and $\|u - x_\delta\|_{C(0,T;H)} < R_2$ in the case of (B2) and (B4), respectively. Thus it is enough to show $u(t) \notin \partial K_\delta$ for all $t \in [0,T]$. First, we consider the case of (T3). Let $g \in C(0,T;H)$. Let $\lambda > 0$ and let v be the $C^1(0,T;H)$-solution of the initial value problem

$$v(0) = u(0), \quad v'(t) + \varepsilon\big(v(t) - x_\delta\big) + A_\lambda v(t) = g(t), \quad 0 \le t \le T. \quad (4.27)$$

Let $t \in [0,T)$ and let $s > 0$ with $t+s \le T$. Since

$$\begin{aligned}|v(t+s)-Pv(t+s)|^2 &\le |v(t+s)-Pv(t)|^2\\ &= |v(t+s)-v(t)|^2 + 2\langle v(t+s)-v(t), v(t)-Pv(t)\rangle\\ &\quad + |v(t)-Pv(t)|^2,\end{aligned} \tag{4.28}$$

we have

$$\begin{aligned}&\frac{|v(t+s)-Pv(t+s)|^2 - |v(t)-Pv(t)|^2}{s}\\ &\quad \le s\left|\frac{v(t+s)-v(t)}{s}\right|^2 + 2\left\langle \frac{v(t+s)-v(t)}{s}, v(t)-Pv(t)\right\rangle.\end{aligned} \tag{4.29}$$

By (i) of Lemma 4.11, we get

$$\begin{aligned}&\overline{\lim_{s\to+0}} \frac{|v(t+s)-Pv(t+s)|^2 - |v(t)-Pv(t)|^2}{s}\\ &\quad \le 2\langle v'(t), v(t)-Pv(t)\rangle\\ &\quad = 2\big(\langle g(t), v(t)-Pv(t)\rangle - \langle A_\lambda v(t), v(t)-Pv(t)\rangle - \varepsilon\langle v(t)-x_\delta, v(t)-Pv(t)\rangle\big)\\ &\quad \le 2\big(\langle g(t), v(t)-Pv(t)\rangle - \varepsilon\langle v(t)-x_\delta, v(t)-Pv(t)\rangle\big)\end{aligned} \tag{4.30}$$

for every $t \in [0,T)$. By Proposition 2.1, we have

$$\begin{aligned}&|v(T)-Pv(T)|^2 - |v(0)-Pv(0)|^2\\ &\quad \le 2\left(\int_0^T \langle g(\tau), v(\tau)-Pv(\tau)\rangle d\tau - \varepsilon\int_0^T \langle v(\tau)-x_\delta, v(\tau)-Pv(\tau)\rangle d\tau\right),\end{aligned} \tag{4.31}$$

which implies, by Theorem 2.4, Lemma 4.11(ii), and $u(0) = u(T)$,

$$\int_0^T \langle u(\tau)-x_\delta, u(\tau)-Pu(\tau)\rangle d\tau \le 0. \tag{4.32}$$

Let $\tilde{P}$ be the metric projection from H onto $K_{\delta/2}$. Since $x_\delta \in K_{\delta/2}$ and $\tilde{P}u(\tau)$ is in the line segment between $u(\tau)$ and $Pu(\tau)$ for all $\tau \in [0,T]$, we have $\langle x_\delta - \tilde{P}u(\tau), u(\tau)-Pu(\tau)\rangle \le 0$ for all $\tau \in [0,T]$. Hence, we obtain

$$0 \ge \int_0^T \langle u(\tau)-\tilde{P}u(\tau), u(\tau)-Pu(\tau)\rangle d\tau \ge \int_0^T |u(\tau)-\tilde{P}u(\tau)|^2 d\tau, \tag{4.33}$$

which implies $u(t) \notin \partial K_\delta$ for all $t \in [0,T]$. Next, we consider the cases of (T1) and (T2). Let $x \in D(A)$, let $g \in C^1(0,T;H)$, and let v be the integral solution of

$$v(0) = x, \quad v'(t) + \varepsilon(v(t)-x_\delta) + Av(t) \ni g(t), \quad 0 \le t \le T. \tag{4.34}$$

By Theorem 2.5, v is everywhere differentiable from the right on $[0,T)$ and there exists $y \in L^\infty(0,T;H)$ such that

$$y(t) \in Av(t), \quad \frac{d^+}{dt}v(t) + \varepsilon\big(v(t) - x_\delta\big) + y(t) - g(t) = 0 \quad \text{for every } 0 \le t < T. \tag{4.35}$$

Then we get

$$\begin{aligned}
&\overline{\lim_{s\to+0}} \frac{\left|v(t+s) - Pv(t+s)\right|^2 - \left|v(t) - Pv(t)\right|^2}{s} \\
&\quad \le 2\left\langle \frac{d^+}{dt}v(t), v(t) - Pv(t)\right\rangle \\
&\quad = 2\big(\langle g(t) - y(t), v(t) - Pv(t)\rangle - \varepsilon\langle v(t) - x_\delta, v(t) - Pv(t)\rangle\big)
\end{aligned} \tag{4.36}$$

for every $t \in [0,T)$. By Proposition 2.1 and Lemma 4.11(ii), (iii), and (iv), we have

$$\begin{aligned}
&\left|v(T) - Pv(T)\right|^2 - \left|v(0) - Pv(0)\right|^2 \\
&\quad \le 2\left(\int_0^T \langle g(\tau) - y(\tau), v(\tau) - Pv(\tau)\rangle d\tau - \varepsilon\int_0^T \langle v(\tau) - x_\delta, v(\tau) - Pv(\tau)\rangle d\tau\right) \\
&\quad \le 2\left(\int_0^T \langle g(\tau) - f(\tau, Pv(\tau)), v(\tau) - Pv(\tau)\rangle d\tau - \varepsilon\int_0^T \langle v(\tau) - x_\delta, v(\tau) - Pv(\tau)\rangle d\tau\right),
\end{aligned} \tag{4.37}$$

which implies (4.32). By the same argument as above, we have $u(t) \notin \partial K_\delta$ for all $t \in [0,T]$. $\square$

Lemma 4.13. *Assume one of the conditions of (B1), (B2), and (B4), and assume also one of the conditions of (T1), (T2), and (T3). Then there exists a T-periodic, integral solution u of*

$$u'(t) + Au(t) \ni f(t, Pu(t)) \quad \text{for } 0 \le t \le T \tag{4.38}$$

which satisfies $u(t) \in K_\delta$ for all $t \in [0,T]$.

Proof. Let ε_δ be a constant obtained in Lemma 4.10. We know that the mapping $(\varepsilon, u) \mapsto Q_\delta(\varepsilon, f(\cdot, Pu))$ from $(0, \varepsilon_\delta] \times X_\delta$ into $C(0,T;H) \cap L^p(0,T;V)$ is continuous and compact by (H4), Lemmas 4.4, and 4.5. By Lemma 4.10 and Theorem 2.2, for each $n \in \mathbb{N}$, there exists $u_n \in X_\delta$ such that $u_n = Q_\delta(1/n, f(\cdot, Pu_n))$. Then u_n is a T-periodic, integral solution of $u_n'(t) + 1/n(u_n(t) - x_\delta) + Au_n(t) \ni f(t, Pu_n(t))$ for $0 \le t \le T$. By Lemmas 4.2 and 4.7, $\{u_n\}$ is bounded in $C(0,T;H) \cap L^p(0,T;V)$, and by Lemma 4.5, $\{u_n\}$ is relatively compact in $C(0,T;H) \cap L^p(0,T;V)$. So we may assume that $\{u_n\}$ converges strongly to some $u \in C(0,T;H) \cap L^p(0,T;V)$. We know that u is T-periodic and $u(t) \in K_\delta$ for all $t \in [0,T]$. Since

$$\left|u_n(t) - x\right|^2 \le \left|u_n(s) - x\right|^2 + 2\int_s^t \left\langle f(\tau, Pu_n(\tau)) - \frac{1}{n}(u_n(\tau) - x_\delta) - y, u_n(\tau) - x\right\rangle d\tau \tag{4.39}$$

for every $(x,y) \in A$, s,t with $0 \le s \le t \le T$, and $n \in \mathbb{N}$, we obtain

$$|u(t)-x|^2 \le |u(s)-x|^2 + 2\int_s^t \langle f(\tau, Pu(\tau)) - y, u(\tau)-x\rangle d\tau \tag{4.40}$$

for every $(x,y) \in A$ and s, t with $0 \le s \le t \le T$, which implies that u is an integral solution of (4.38). □

Proof of Theorem 3.2. By Lemma 4.13, for each $n \in \mathbb{N}$, there exists a T-periodic, integral solution u_n of (4.38) satisfying $u_n(t) \in K_{1/n}$ for all $t \in [0,T]$. We know that $\{u_n\}$ is bounded in $L^p(0,T;V)$ by Remark 4.8, and $\{u_n\}$ is bounded in $C(0,T;H)$ by Remark 4.3. We also know that $\{u_n\}$ is relatively compact by Remark 4.6. So we may assume that $\{u_n\}$ converges strongly to some $u \in C(0,T;H) \cap L^p(0,T;V)$. It is easy to see that u is T-periodic and $u(t) \in K$ for all $t \in [0,T]$. By similar lines as those in the proof of Lemma 4.13, there holds (4.40) for every $(x,y) \in A$ and s, t with $0 \le s \le t \le T$. Since $Pu(t) = u(t)$ for all $t \in [0,T]$, u is a desired solution. □

Proof of Corollary 3.5. From Theorem 3.2, for each $n \in \mathbb{N}$, there exists a $1/2^n$-periodic, integral solution u_n of

$$u_n(t) \in K, \quad u_n'(t) + Au_n(t) \ni f(u_n(t)) \quad \text{for } 0 \le t \le 1. \tag{4.41}$$

By Remarks 4.3 and 4.8, $\{u_n\}$ is bounded in $C(0,1;H) \cap L^p(0,1;V)$. So $\{u_n\}$ is relatively compact in $C(0,1;H) \cap L^p(0,1;V)$ by Remark 4.6. Hence, there exists a constant function $u(t) \equiv x \in K \cap V$, which is a cluster point of $\{u_n\}$. Since

$$\langle f(x) - w, x-z\rangle = \int_0^1 \langle f(u(t)) - w, u(t)-z\rangle dt \ge |u(1)-z| - |u(0)-z| = 0 \tag{4.42}$$

for every $(z,w) \in A$, we have $(x, f(x)) \in A$. □

Next, we give the proof of Theorem 3.1. We show the following proposition concerning the existence of local solutions for the initial value problem.

Proposition 4.14. *Assume (H1), (H2), (H3), and (H4) and one of the conditions of (T1), (T2), and (T3). Then for each $x \in K \cap \overline{D(A)}$, there exists $T_0 \in (0,T]$ and an integral solution u of*

$$u(0) = x, \quad u'(t) + Au(t) \ni f(t,u(t)) \quad \textit{for } 0 \le t \le T_0, \tag{4.43}$$

which satisfies $u(t) \in K$ for all $t \in [0,T_0]$.

Proof. Fix $x \in K \cap \overline{D(A)}$. For each $T_0 \in (0,T]$, we define a mapping $G_{T_0} : L^1(0,T_0;H) \to C(0,T_0;H) \cap L^p(0,T_0;V)$ by $G_{T_0}g = u$ for $g \in L^1(0,T_0;H)$, where $u \in C(0,T_0;H) \cap L^p(0,T_0;V)$ is the unique integral solution of the initial value problem

$$u(0) = x, \quad u'(t) + Au(t) \ni g(t) \quad \text{for } 0 \le t \le T_0. \tag{4.44}$$

For each $T_0 \in (0,T]$, we also define a subset X_{T_0} of $C(0,T_0;H) \cap L^p(0,T_0;V)$ by

$$X_{T_0} = \Big\{ u \in C(0,T_0;H) \cap L^p(0,T_0;V) : \\ u(0) = x, \sup_{0 \le t \le T_0} |u(t) - S(t)x| \le 1, \int_0^{T_0} \|u(t) - S(t)x\|^p dt \le 1 \Big\}. \tag{4.45}$$

By similar arguments as those in the case of the periodic problem, it is easy to see that the mapping $v \mapsto G_{T_0}(f(\cdot,Pv))$ is compact and continuous from X_{T_0} into $C(0,T_0;H) \cap L^p(0,T_0;V)$ for each $T_0 \in (0,T]$. It is also easy to see that if $T_0 > 0$ is sufficiently small, then $G_{T_0}(f(\cdot,Pv)) \in X_{T_0}$ for all $v \in X_{T_0}$. Fix such $T_0 \in (0,T]$. By Schauder's fixed point theorem, there exists $u \in X_{T_0}$ with $G_{T_0}(f(\cdot,Pu)) = u$, that is, u is an integral solution of the problem $u(0) = x$ and $u'(t) + Au(t) \ni f(t,Pu(t))$ for $0 \le t \le T_0$. By similar lines as those in the proof of Lemma 4.12, we can show that $t \mapsto |u(t) - Pu(t)|^2$ is decreasing on $[0,T_0]$. Hence, $u(t) \in K$ for all $t \in [0,T_0]$ and u is an integral solution of (4.43). □

Remark 4.15. Intuitively, (T3) seems to imply (T1) in the case of $K \subset \overline{D(A)}$, and (T2) seems to imply (T1). But it seems to be difficult to give a proof even after we obtain the proposition above.

Proof of Theorem 3.1. Set $T_* = \sup\{T_0 \in (0,T] :$ there is an integral solution of (4.43)$\}$. By Theorem 2.6, Proposition 4.14, and a diagonal process argument, we can obtain a function $u : [0,T_*) \to K$ such that for each $T_0 \in (0,T_*)$, $u|_{[0,T_0]} \in C(0,T_0;H) \cap L^p(0,T_0;V)$ and $u|_{[0,T_0]}$ is an integral solution of (4.43). We know $u \in L^p(0,T_*;V)$ by similar lines as those in the proof of Lemma 4.7 and $\sup_{0 \le t < T_*} |u(t)| < \infty$ by $|u(t) - S(t)x| \le \int_0^t |f(\tau,u(\tau))| d\tau$ for every $t \in [0,T_*)$. We will show that $\lim_{t \to T_* - 0} u(t)$ exists with respect to the topology of H. Let $\varepsilon > 0$. There exists $t_0 \in [0,T_*)$ with $\int_{t_0}^{T_*} |f(\tau,u(\tau))| d\tau \le \varepsilon$ and there exists $\delta > 0$ such that $|S(s)u(t_0) - u(t_0)| \le \varepsilon$ for all $s \in [0,\delta]$. Then for each $t,s \in [\max\{t_0, T_* - \delta\}, T_*)$, we have $|u(t) - u(s)| \le 4\varepsilon$, since

$$|u(t) - S(T_* - t_0)u(t_0)| \le |u(t) - S(t - t_0)u(t_0)| + |u(t_0) - S(T_* - t)u(t_0)| \le 2\varepsilon. \tag{4.46}$$

So, there exists $z \in K$ such that $|u(t) - z| \to 0$ as $t \to T_* - 0$, and hence we can think $u(T_*) = z$ and $u \in C(0,T_*;H)$. It is easy to see that u is an integral solution of (4.43) on $[0,T_*]$. We know $T_* = T$. Indeed, if $T_* < T$, we can derive a contradiction by similar lines as those in Proposition 4.14. Therefore, u is a desired solution. □

5. An example

We denote by Ω a bounded domain in $\mathbb{R}^N$ whose boundary $\partial\Omega$ is of class $C^{2,\gamma}$ with $0 < \gamma < 1$. We define $\langle u,v \rangle = \int_\Omega u(x)v(x)dx$ for $u,v \in L^2(\Omega)$ and $\|u\| = \big(\int_\Omega |\nabla u(x)|^2 dx\big)^{1/2}$ for $u \in H_0^1(\Omega)$. We also define $u^+(x) = \max\{u(x),0\}$ and $u^-(x) = \max\{-u(x),0\}$ for $u \in L^2(\Omega)$.

Let $a_j \in C^{1,\gamma}(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N)$ for each $j = 1,\dots,N$ such that $a_j(x,0,0) = 0$ for every $x \in \overline{\Omega}$ and $j = 1,\dots,N$, and there exists an increasing function $\mu : \mathbb{R}_+ \to \mathbb{R}_+$ such that $|\partial a_j/\partial x_i(x,u,z)| \le \mu(|u|)|z|^2$, $|\partial a_j/\partial u(x,u,z)| \le \mu(|u|)|z|$, and $|\partial a_j/\partial z_i(x,u,z)| \le \mu(|u|)$ for every $(x,u,z) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N$ and $i, j = 1,\dots,N$. We assume that there exists $\omega > 0$ such that

$$\sum_{j=1}^{N} \big(a_j(x,u,z) - a_j(x,v,w)\big)\big(z_j - w_j\big) \ge \omega \sum_{j=1}^{N} \big|z_j - w_j\big|^2 \tag{5.1}$$

for every $x \in \overline{\Omega}$, $u, v \in \mathbb{R}$, and $z, w \in \mathbb{R}^N$. We put

$$Au = -\sum_{j=1}^{N} \frac{\partial}{\partial x_j} a_j\big(x, u(x), \nabla u(x)\big), \quad u \in D(A) \equiv \{u \in H_0^1(\Omega) : Au \in L^2(\Omega)\}. \tag{5.2}$$

Then A satisfies

$$\langle Au - Av, u - v\rangle \ge \omega \|u - v\|^2 \quad \text{for every } u, v \in D(A). \tag{5.3}$$

We will show that A is maximal monotone in $L^2(\Omega) \times L^2(\Omega)$. Let $\lambda > 0$ and $\tilde{u} \in C^1(\overline{\Omega})$. By the solvability in Hölder space [12, Theorem 15.11], there is $u \in C^{2,\gamma}(\overline{\Omega})$ with $u + \lambda A u = \tilde{u}$. Since $C^1(\overline{\Omega})$ is dense in $L^2(\Omega)$, and $(I + \lambda A)^{-1}$ is nonexpansive on $R(I + \lambda A)$, we obtain $R(I + \lambda A) = L^2(\Omega)$. Thus A is maximal monotone. Next, we will show

$$J_\lambda K \subset K \quad \text{for every } \lambda > 0, \tag{5.4}$$

where $K = \{u \in L^2(\Omega) : u \ge 0\}$. By comparison principle [12, Theorem 10.7], for each $u, v \in C^1(\overline{\Omega})$ with $u \ge v$ and $\lambda > 0$, there holds $J_\lambda u \ge J_\lambda v$. Since $C^1(\overline{\Omega})$ is dense in $L^2(\Omega)$, and the resolvents are continuous on $L^2(\Omega)$, we know that the resolvents are order preserving on $L^2(\Omega)$. By $A0 = 0$ and the order-preserving property of the resolvents, we have shown $J_\lambda K \subset K$ for each $\lambda > 0$. We set $Pu = u^+$ for each $u \in L^2(\Omega)$. Then P is the metric projection from $L^2(\Omega)$ onto K, $P(H_0^1(\Omega)) \subset H_0^1(\Omega)$, $P : H_0^1(\Omega) \to H_0^1(\Omega)$ is continuous, and $\|Pu\| \le \|u\|$ for each $u \in H_0^1(\Omega)$. Let $g : \mathbb{R} \times \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ be a continuous function, which is T-periodic in its first variable. We assume that

$$\big|g(t,x,u,z)\big| \le c\big(|u| + 1\big) \quad \text{for every } (t,x,u,z) \in \mathbb{R} \times \Omega \times \mathbb{R} \times \mathbb{R}^N \tag{5.5}$$

with some constant $c > 0$, and

$$\lim_{|u| \to \infty} \frac{g(t,x,u,z)}{u} < \omega \lambda_1 \quad \text{uniformly in } (t,x,z) \in \mathbb{R} \times \Omega \times \mathbb{R}^N, \tag{5.6}$$

where λ_1 is the first eigenvalue of the operator $-\Delta$ with homogeneous Dirichlet boundary condition. We also assume that

$$g(t,x,0,z) \geq 0 \quad \text{for each } (t,x,z) \in \mathbb{R} \times \Omega \times \mathbb{R}^N. \tag{5.7}$$

We set $f(t,u)(x) = g(t,x,u(x),\nabla u(x))$ for each $(t,u) \in \mathbb{R} \times H_0^1(\Omega)$. By (5.5), (5.6), and (5.7), there exist constants $C, \varepsilon > 0$, which satisfy

$$\begin{aligned}\langle f(t,Pu),u\rangle &= \langle g(t,x,u^+,\nabla u^+),u^+ - u^-\rangle \leq \langle g(t,x,u^+,\nabla u^+),u^+\rangle \\ &\leq (\omega-\varepsilon)\lambda_1|u|^2 + C \leq (\omega-\varepsilon)\|u\|^2 + C\end{aligned} \tag{5.8}$$

for every $u \in H_0^1(\Omega)$. Then we find

$$\langle Au - f(t,Pu),u\rangle \geq \varepsilon\|u\|^2 - C \quad \text{for every } (t,u) \in \mathbb{R} \times H_0^1(\Omega), \tag{5.9}$$

which implies (B4). On the other hand, we have (T3). Indeed, by (5.7) and Lebesgue's convergence theorem, we have

$$\lim_{s\to+0}\frac{1}{s}\left(\int_\Omega \left|(u+sg(t,x,u,\nabla u))^-\right|^2 dx\right)^{1/2} = 0 \quad \text{for each } (t,u) \in \mathbb{R} \times (K \cap H_0^1(\Omega)). \tag{5.10}$$

Hence, Theorem 3.2 says that the problem

$$\begin{gathered}\frac{\partial u}{\partial t} - \sum_{j=1}^N \frac{\partial}{\partial x_j} a_j(x,u,\nabla u) = g(t,x,u,\nabla u) \quad \text{in } \mathbb{R} \times \Omega, \\ u(t,x) \geq 0 \quad \text{in } \mathbb{R} \times \Omega, \\ u(t,x) = 0 \quad \text{on } \mathbb{R} \times \partial\Omega,\end{gathered} \tag{5.11}$$

has a T-periodic solution.

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Norimichi Hirano: Department of Mathematics, Graduate School of Environment and Information Sciences, Yokohama National University, Tokiwadai, Hodogaya-ku, Yokohama 240-8501, Japan
E-mail address: hirano@math.sci.ynu.ac.jp

Naoki Shioji: Department of Mathematics, Graduate School of Environment and Information Sciences, Yokohama National University, Tokiwadai, Hodogaya-ku, Yokohama 240-8501, Japan
E-mail address: shioji@math.sci.ynu.ac.jp