A THREE-POINT BOUNDARY VALUE PROBLEM WITH AN INTEGRAL CONDITION FOR A THIRD-ORDER PARTIAL DIFFERENTIAL EQUATION

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We prove the existence and uniqueness of a strong solution for a linear third-order equation with integral boundary conditions. The proof uses energy inequalities and the density of the range of the operator generated.

1. Introduction

In the rectangle $\Omega = (0,1) \times (0,T)$, we consider the equation

$$f(x,t) = \frac{\partial^3 u}{\partial t^3} + \frac{\partial}{\partial x}\left(a(x,t)\frac{\partial u}{\partial x}\right) \tag{1.1}$$

with the initial conditions

$$u(x,0) = 0, \quad \frac{\partial u}{\partial t}(x,0) = 0, \quad x \in (0,1), \tag{1.2}$$

the final condition

$$\frac{\partial^2 u}{\partial t^2}(x,T) = 0, \quad x \in (0,1), \tag{1.3}$$

the Dirichlet condition

$$u(0,t) = 0 \quad \forall t \in (0,T), \tag{1.4}$$

and the integral condition

$$\int_l^1 u(x,t)dx = 0, \quad 0 \le l < 1,\ t \in (0,T). \tag{1.5}$$

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In addition, we assume that the function $a(x,t)$ and its derivatives satisfy the conditions

$$\begin{gathered} 0 < a_0 < a(x,t) < a_1 \quad \forall x,t \in \Omega, \\ \left| \frac{\partial a}{\partial x} \right| \leq b \quad \forall x,t \in \Omega, \\ c'_k < \frac{\partial^k u}{\partial t^k}(x,t) < c_k \quad \forall x,t \in \Omega,\ k = \overline{1,3},\ \text{with } c'_1 > 0. \end{gathered} \tag{1.6}$$

Over the last few years, many physical phenomena were formulated into nonlocal mathematical models with integral boundary conditions [1, 9, 10, 11]. The reader should refer to [13, 14] and the references therein. The importance of these kinds of problems has also been pointed out by Samarskii [22]. This type of boundary value problems has been investigated in [2, 3, 4, 6, 7, 8, 12, 18, 19, 20, 23, 25] for parabolic equations, in [21, 24] for hyperbolic equations, and in [15, 16, 17] for mixed-type equations. The basic tool in [5, 15, 16, 17, 20, 25] is the energy inequality method which, of course, requires appropriate multipliers and functional spaces. In this paper, we extend this method to the study of a linear third-order partial differential equation.

2. Preliminairies

In this paper, we prove the existence and uniqueness of a strong solution of the problem (1.1)–(1.5). For this, we consider the solution of problem (1.1)–(1.5) as a solution of the operator equation

$$Lu = \mathcal{F}, \tag{2.1}$$

where the operator L has domain of definition $D(L)$ consisting of functions $u \in L^2(\Omega)$ such that $(\partial^{k+1}u/\partial t^k \partial x)(x,t) \in L^2(\Omega)$, $k = \overline{1,3}$ and satisfing the conditions (1.4)-(1.5).

The operator L is considered from E to F, where E is the Banach space consisting of function $u \in L^2(\Omega)$, with the finite norm

$$\begin{aligned} \|u\|_E^2 = {} & \int_\Omega \Theta(x)\left[\left|\frac{\partial^3 u}{\partial t^3}\right|^2 + \left|\frac{\partial^2 u}{\partial x^2}\right|^2\right] dx\,dt \\ & + \int_\Omega \Theta(x)\left[\left|\frac{\partial u}{\partial x}\right|^2 + \left|\frac{\partial^2 u}{\partial t \partial x}\right|^2\right] dx\,dt \\ & + \int_\Omega \Phi(x)\left[\left|\frac{\partial u}{\partial t}\right|^2 + |u|^2\right] dx\,dt. \end{aligned} \tag{2.2}$$

F is the Hilbert space of functions $\mathcal{F} = (f,0,0,0)$, $f \in L^2(\Omega)$, with the finite norm

$$\|\mathcal{F}\|_F^2 = \int_\Omega \Theta(x)\left|f(x,t)\right|^2 dx\,dt, \tag{2.3}$$

where

$$\begin{aligned}\Theta(x) &= \begin{cases}(1-l)^2, & 0<x\le l,\\ (1-x)^2, & l\le x<1,\end{cases}\\ \Phi(x) &= \begin{cases}0, & 0<x<l,\\ 1, & l\le x<1.\end{cases}\end{aligned} \tag{2.4}$$

3. An energy inequality and its application

THEOREM 3.1. *For any function $u\in D(L)$, the a priori estimate*

$$\|u\|_E \le k\|Lu\|_F \quad \textit{for } u\in D(L), \tag{3.1}$$

where $k^2 = 40\exp(cT)/k_1$ with $k_1 = \inf\{1/4,(c_3' - 3cc_1' + 3c^2c_1' - c^3a_1 - b^2)/2,\ a_0^2/2,\ (3/2)(ca_0 - c_1)\}$. The constant c satisfies

$$\begin{gathered}\sup_{(x,t)\in\Omega}\left(\frac{1}{a}\frac{\partial a}{\partial t}\right) < c < \inf_{(x,t)\in\Omega}\left(\frac{1}{a}\frac{\partial a}{\partial t}+1\right),\\ c_3' - 3cc_1' + 3c^2c_1' - c^3a_1 - b^2 > 0,\\ c_2' - 2cc_1' + c^2a_1^2 + ca_0 - c_1 > 0.\end{gathered} \tag{3.2}$$

Proof. Let

$$Mu = \begin{cases}(1-l)^2\dfrac{\partial^3 u}{\partial t^3}, & 0<x<l,\\[2ex] (1-x)^2\dfrac{\partial^3 u}{\partial t^3} + 2(1-x)J_x\dfrac{\partial^3 u}{\partial t^3}, & l<x<1,\end{cases} \tag{3.3}$$

where $J_x u = \int_l^x u(x,t)dx$.

We consider the quadratic form obtained by multiplying (1.1) by $\exp(-ct)\overline{Mu}$, with the constant c satisfying (3.2), integrating over $\Omega = (0,1)\times(0,T)$, and taking the real part:

$$\Phi(u,u) = \operatorname{Re}\int_\Omega \exp(-ct)f(x,t)\overline{Mu}\,dx\,dt. \tag{3.4}$$

By substituting the expression of Mu in (3.4), integrating with respect to x, and using the Dirichlet and integral conditions, we obtain

$$
\begin{aligned}
\operatorname{Re}&\int_{\Omega}\exp(-ct)f(x,t)\overline{Mu}\,dx\,dt\\
&=\int_0^T\int_0^1\Theta(x)\exp(-ct)\left|\frac{\partial^3 u}{\partial t^3}\right|^2dx\,dt\\
&\quad-\frac{3}{2}\int_0^T\int_0^1\Theta(x)\exp(-ct)\left[\frac{\partial a}{\partial t}-ca\right]\left|\frac{\partial^2 u}{\partial x\partial t}\right|^2dx\,dt\\
&\quad+\int_0^T\int_0^1\frac{\Theta(x)}{2}\exp(-ct)\left[\frac{\partial^3 a}{\partial t^3}-3c\frac{\partial^2 a}{\partial t^2}+3c\frac{\partial a}{\partial t}-c^3a\right]\left|\frac{\partial u}{\partial x}\right|^2dx\,dt\\
&\quad+\int_0^T\int_l^1\exp(-ct)\left|J_x\frac{\partial^3 u}{\partial t^3}\right|^2dx\,dt\\
&\quad-2\operatorname{Re}\int_0^T\int_l^1\exp(-ct)a(x,t)u\overline{\frac{\partial^3 u}{\partial t^3}}dx\,dt\\
&\quad+\int_0^1\Theta(x)\exp(-ct)a(x,t)\left|\frac{\partial^2 u}{\partial x\partial t}\right|^2dx|_{t=T}\\
&\quad-\int_0^1\Theta(x)\exp(-ct)\left(\frac{\partial a}{\partial t}-ca\right)\frac{\partial u}{\partial x}\overline{\frac{\partial^2 u}{\partial x\partial t}}dx|_{t=T}\\
&\quad-\int_0^1\frac{\Theta(x)}{2}\exp(-ct)\left[\frac{\partial^2 a}{\partial t^2}-2c\frac{\partial a}{\partial t}+c^2a\right]\left|\frac{\partial u}{\partial x}\right|^2dx|_{t=T}\\
&\quad-2\operatorname{Re}\int_0^T\int_l^1\exp(-ct)\frac{\partial a}{\partial x}uJ_x\overline{\frac{\partial^3 u}{\partial t^3}}dx\,dt.
\end{aligned}
\tag{3.5}
$$

Integrating by parts $-2\operatorname{Re}\int_0^T\int_l^1\exp(-ct)a(x,t)u(\overline{\partial^3 u}/\partial t^3)dx\,dt$ with respect to t, and using the initial conditions, the final conditions, and the elementary inequalities, we obtain

$$
\begin{aligned}
\int_0^T&\int_0^1\frac{\Theta(x)}{2}\exp(-ct)\left|\frac{\partial^3 u}{\partial t^3}\right|^2dx\,dt\\
&\quad-\frac{3}{2}\int_0^T\int_0^1\Theta(x)\exp(-ct)\left[\frac{\partial a}{\partial t}-ca\right]\left|\frac{\partial^2 u}{\partial x\partial t}\right|^2dx\,dt\\
&\quad+\int_0^T\int_0^1\frac{\Theta(x)}{2}\exp(-ct)\left[\frac{\partial^3 a}{\partial t^3}-3c\frac{\partial^2 a}{\partial t^2}+3c\frac{\partial a}{\partial t}-c^3a\right]\left|\frac{\partial u}{\partial x}\right|^2dx\,dt\\
&\quad+\int_0^T\int_l^1\exp(-ct)\left|J_x\frac{\partial^3 u}{\partial t^3}\right|^2dx\,dt\\
&\quad+\int_0^T\int_l^1\exp(-ct)\left[\frac{\partial^3 a}{\partial t^3}-3c\frac{\partial^2 a}{\partial t^2}+3c\frac{\partial a}{\partial t}-c^3a\right]|u|^2dx\,dt\\
&\quad-\frac{3}{2}\int_0^T\int_l^1\exp(-ct)\left[\frac{\partial a}{\partial t}-ca\right]\left|\frac{\partial u}{\partial t}\right|^2dx\,dt\\
&\quad+\int_0^1\frac{\Theta(x)}{2}\exp(-ct)\left[a-\left|\frac{\partial a}{\partial t}-ca\right|\right]\left|\frac{\partial^2 u}{\partial x\partial t}\right|^2dx|_{t=T}
\end{aligned}
$$

$$
\begin{aligned}
&-\int_0^1 \frac{\Theta(x)}{2}\exp(-ct)\left[\frac{\partial^2 a}{\partial t^2}-2c\frac{\partial a}{\partial t}+c^2a+\left|\frac{\partial a}{\partial t}-ca\right|\right]\left|\frac{\partial u}{\partial x}\right|^2 dx|_{t=T}\\
&+\int_0^1 \Phi(x)\exp(-ct)\left[a-\left|\frac{\partial a}{\partial t}-ca\right|\right]\left|\frac{\partial u}{\partial t}\right|^2 dx|_{t=T}\\
&-\int_0^1 \Phi(x)\exp(-ct)\left[\frac{\partial^2 a}{\partial t^2}-2c\frac{\partial a}{\partial t}+c^2a+\left|\frac{\partial a}{\partial t}-ca\right|\right]|u|^2 dx|_{t=T}\\
\leq{}& 17\int_0^T\int_l^1 \Theta(x)\exp(-ct)|f|^2dx\,dt.
\end{aligned}
\tag{3.6}
$$

From (1.1), we get

$$
\begin{aligned}
&\int_\Omega \Theta(x)a^2\left|\frac{\partial^2 u}{\partial x^2}\right|^2 dx\,dt\\
&\quad\leq 2\int_\Omega \Theta(x)\left|\frac{\partial^3 u}{\partial t^3}\right|^2 dx\,dt+2\int_\Omega \Theta(x)\left(\frac{\partial a}{\partial x}\right)^2\left|\frac{\partial u}{\partial x}\right|^2 dx\,dt\\
&\qquad+4\int_\Omega \Theta(x)|f|^2dx\,dt.
\end{aligned}
\tag{3.7}
$$

Combining this last inequality with (3.6) and using the conditions (3.2) yield

$$
\begin{aligned}
&\int_\Omega \Theta(x)\left[\left|\frac{\partial^3 u}{\partial t^3}\right|^2+\left|\frac{\partial^2 u}{\partial x^2}\right|^2\right]dx\,dt\\
&\quad+\int_\Omega \Theta(x)\left[\left|\frac{\partial u}{\partial x}\right|^2+\left|\frac{\partial^2 u}{\partial t\partial x}\right|^2\right]dx\,dt+\int_\Omega \Phi(x)\left[\left|\frac{\partial u}{\partial t}\right|^2+|u|^2\right]dx\,dt\\
&\leq k\int_\Omega \Theta(x)\left|f(x,t)\right|^2dx\,dt,
\end{aligned}
\tag{3.8}
$$

which is the desired inequality. □

It can be proved in a standard way that the operator $L: E \to F$ is closable. Let $\overline{L}$ be the closure of this operator, with the domain of definition $D(\overline{L})$.

Definition 3.2. A solution of the operator equation $\overline{L}u = \mathcal{F}$ is called a strong solution of problem (1.1)–(1.5).

The a priori estimate (3.1) can be extended to strong solutions, that is, we have the estimate

$$
\|u\|_E \leq c\|\overline{L}u\|_F \quad \forall u \in D(\overline{L}). \tag{3.9}
$$

This last inequality implies the following corollaries.

Corollary 3.3. *A strong solution of (1.1)–(1.5) is unique and depends continuously on $\mathcal{F}$.*

Corollary 3.4. *The range $R(\overline{L})$ of $\overline{L}$ is closed in F and $\overline{R(L)} = R(\overline{L})$.*

Corollary 3.4 shows that to prove that problem (1.1)–(1.5) has a strong solution for arbitrary $\mathcal{F}$, it suffices to prove that set $R(L)$ is dense in F.

4. Solvability of problem (1.1)–(1.5)

To prove the solvability of problem (1.1)–(1.5) it is sufficient to show that $R(L)$ is dense in F. The proof is based on the following lemma.

LEMMA 4.1. *Suppose that the function $a(x,t)$ and its derivatives are bounded. Let $u \in D_0(L) = \{u \in D(L),\ u(x,0) = 0,\ (\partial u/\partial t)(x,0) = 0,\ (\partial^2 u/\partial t^2)(x,T) = 0\}$. If for $u \in D_0(L)$ and some functions $w(x,t) \in L^2(\Omega)$,*

$$\int_\Omega h(x) f \overline{w} dx\, dt = 0, \tag{4.1}$$

where

$$h(x) = \begin{cases} 1-l, & 0 < x < l, \\ 1-x, & l < x < 1, \end{cases} \tag{4.2}$$

holds, for arbitrary $u \in D_0(L)$, and then $w = 0$.

Proof. The equality (4.1) can be written as follows:

$$\int_\Omega h(x) \frac{\partial^3 u}{\partial t^3} \overline{w} dx\, dt = \int_\Omega A(t) u \overline{v} dx\, dt, \tag{4.3}$$

for a given $w(x,t)$, where

$$\begin{gathered} v = \begin{cases} (1-l)w, & 0 < x < l, \\ w - \displaystyle\int_l^x \frac{w}{1-\zeta} d\zeta, & l < x < 1, \end{cases} \\ A(t)u = \frac{\partial}{\partial x}\left(h(x) a(x,t) \frac{\partial u}{\partial x}\right), \\ Nv = \begin{cases} (1-l)v, & 0 < x < l, \\ (1-x)v + J_x v, & l < x < 1. \end{cases} \end{gathered} \tag{4.4}$$

For $v = w - \int_l^x (w/(1-\zeta)) d\zeta$, $l < x < 1$ we deduce $\int_l^x v(\zeta,t) d\zeta = (1-x) \int_l^x (w/(1-\zeta)) d\zeta$, then $\int_l^1 v(\zeta,t) d\zeta = 0$.

Following [25], we introduce the smoothing operators with respect to t, $(J_\epsilon^{-1}) = (I - \epsilon(\partial^3/\partial t^3))^{-1}$, and $(J_\epsilon^{-1})^* = (I + \epsilon(\partial^3/\partial t^3))^{-1}$ which provide the solution of the respective problems:

$$\begin{aligned} &u_\epsilon - \epsilon \frac{\partial^3 u_\epsilon}{\partial t^3} = u, && u_\epsilon(x,0) = 0, && \frac{\partial u_\epsilon}{\partial t}(x,0) = 0, && \frac{\partial^2 u_\epsilon}{\partial t^2}(x,T) = 0, \\ &v_\epsilon^* + \epsilon \frac{\partial^3 v_\epsilon^*}{\partial t^3} = v, && v_\epsilon^*(x,0) = 0, && \frac{\partial v_\epsilon^*}{\partial t}(x,T) = 0, && \frac{\partial^2 v_\epsilon^*}{\partial t^2}(x,T) = 0. \end{aligned} \tag{4.5}$$

And also, we have the following properties: for any $u \in L^2(0,T)$, the function $J_\epsilon^{-1}u \in W_2^3(0,T)$, $(J_\epsilon^{-1})^* u \in W_2^3(0,T)$. If $u \in D(L)$, $J_\epsilon^{-1}u \in D(L)$.

$$\lim_{\epsilon\to 0} \|J_\epsilon^{-1}u - u\|_{L^2(0,T)} = 0, \qquad \lim_{\epsilon\to 0} \|(J_\epsilon^{-1})^* u - u\|_{L^2(0,T)} = 0. \tag{4.6}$$

Substituting the function u in (4.3) by the smoothing function u_ϵ and using the relation $A(t)u_\epsilon = J_\epsilon^{-1}A(t)u + \epsilon J_\epsilon^{-1}B_\epsilon(t)u$, where $B_\epsilon(t) = (3\partial/\partial t)((\partial A(t)/\partial t)(\partial u_\epsilon/\partial t)) + (\partial^3 A(t)/\partial t^3)u_\epsilon$, we obtain

$$\int_\Omega uN\overline{\frac{\partial^3 v_\epsilon^*}{\partial t^3}}\,dx\,dt = \int_\Omega A(t)u\overline{v_\epsilon^*}\,dx\,dt - \epsilon\int_\Omega B_\epsilon(t)u\overline{v_\epsilon^*}\,dx\,dt. \tag{4.7}$$

The operator $A(t)$ has a continuous inverse in $L^2(0,1)$ defined by

$$A^{-1}(t)g = \begin{cases} -\dfrac{1}{1-l}\displaystyle\int_0^x \frac{d\zeta}{a(\zeta,t)}\int_0^\zeta g(\eta)d\eta + \frac{C_1(t)}{1-l}\int_0^x \frac{d\zeta}{a(\zeta,t)}, & 0<x<l, \\ \displaystyle\int_l^x \frac{-d\zeta}{(1-\zeta)a(\zeta,t)}\int_l^\zeta g(\eta)d\eta + C_2(t)\int_l^x \frac{d\zeta}{(1-\zeta)a(\zeta,t)} + u(l), & l<x<1, \end{cases} \tag{4.8}$$

where

$$\begin{aligned} C_1(t) &= \frac{(1-l)u(l) + \int_0^l (d\zeta/a(\zeta,t))\int_0^\zeta g(\eta)d\eta}{\int_0^l (d\zeta/a(\zeta,t))}, \\ C_2(t) &= \frac{-(1-l)u(l) + \int_l^1 (d\zeta/a(\zeta,t))\int_l^\zeta g(\eta)d\eta}{\int_l^1 (d\zeta/a(\zeta,t))}. \end{aligned} \tag{4.9}$$

Then we have $\int_l^1 A^{-1}(t)u = 0$, hence, the function $J_\epsilon^{-1}u = u_\varepsilon$ can be represented in the form

$$u_\varepsilon = J_\epsilon^{-1}A^{-1}(t)A(t)u. \tag{4.10}$$

The adjoint of $B_\epsilon(t)$ has the form

$$\begin{aligned} B_\epsilon^*(t)v = {} & \frac{1}{a}(J_\epsilon^{-1})^*\frac{\partial^3 a}{\partial t^3}v + \frac{3}{a}(J_\epsilon^{-1})^*\frac{\partial}{\partial t}\left(\frac{\partial a}{\partial t}\frac{\partial v}{\partial t}\right) - G_\epsilon(v)(x) \\ & + \frac{\int_0^x (d\zeta/a(\zeta,t))}{\int_0^1 (d\zeta/a(\zeta,t))}G_\epsilon(v)(1), \end{aligned} \tag{4.11}$$

where

$$\begin{aligned} G_\epsilon(v)(x) = \int_0^x \Bigg[& \frac{3}{a}(J_\epsilon^{-1})^*\frac{\partial}{\partial t}\left(\frac{\partial^2 a}{\partial t\partial\zeta}\frac{\partial v}{\partial t}\right) - \frac{3}{a^2}\frac{\partial a}{\partial\zeta}(J_\epsilon^{-1})^*\frac{\partial}{\partial t}\left(\frac{\partial a}{\partial t}\frac{\partial v}{\partial t}\right) \\ & + \frac{1}{a}(J_\epsilon^{-1})^*\frac{\partial}{\partial t}\left(\frac{\partial^4 a}{\partial t^3\partial\zeta}v\right) - \frac{1}{a^2}\frac{\partial a}{\partial\zeta}(J_\epsilon^{-1})^*\left(\frac{\partial^3 a}{\partial t^3}v\right)\Bigg]d\zeta. \end{aligned} \tag{4.12}$$

Consequently, equality (4.7) becomes

$$\int_\Omega uN\overline{\frac{\partial^3 v_\epsilon^*}{\partial t^3}}\,dx\,dt = \int_\Omega A(t)u\overline{h_\epsilon}\,dx\,dt, \tag{4.13}$$

where $h_\epsilon = v_\epsilon^* - \epsilon B_\epsilon^*(t)v_\epsilon^*$.

The left-hand side of (4.13) is a continuous linear functional of u, hence the function h_ϵ has the derivatives $\partial h_\epsilon/\partial x$, $(1-x)(\partial h_\epsilon/\partial x) \in L^2(\Omega)$, and the condition $h_\epsilon(0,t) = 0$ is satisfied.

From the equality

$$(1-x)\frac{\partial h_\epsilon}{\partial x} = \left[I - \epsilon\frac{1}{a}(J_\epsilon^{-1})^*\left(\frac{\partial^3 a}{\partial t^3}\right)\right](1-x)\frac{\partial v_\epsilon^*}{\partial x} - 3\epsilon\frac{1}{a}(J_\epsilon^{-1})^*\frac{\partial}{\partial t}\left(\frac{\partial a}{\partial t}\frac{\partial}{\partial t}(1-x)\frac{\partial v_\epsilon^*}{\partial x}\right), \tag{4.14}$$

and since the operator $(J_\epsilon^{-1})^*$ is bounded in $L^2(\Omega)$, for sufficiently small ϵ, we have $\|\epsilon(1/a)(J_\epsilon^{-1})^*(\partial^3 a/\partial t^3)\| < 1$. Hence, the operator $I - \epsilon(1/a)(J_\epsilon^{-1})^*(\partial^3 a/\partial t^3)$ has a bounded inverse in $L^2(\Omega)$. We conclude that $(1-x)(\partial v_\epsilon^*/\partial x) \in L^2(\Omega)$. Similarly, we conclude that $(\partial/\partial x)((1-x)(\partial v_\epsilon^*/\partial x))$ exists and belongs to $L^2(\Omega)$, and the condition $v_\epsilon^*(0,t) = 0$ is satisfied.

Putting $u = \int_0^t\int_0^\zeta\int_\eta^T \exp(c\tau)v_\epsilon^*\,d\tau\,d\eta\,d\zeta$ in (4.3), where the constant c satisfies (3.2) and using the proprieties of smoothing operator, we obtain

$$\int_\Omega \exp(ct)v_\varepsilon^*\overline{Nv}\,dx\,dt = -\int_\Omega A(t)u\overline{v_\varepsilon^*}\,dx\,dt - \varepsilon\int_\Omega A(t)u\overline{\frac{\partial^3 v_\epsilon^*}{\partial t^3}}\,dx\,dt, \tag{4.15}$$

and from

$$\begin{aligned}
&-\varepsilon\int_\Omega A(t)u\overline{\frac{\partial^3 v_\epsilon^*}{\partial t^3}}\,dx\,dt\\
&\quad= 3\int_\Omega h(x)\exp(-ct)\frac{\partial^2 a}{\partial t^2}\left|\frac{\partial^3 u}{\partial t^2\partial x}\right|^2 dx\,dt\\
&\qquad- 3\int_\Omega h(x)\exp(-ct)\left[\frac{\partial^3 a}{\partial t^3} - c\frac{\partial^2 a}{\partial t^2}\right]\frac{\partial^3 u}{\partial t^2\partial x}\overline{\frac{\partial^2 u}{\partial t\partial x}}\,dx\,dt\\
&\qquad+ 3\int_0^1 \frac{h(x)}{2}\exp(-ct)\frac{\partial a}{\partial t}\left|\frac{\partial^3 u}{\partial t^2\partial x}\right|^2 dx|_{t=T}\\
&\qquad+ 3\int_0^1 \frac{h(x)}{2}\exp(-ct)\left[\frac{\partial^2 a}{\partial t^2} - c\frac{\partial a}{\partial t}\right]\left|\frac{\partial^2 u}{\partial t\partial x}\right|^2 dx|_{t=T}\\
&\qquad- \int_\Omega h(x)\exp(-ct)a\left|\frac{\partial^3 v_\epsilon^*}{\partial t^3}\right|^2 dx\,dt\\
&\qquad- \int_\Omega h(x)\exp(-ct)\frac{\partial^3 a}{\partial t^3}\frac{\partial u}{\partial x}\overline{\frac{\partial^3 u}{\partial t^2\partial x}}\,dx\,dt,
\end{aligned} \tag{4.16}$$

we have

$$
\begin{aligned}
&-\varepsilon \operatorname{Re}\int_{\Omega} A(t)u\overline{\frac{\partial^3 v_\epsilon^*}{\partial t^3}}\,dx\,dt \\
&\quad\leq \varepsilon\bigg\{3\int_{\Omega} h(x)\exp(-ct)\bigg[\frac{\partial^2 a}{\partial t^2}+\frac{1}{2}\bigg|\frac{\partial^3 a}{\partial t^3}-c\frac{\partial^2 a}{\partial t^2}\bigg|\bigg]\bigg|\frac{\partial^3 u}{\partial t^2\partial x}\bigg|^2 dx\,dt \\
&\qquad+\frac{3}{2}\int_{\Omega} h(x)\exp(-ct)\bigg[\frac{\partial^2 a}{\partial t^2}-c\frac{\partial a}{\partial t}+\bigg|\frac{\partial^3 a}{\partial t^3}-c\frac{\partial^2 a}{\partial t^2}\bigg|\bigg]\bigg|\frac{\partial^2 u}{\partial t\partial x}\bigg|^2 dx\,dt \\
&\qquad-\int_{\Omega} h(x)\exp(-ct)a\bigg|\frac{\partial^3 v_\epsilon^*}{\partial t^3}\bigg|^2 dx\,dt \\
&\qquad+\frac{3}{2}\int_{\Omega} h(x)\exp(-ct)\bigg|\frac{\partial^3 a}{\partial t^3}\bigg|\bigg|\frac{\partial u}{\partial x}\bigg|^2 dx\,dt \\
&\qquad+\frac{1}{2}\int_{\Omega} h(x)\exp(-ct)\bigg|\frac{\partial^3 a}{\partial t^3}\bigg|\bigg|\frac{\partial^4 u}{\partial t^3\partial x}\bigg|^2 dx\,dt \\
&\qquad+\frac{1}{2}\int_{\Omega} h(x)\exp(-ct)\frac{\partial a}{\partial t}\bigg|\frac{\partial^3 u}{\partial t^2\partial x}\bigg|^2 dx\,dt\bigg\}.
\end{aligned}
\tag{4.17}
$$

Integrating the first term on the right-hand side by parts in (4.15), we obtain

$$
\begin{aligned}
&-\varepsilon \operatorname{Re}\int_{\Omega} A(t)u\overline{v_\varepsilon^*}\,dx\,dt \\
&\quad=\frac{3}{2}\int_{\Omega} h(x)\exp(-ct)\bigg[\frac{\partial a}{\partial t}-ca\bigg]\bigg|\frac{\partial^2 u}{\partial t\partial x}\bigg|^2 dx\,dt \\
&\qquad-\int_{\Omega} h(x)\exp(-ct)\bigg\{\frac{\partial^3 a}{\partial t^3}-3c\frac{\partial^2 a}{\partial t^2}+3c^2\frac{\partial a}{\partial t}-c^3 a\bigg\}\bigg|\frac{\partial u}{\partial x}\bigg|^2 dx\,dt \\
&\qquad-\int_0^1 \frac{1}{2}h(x)\exp(-ct)a\bigg|\frac{\partial^2 u}{\partial t\partial x}\bigg|^2 dx|_{t=T} \\
&\qquad+\int_0^1 \frac{1}{2}h(x)\exp(-ct)\bigg\{\frac{\partial^2 a}{\partial t^2}-2c\frac{\partial a}{\partial t}+c^2 a\bigg\}\bigg|\frac{\partial u}{\partial x}\bigg|^2 dx|_{t=T} \\
&\qquad-\int_0^1 h(x)\exp(-ct)\bigg\{\frac{\partial a}{\partial t}-ca\bigg\}\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial t\partial x}dx|_{t=T}.
\end{aligned}
\tag{4.18}
$$

This last equality gives

$$
\begin{aligned}
&-\varepsilon \operatorname{Re}\int_{\Omega} A(t)u\overline{v_\varepsilon^*}\,dx\,dt \\
&\quad\leq-\int_0^1 h(x)\exp(-ct)\bigg|\frac{\partial a}{\partial t}+a-ca\bigg|\bigg|\frac{\partial^2 u}{\partial x\partial t}\bigg|^2 dx|_{t=T} \\
&\qquad+\int_0^1 \frac{1}{2}h(x)\exp(-ct)\bigg\{\frac{\partial^2 a}{\partial t^2}-2c\frac{\partial a}{\partial t}+c^2 a+ca-\frac{\partial a}{\partial t}\bigg\}\bigg|\frac{\partial u}{\partial x}\bigg|^2 dx|_{t=T}.
\end{aligned}
\tag{4.19}
$$

By using the conditions (3.2), inequalities (4.17) and (4.19), we obtain

$$
\operatorname{Re}\int_{\Omega}\exp(ct)v_\varepsilon^*\overline{Nv}\,dx\,dt\leq 0\quad\text{as }\epsilon\longrightarrow 0. \tag{4.20}
$$

This implies $\operatorname{Re}\int_\Omega \exp(ct)(v_\varepsilon^* - v)\overline{Nv}\,dx\,dt + \operatorname{Re}\int_\Omega \exp(ct)v\overline{Nv}\,dx\,dt \le 0$, that is,

$$\begin{aligned}
&\int_0^T\int_0^l \exp(-ct)(1-l)|v|^2 dx\,dt \\
&\quad + \int_0^T\int_l^1\int_0^l \exp(-ct)(1-x)|v|^2 dx\,dt + \int_0^T\int_l^1 \exp(-ct)\left|J_x v\right|^2 dx\,dt \\
&\quad + \int_0^T\int_0^l \frac{1-l}{2l}\exp(-ct)\left|J_x v\right|^2 dx\,dt \le 0.
\end{aligned} \tag{4.21}$$

Then $v = 0$.

Finally from (4.4), we conclude $w = 0$. □

Theorem 4.2. *The range $R(\overline{L})$ of $\overline{L}$ coincides with F.*

Proof. Since F is Hilbert space, then $R(\overline{L}) = F$ if and only if the relation

$$\int_\Omega \Theta(x) f\overline{g}\,dx\,dt = 0 \tag{4.22}$$

holds.

Arbitrary $u \in D_0(L)$ and $\mathcal{F} = (f,0,0,0) \in F$ implies $f = 0$. Taking in (4.22), $u \in D_0(L)$, and using Lemma 4.1, we obtain

$$w = \begin{cases} (1-l)g, & 0 < x < l, \\ (1-x)g, & l < x < 1, \end{cases} \tag{4.23}$$

then $g = 0$. □

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