ACCURATE SOLUTION ESTIMATES FOR NONLINEAR NONAUTONOMOUS VECTOR DIFFERENCE EQUATIONS

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The paper deals with the vector discrete dynamical system $x_{k+1} = A_k x_k + f_k(x_k)$. The well-known result by Perron states that this system is asymptotically stable if $A_k \equiv A = \text{const}$ is stable and $f_k(x) \equiv \tilde{f}(x) = o(\|x\|)$. Perron's result gives no information about the size of the region of asymptotic stability and norms of solutions. In this paper, accurate estimates for the norms of solutions are derived. They give us stability conditions for (1.1) and bounds for the region of attraction of the stationary solution. Our approach is based on the "freezing" method for difference equations and on recent estimates for the powers of a constant matrix. We also discuss applications of our main result to partial reaction-diffusion difference equations.

1. Introduction and notation

Let $\mathbb{C}^n$ be the set of n-complex vectors endowed with the Euclidean norm $\|\cdot\|$. Consider in $\mathbb{C}^n$ the equation

$$x_{k+1} = A_k x_k + f_k(x_k) \quad (k = 0, 1, 2, \dots), \tag{1.1}$$

where A_k $(k = 0, 1, 2, \dots)$ are $n \times n$-complex matrices and $f_k : \mathbb{C}^n \to \mathbb{C}^n$ are given functions. A well-known result of Perron which dates back to 1929 (see [11, page 270], [8, Theorem 9.14], and [6]) states that (1.1) is asymptotically stable if $A_k \equiv A = \text{const}$ is stable (i.e., the spectral radius $r_s(A)$ of A is less than 1) and $f_k(x) = \tilde{f}(x) = o(\|x\|)$. Clearly, this result is purely local. It gives no information about the size of the region of asymptotic stability and norms of solutions.

In this paper, we derive accurate estimates for the norms of solutions. Our approach is based on the "freezing" method for difference equations and on recent estimates for the powers of a constant matrix. Note that the "freezing" method for difference equations was developed in [5]. It is based on the relevant ideas for differential equations (cf. [2, 3, 7]).

Firstly, we consider the linear difference equation

$$x_{k+1} = A_k x_k \quad (k = 0, 1, 2, \dots). \tag{1.2}$$

As it is well known, the fundamental matrix $X(k)$ of (1.2) can be expressed as

$$X(m) = A_m A_{m-1} \cdots A_0. \tag{1.3}$$

But such a representation does not yield much information about the asymptotic value of solutions, except in the case of constant coefficients $A_k = A$ $(k = 0, 1, 2, \ldots)$, when $X(k) = A^k$ and the Jordan canonical form of A determines the asymptotic behavior of the solutions. To prove the stability of (1.2) is equivalent to proving the boundedness of the sequence $\{\|X(m)\|\}_1^\infty$. This problem is easy to solve under the condition $\sup_k \|A_k\| \le 1$. But it is rather restrictive. The "freezing" method allows us to avoid this condition in the case

$$\|A_k - A_j\| \le q_{k-j} \quad (q_k = q_{-k} = \text{const} \ge 0, \; q_0 = 0; \; j, k = 0, 1, 2, \ldots). \tag{1.4}$$

For example, if $A_k = \sin(ck)B$ ($c = \text{const} > 0$), where B is a constant matrix, then condition (1.4) holds with $q_k = 2\|B\| |\sin(ck/2)|$, since

$$\sin\alpha - \sin\beta = 2\sin\left(\frac{\alpha - \beta}{2}\right)\cos\left(\frac{\alpha + \beta}{2}\right) \tag{1.5}$$

for real α, β.

For an $n \times n$-matrix A, denote

$$g(A) = \left[N^2(A) - \sum_{j=1}^{n} |\lambda_j(A)|^2\right]^{1/2}, \tag{1.6}$$

where $N(A)$ is the Frobenius (Hilbert-Schmidt) norm of a matrix $A : N^2(A) = \text{Trace}(AA^*)$, and $\lambda_1(A), \lambda_2(A), \ldots, \lambda_n(A)$ are the eigenvalues of A including their multiplicities. The relations

$$g(A) \le [N^2(A) - |\text{Trace}(A^2)|]^{1/2},$$
$$g(A) \le \sqrt{\frac{1}{2}N(A^* - A)} \tag{1.7}$$

are true [3, Section 2.1]. Here A^* is the adjoint matrix. If A is a normal matrix: $A^*A = AA^*$, then $g(A) = 0$. If $A = (a_{ij})$ is a triangular matrix such that $a_{ij} = 0$ for $1 \le j \le i \le n$, then

$$g^2(A) = \sum_{1 \le i \le j \le n} |a_{ij}|^2. \tag{1.8}$$

For a natural number $n > 1$, introduce the numbers

$$\gamma_{n,p} = \sqrt{\frac{C_{n-1}^p}{(n-1)^p}} \tag{1.9}$$

for $p = 1, 2, \ldots, n-1$ and $\gamma_{n,0} = 1$. Here and below,

$$\mathbb{C}_m^k = \frac{m!}{(m-k)!k!} \quad (k = 0,1,2,\ldots,m; \; m = 1,2,\ldots) \tag{1.10}$$

are the binomial coefficients. Evidently, for $n > 2$,

$$\gamma_{n,p}^2 = \frac{(n-2)(n-3)\cdots(n-p)}{(n-1)^{p-1}p!} \le \frac{1}{p!}. \tag{1.11}$$

Due to [4, Theorem 1.2.1], for any $n \times n$-matrix A, the inequality

$$\begin{aligned} \|A^m\| &\le \sum_{k=0}^{m_1} \frac{m! r_s^{m-k}(A) g^k(A) \gamma_{n,k}}{(m-k)!k!} \\ &= \sum_{k=0}^{m_1} \mathbb{C}_m^k r_s^{m-k}(A) g^k(A) \gamma_{n,k} \quad (m_1 = \min\{n-1,m\}) \end{aligned} \tag{1.12}$$

holds for every integer m, where $r_s(A)$ is the spectral radius of A.

2. Preliminary facts

Firstly, we recall a boundedness result for (1.2) which is proven in [5, Lemma 1.1], namely, we recall the following lemma.

LEMMA 2.1. *Under condition (1.4), let*

$$\zeta_0 \equiv \sum_{k=1}^{\infty} q_k \sup_{l=1,2,\ldots} \|A_l^k\| < 1. \tag{2.1}$$

Then, every solution $\{x_k\}$ of (1.2) satisfies the inequality

$$\sup_{k=1,2,\ldots} \|x_k\| \le \beta_0 \|x_0\| (1 - \zeta_0)^{-1}, \tag{2.2}$$

where

$$\beta_0 = \sup_{k,l=0,1,2,\ldots} \|A_l^k\|. \tag{2.3}$$

As a consequence, it is possible to establish the next corollaries.

COROLLARY 2.2. *Let the conditions*

$$\|A_k - A_{k+1}\| \le \tilde{q} \quad (k = 1,2,\ldots; \; \tilde{q} = \text{const} > 0), \tag{2.4}$$

$$\theta_0 \equiv \sum_{k=1}^{\infty} \sup_{l=1,2,\ldots} \|A_l^k\| k < \tilde{q}^{-1} \tag{2.5}$$

hold. Then, every solution $\{x_k\}$ of (1.2) satisfies the inequality

$$\|x_k\| \le \beta_0 \|x_0\| (1 - \tilde{q}\theta_0)^{-1} \quad (k = 1,2,\ldots). \tag{2.6}$$

Indeed, under condition (2.4), we have

$$||A_k - A_j|| \leq \tilde{q}|k - j| \quad (j,k = 0,1,2,\ldots). \tag{2.7}$$

So $\zeta_0 \leq \tilde{q}\theta_0$. Now, the required result follows from Lemma 2.1.

COROLLARY 2.3. *Let condition (2.4) hold. In addition, for a constant $v > 0$, let*

$$\theta(v) \equiv \sum_{k=1}^{\infty} v^{-k-1} \sup_{l=1,2,\ldots} ||A_l^k|| k < \tilde{q}^{-1}. \tag{2.8}$$

Then, every solution $\{x_k\}$ of (1.2) satisfies the inequality

$$||x_k|| \leq v^k m(v)||x_0||(1 - \tilde{q}\theta(v))^{-1} \quad (k = 1,2,\ldots), \tag{2.9}$$

where

$$m(v) \equiv \sup_{l,k=0,1,2,\ldots} v^{-k}||A_l^k||. \tag{2.10}$$

Indeed, due to condition (2.8), $m(v) < \infty$. Putting $x_k = v^k z_k$ in (1.2), we get

$$z_{k+1} = v^{-1}A_k z_k. \tag{2.11}$$

Corollary 2.2 and condition (2.8) imply

$$\sup_{k=1,2,\ldots} ||z_k|| \leq m(v)||z_0||(1 - \tilde{q}\theta(v))^{-1} \quad (k = 1,2,\ldots). \tag{2.12}$$

Hence, the required estimate follows. Recall also the following result from [5].

THEOREM 2.4. *Under condition (1.4), let*

$$\rho_0 \equiv \sup_{l=1,2,\ldots} r_s(A_l) < 1, \qquad v_0 \equiv \sup_{l=0,1,2,\ldots} g(A_l) < \infty, \tag{2.13}$$

$$\xi \equiv \sum_{m=1}^{\infty} \sum_{k=0}^{n-1} \mathbb{C}_m^k \rho_0^{m-k} v_0^k \gamma_{n,k} q_m < 1. \tag{2.14}$$

Then, every solution $\{x_k\}$ of (1.2) is bounded. Moreover,

$$\sup_{k=1,2,\ldots} ||x_k|| \leq M_0||x_0||(1 - \xi)^{-1}, \tag{2.15}$$

where

$$M_0 = \sum_{k=0}^{n-1} v_0^k \gamma_{n,k} (\psi_k + k)^k \rho_0^{\psi_k} \tag{2.16}$$

with $\psi_k = \max\{0, -k(1 + \log\rho_0)/\log\rho_0\}$.

3. The main result

The previous estimates give us a possibility to investigate (1.1) as a nonlinear perturbation of (1.2). For a positive $r \leq \infty$, denote the ball

$$B_r = \{x \in \mathbb{C}^n : \|x\| \leq r\} \tag{3.1}$$

and assume that there are constants $\mu, \nu \geq 0$, such that

$$\|f_k(x)\| \leq \nu \|x\| + \mu \quad (x \in B_r; \ k = 0, 1, 2, \dots). \tag{3.2}$$

Recall that the quantities ρ_0, ν_0, and M_0 are defined by (2.13) and (2.16). Let

$$\psi(A) \equiv \sum_{k=0}^{\infty} \sum_{j=0}^{n-1} \mathbb{C}_k^j \rho_0^{k-j} \nu_0^j \gamma_{n,j}. \tag{3.3}$$

Now we are in a position to formulate the main result of the paper.

THEOREM 3.1. *Under the conditions (1.4), (2.13), and (3.2), let*

$$S(f;A) \equiv \sum_{k=0}^{\infty} \sum_{j=0}^{n-1} \mathbb{C}_k^j \rho_0^{k-j} \nu_0^j \gamma_{n,j} (q_k + \nu) < 1. \tag{3.4}$$

Then, any solution $\{x_k\}_{=0}^{\infty}$ of (1.1) satisfies the inequality

$$\sup_{k=1,2,\dots} \|x_k\| \leq \frac{M_0 \|x_0\| + \mu \psi(A)}{1 - S(f;A)}, \tag{3.5}$$

provided that

$$\frac{M_0 \|x_0\| + \mu \psi(A)}{1 - S(f;A)} \leq r. \tag{3.6}$$

The proof of this theorem is given afterwards.
Recall that

$$\beta_0 = \sup_{k,l=0,1,\dots} \|A_l^k\| \tag{3.7}$$

and let

$$\theta_1 \equiv \sum_{k=0}^{\infty} \sup_{l=0,1,\dots} \|A_l^k\|. \tag{3.8}$$

LEMMA 3.2. *Under conditions (1.4) and (3.2), let*

$$S_0 \equiv \sum_{k=0}^{\infty} (q_k + \nu) \sup_{l=0,1,2,\dots} \|A_l^k\| < 1. \tag{3.9}$$

Then, every solution $\{x_k\}$ of (1.1) satisfies the inequality

$$||x_k|| \le [\beta_0||x_0|| + \theta_1\mu](1 - S_0)^{-1} \quad (k = 1, 2, \ldots), \tag{3.10}$$

provided that

$$[\beta_0||x_0|| + \theta_1\mu](1 - S_0)^{-1} \le r. \tag{3.11}$$

Proof. Rewrite (1.1) as

$$x_{k+1} - A_l x_k = (A_k - A_l)x_k + f_k(x_k) \tag{3.12}$$

with a fixed integer l. The variation of parameters formula yields

$$x_{l+1} = A_l^{l+1}x_0 + \sum_{j=0}^{l} A_l^{l-j}[(A_j - A_l)x_j + f_j(x_j)]. \tag{3.13}$$

There are two cases to consider: $r = \infty$ and $r < \infty$. First, assume that (3.2) is valid with $r = \infty$, then, by (1.4),

$$
\begin{aligned}
||x_{l+1}|| &\le \beta_0||x_0|| + \sum_{j=0}^{l} ||A_l^{l-j}||[[q_{l-j}||x_j|| + \nu||x_j|| + \mu] \\
&\le \beta_0||x_0|| + \sum_{j=0}^{l} ||A_l^{l-j}||(q_{l-j} + \nu)||x_j|| + \theta_1\mu \\
&\le \beta_0||x_0|| + \max_{k=0,\ldots,l} ||x_k|| \sum_{k=0}^{l} ||A_l^k||(q_k + \nu) + \mu\theta_1 \\
&\le \beta_0||x_0|| + \max_{k=1,\ldots,l} ||x_k|| \left(\sum_{k=0}^{\infty} (q_k + \nu) \sup_{l=0,1,2,\ldots} ||A_l^k|| \right) + \mu\theta_1.
\end{aligned} \tag{3.14}
$$

Consequently,

$$\max_{k=1,2,\ldots,l+1} ||x_k|| \le \beta_0||x_0|| + S_0 \max_{k=0,1,\ldots,l+1} ||x_k|| + \mu\theta_1. \tag{3.15}$$

But $\beta_0 \ge 1$. So

$$\max_{k=0,1,2,\ldots,l+1} ||x_k|| \le \beta_0||x_0|| + S_0 \max_{k=0,1,\ldots,l+1} ||x_k|| + \mu\theta_1. \tag{3.16}$$

Hence,

$$\sup_{k=0,1,2,\ldots} ||x_k|| \le \frac{\beta_0||x_0|| + \mu\theta_1}{1 - S_0}. \tag{3.17}$$

Let now $r < \infty$. Define the function

$$\tilde{f}_k(x) = \begin{cases} f_k(x), & ||x|| \le r, \\ 0, & ||x|| > r. \end{cases} \tag{3.18}$$

Since

$$\left\|\widetilde{f}_k(x)\right\| \leq \nu\|x\| + \mu, \quad k = 0,1,\dots ; \ x \in B_\infty, \tag{3.19}$$

then the sequence $\{\widetilde{x_k}\}_{=0}^\infty$ defined by

$$\widetilde{x_0} = x_0, \qquad \widetilde{x_{k+1}} = A_k\widetilde{x_k} + \widetilde{f}_k(\widetilde{x_k}), \quad k = 0,1,\dots, \tag{3.20}$$

satisfies the inequality

$$\sup_{k=0,1,\dots} \|\widetilde{x_k}\| \leq \frac{\beta_0\|x_0\| + \mu\theta_1}{1 - S_0} \leq r \tag{3.21}$$

according to the above arguments and condition (3.11). But f and $\widetilde{f}_k(x)$ coincide on B_r. So $x_k = \widetilde{x_k}$ for $k = 0,1,2,\dots$. Therefore, (3.10) is satisfied, thus concluding the proof. $\square$

proof of Theorem 3.1. As it was proved in [5, Lemma 2.2], $\beta_0 \leq M_0$. Moreover, due to (1.12), we have $\theta_1 \leq \psi(A)$ and $S_0 \leq S(f : A)$. Now the result is due to Lemma 3.2. $\square$

Remarks 3.3. (a) Under (3.2) with $\mu = 0$, $f_k(0) = 0$ so that $\{0\}$ is a solution of (1.1). Under condition $S(f,A) < 1$, Theorem 3.1 asserts that the trivial solution is stable, and that any initial vector $x_0 \in B_r$, satisfying the condition

$$\|x_0\| \leq \frac{(1 - S(f,A))r}{M_0}, \tag{3.22}$$

belongs to the region of attraction.

(b) If $\nu \equiv 0$, then every solution of (1.1) with the initial vector x_0 satisfying

$$\|x_0\|M_0 + \mu\psi(A) \leq (1 - \xi)r \tag{3.23}$$

is bounded.

4. Applications

In this section, we will illustrate our main results by considering a partial difference equation. We consider a simple three-level discrete reaction-diffusion equation of the form

$$u_i^{(j+1)} = a_j u_{i-1}^{(j)} + b_j u_i^{(j)} + c_j u_{i+1}^{(j)} + g_i^{(j)} + f_j\left(u_i^{(j)}\right), \tag{4.1}$$

defined on $\Omega = \{(i,j) : i = 0,1,\dots,n+1; \ j = 0,1,\dots\}$, where $\{a_j\}$, $\{b_j\}$, and $\{c_j\}$ are real sequences, $g = \{g_i^{(j)}\}$ is a complex function defined on Ω, and $f_j : \mathbb{C} \to \mathbb{C}$ ($j = 0,1,\dots$) are given functions. Assume that the side conditions

$$u_0^{(j)} = \delta_j \in \mathbb{C}, \quad j = 0,1,\dots, \tag{4.2}$$

$$u_{n+1}^{(j)} = \gamma_j \in \mathbb{C}, \quad j = 0,1,\dots, \tag{4.3}$$

$$u_i^{(0)} = \tau_j \in \mathbb{C}, \quad i = 1,2,\dots,n, \tag{4.4}$$

are imposed, where $\tau = (\tau_1, \tau_2, \ldots, \tau_n) \in \mathbb{C}^n$. A solution of problem (4.1), (4.2), (4.3), and (4.4) is a discrete function $u = \{u_i^{(j)}\}_{(i,j)\in\Omega}$ which satisfies relations (4.1), (4.2), (4.3), and (4.4). The existence and uniqueness of solutions to that problem is obvious, provided that f_j is one-one valued. With the notation

$$u^{(j)} = \left(u_1^{(j)}, u_2^{(j)}, \ldots, u_n^{(j)}\right), \tag{4.5}$$

the sequence $\{u^{(j)}\}_{j=0}^{\infty}$ satisfies the vector equation

$$u^{(j+1)} = A_j u^{(j)} + G_j + F_j\left(u^{(j)}\right), \quad j = 0, 1, \ldots, \tag{4.6}$$

and the initial condition

$$u^{(0)} = \tau, \tag{4.7}$$

where

$$A_j = \begin{bmatrix} b_j & c_j & 0 & \cdots & \cdots & 0 \\ a_j & b_j & c_j & 0 & \cdots & 0 \\ 0 & a_j & b_j & c_j & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \cdots & 0 & a_j & b_j \end{bmatrix}, \quad j = 0, 1, 2, \ldots, \tag{4.8}$$

$$G_j = \left(g_1^{(j)}, \ldots, g_n^{(j)}\right) + (a_j \delta_j, 0, \ldots, 0, c_j \gamma_j),$$

$$F_j(x) = (f_j(x_1), \ldots, f_j(x_n)), \quad x = (x_1, x_2, \ldots, x_n).$$

Thus, we can write problem (4.1), (4.2), (4.3), and (4.4) as (1.1) with

$$f_j(x) = F_j(x) + G_j. \tag{4.9}$$

Assume that there are nonnegative constants μ_1 and ν such that

$$\|F_j(x)\| \le \nu \|x\| + \mu_1 \quad (x \in B_r; \ j = 1, 2, \ldots). \tag{4.10}$$

In addition,

$$\mu_2 \equiv \sum_{j=0}^{\infty} \|G_j\| < \infty. \tag{4.11}$$

So condition (3.2) holds with $\mu_0 = \mu_1 + \mu_2$. As a direct consequence of Theorem 3.1, we get the following theorem.

THEOREM 4.1. *Let conditions (1.4), (4.2), (4.10), and (4.11) hold with $\mu = \mu_1 + \mu_2$ and $x_0 = \tau$. Then, the unique solution $x_j = \{u_i^{(j)}\}_{(i,j)\in\Omega}$ of problem (4.1), (4.2), (4.3), and (4.4) satisfies inequality (3.5).*

Remarks 4.2. Comparing Theorem 4.1 with [10, Theorems 1 and 2], we point out that the hypotheses of Theorem 4.1 can be checked more easily. In this paper, we have used a different approach. Our results do not overlap with those from [9, 10]. Other related works can be found in [1, pages 237–245].

Acknowledgments

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