CONTINUOUS DEPENDENCE ON DATA FOR QUASIAUTONOMOUS NONLINEAR BOUNDARY VALUE PROBLEMS

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Received 20 December 2003

We devote this paper to quasiautonomous second-order differential equations in Hilbert spaces governed by maximal monotone operators. Some bilocal boundary conditions are associated. We discuss the continuous dependence of the solution both on the operator and on the boundary values. One uses the methods of nonlinear analysis. Some applications to internal approximate schemes are given.

1. Introduction

The main purpose of this paper is to prove the continuous dependence on A, a, b, f of the solution of the second-order evolution equation

$$pu''(t)+ru'(t)\in Au(t)+f, \quad \text{a.e. } t\in(0,T), \tag{1.1}$$

subject to the two-point boundary condition

$$u(0)=a, \qquad u(T)=b. \tag{1.2}$$

Here $A:D(A)\subseteq H\to H$ is a maximal monotone operator (possibly multivalued) in a real Hilbert space H, $D(A)$ is its domain, $a,b\in D(A)$, $f\in L^2(0,T;H)$, and p, r are two continuous functions from $[0,T]$ to $\mathbb{R}$.

In [10, 11], Barbu proved the existence of the solution in the case $p\equiv 1$, $r\equiv 0$. The author considered the boundary value problems

$$\begin{gathered} u''(t)\in Au(t)+f(t), \quad \text{a.e. } t\in(0,T), \\ u(0)=a, \qquad u(T)=b, \end{gathered} \tag{1.3}$$

$$\begin{gathered} u''(t)\in Au(t), \quad \text{a.e. } t\in(0,\infty), \\ u(0)=a, \qquad \sup_{t\geq 0}\|u(t)\|<\infty, \end{gathered} \tag{1.4}$$

where $a,b\in D(A)$ and $f\in L^2(0,T;H)$. Denoting by u_a the solution of (1.4) and by $S_{1/2}(t)$ the extension to the closure $\overline{D(A)}$ of $D(A)$ of the mapping $a\mapsto u_a(t)$, one obtains a semigroup of nonlinear contractions $\{S_{1/2}(t),\ t\geq 0\}$ on $\overline{D(A)}$. Let $-A^0_{1/2}$ be its infinitesimal

Abstract and Applied Analysis 2005:1 (2005) 67–86
DOI: 10.1155/AAA.2005.67

generator and $A_{1/2}$ the unique extension of $A^0_{1/2}$ to a maximal monotone operator. The operator $A_{1/2}$ is called the square root of A. Regularity properties of $S_{1/2}(t)$ are presented in [10, 11, 13].

Brézis [14] replaced the condition $u(0) = a$ in (1.4) by $u'(0) \in \partial j(u(0) - a)$, where ∂j is the subdifferential of a lower-semicontinuous, convex, and proper function $j : H \to \overline{\mathbb{R}}$. Problems (1.3) and (1.4) were also studied in Banach spaces [21, 22, 23]. The semigroup $\{S_{1/2}(t),\ t \geq 0\}$ is defined in this case too [22]. Other contributions to the theory of second-order differential equations (1.3) and (1.4) can be found in [16, 18, 19, 24, 25, 26].

Aftabizadeh and Pavel [1, 2] generalized problem (1.3) to (1.1) with the boundary condition

$$u'(0) \in \alpha(u(0) - a), \qquad -u'(T) \in \beta(u(T) - b) \tag{1.5}$$

with α, β being maximal monotone operators in H. If we take $\alpha = \beta = \partial j$, where $j : H \to \overline{\mathbb{R}}$,

$$j(x) = \begin{cases} 0, & x = 0, \\ +\infty, & \text{otherwise}, \end{cases} \tag{1.6}$$

then $D(\partial j) = \{0\}$ and thus (1.5) becomes (1.2). A more general boundary condition can be found in [4]. Antiperiodic solutions for a particular case of (1.1) are given in [3]. In [12], another extension of the equation in (1.3) is studied under a boundary condition of subdifferential type.

As a consequence of [1, Theorem 3.2], it follows that if A is maximal monotone in H, $a, b \in D(A)$, $f \in L^2(0,T;H)$ and $p, r : [0,T] \to \overline{\mathbb{R}}$ are continuous functions, $p(t) \geq c > 0$ on $[0,T]$, then problem (1.1)-(1.2) has a unique solution $u \in W^{2,2}(0,T;H)$.

Discrete variants of (1.3) and (1.4) are studied in [20].

In [5], it is shown that the application which associates to $\{A, a, b\}$ the unique solution u of (1.3) with $f \equiv 0$ is continuous in the following sense. Consider the boundary value problem (1.3) (with $f \equiv 0$) and the sequence of problems

$$\begin{aligned} u_n''(t) &\in A^n u_n(t), \quad \text{a.e. } t \in (0,T), \\ u_n(0) &= a_n, \qquad u_n(T) = b_n, \end{aligned} \tag{1.7}$$

where A, A^n are maximal monotone operators in H, $a, b \in \overline{D(A)}$, $a_n, b_n \in \overline{D(A^n)}$ with $0 \in A0 \cap A^n 0$. If $a_n \to a$, $b_n \to b$ in H and

$$(I + \lambda A^n)^{-1}\xi \longrightarrow (I + \lambda A)^{-1}\xi \quad \text{as } n \longrightarrow \infty, \tag{1.8}$$

for all $\xi \in H$, and for all $\lambda > 0$, then the solution u_n of (1.7) converges to the solution u of (1.3) (with $f \equiv 0$), uniformly on $[0,T]$.

In [6], we have a similar result on $(0,\infty)$. The case of the first-order differential equations is analyzed in [7, 15]. The continuous dependence on data for the antiperiodic solutions to a class of second-order evolution equations with constant coefficients is given in [3].

If A^n and A satisfy condition (1.8), we say that A^n converges to A in the sense of the resolvent. This and other types of convergences of the sequences of operators can be found in [8]. They are of physical interest because of their applications in the homogenization theory, singular perturbation problems, convergence problems in optimal control, stochastic optimization, and so forth. In [22], the authors show that in Banach spaces with some specific properties, the convergence in the sense of resolvent of a sequence (A^n) to A implies the convergence of $(A^n_{1/2})$ to $A_{1/2}$ in the same sense.

In the present paper, we prove that the unique solution u of problem (1.1)-(1.2) depends strongly continuous on the data A, a, b, f. More exactly, we take the sequence of evolution equations

$$pu_n''(t)+ru_n'(t)\in A^n u_n(t)+f_n, \quad \text{a.e. } t\in(0,T), \tag{1.9}$$

subject to the boundary conditions

$$u_n(0)=a_n, \qquad u_n(T)=b_n. \tag{1.10}$$

Here (A^n) is a sequence of maximal monotone operators in H, $a_n,b_n\in D(A^n)=$ the domain of A^n, $f_n\in L^2(0,T;H)$. We show that, under some additional conditions, if $a_n\to a$, $b_n\to b$ in H, $f_n\to f$ in $L^2(0,T;H)$, and (A^n) converges to A in the sense of the resolvent, then the solution u_n of (1.9)-(1.10) converges in $C([0,T];H)$ to the solution u of (1.1)-(1.2).

Using an idea from [1, 2], in the next sections, one uses the weighted space $\mathscr{L}=L^2_{\tilde{r}/p}(0,T;H)$, where

$$\tilde{r}(t)=\exp\left(\int_0^t \frac{r(s)}{p(s)}ds\right), \quad t\in[0,T]. \tag{1.11}$$

Therefore, the scalar product in $\mathscr{L}$ is

$$\langle u,v\rangle=\int_0^T \frac{\tilde{r}(t)}{p(t)}(u(t),v(t))\,dt \quad \forall u,v\in L^2(0,T;H), \tag{1.12}$$

and the corresponding norm is

$$|u|=\left(\int_0^T \frac{\tilde{r}(t)}{p(t)}\|u(t)\|^2dt\right)^{1/2} \quad \forall u\in L^2(0,T;H), \tag{1.13}$$

where $(\cdot,\cdot)$ and $\|\cdot\|$ are the scalar product and the norm of H, respectively. Actually, the spaces $L^2(0,T;H)$ and $\mathscr{L}$ contain the same functions and have equivalent norms. The difference between them is that the operator

$$\begin{gathered} Bu=-pu''-ru'=-\frac{p}{\tilde{r}}(\tilde{r}u')', \\ D(B)=\{u\in W^{2,2}(0,T;H),\ u(0)=a,\ u(T)=b\} \end{gathered} \tag{1.14}$$

is maximal monotone only in $\mathscr{L}$ (see [1]). Taking into account this remark, we may write (1.1) in the form

$$\frac{p}{\tilde{r}}(\tilde{r}u')' \in Au + f, \quad \text{a.e. } t \in (0,T). \tag{1.15}$$

In Section 2, we recall some definitions and results from the theory of maximal monotone operators. The main result is stated in Section 3 and proved in Section 4. The proof combines an idea related to the case $p \equiv 1$, $r \equiv 0$, $f \equiv 0$ (see [5]) with some techniques from the existence theory (see [1, 2]). In the last section, we give a numerical approximation of (1.1)-(1.2) with $f \equiv 0$ by an internal approximating scheme (see [9]).

2. Preliminaries

Throughout this paper, H is a real Hilbert space of norm $\|\cdot\|$ and scalar product $(\cdot,\cdot)$. Denote by "$\to$" and "$\rightharpoonup$" the strong and the weak convergence in all the involved spaces, respectively.

The nonlinear multivalued operator A with the domain $D(A)$ and the range $R(A)$ is said to be *monotone* if $(y_1 - y_2, x_1 - x_2) \geq 0$ for all $y_i \in Ax_i$, $x_i \in D(A)$, $i = 1,2$. The operator A is called *maximal monotone* if it is monotone and it has not any proper monotone extension. It is known that A is maximal monotone if and only if A is monotone and $R(A+\lambda I) = H$ for all $\lambda > 0$ (or equivalently, for some $\lambda > 0$) (see [13, Theorem 1.2, page 39]). For $x \in D(A)$, let A^0x be *the element of least norm* in Ax, that is,

$$||A^0x|| = \inf\{\|y\|,\ y \in Ax\}. \tag{2.1}$$

The single-valued operator A^0 which associates to each $x \in D(A)$ the element A^0x is called the *minimal section* of A. For every maximal monotone operator A, one can define the *resolvent* J_λ and the *Yosida approximation* A_λ of A, namely,

$$J_\lambda = (I+\lambda A)^{-1}, \quad A_\lambda = \frac{I - J_\lambda}{\lambda}, \quad \lambda > 0. \tag{2.2}$$

The *realization* of A in $L^2(0,T;H)$ is the operator $\mathscr{A}$ given by

$$D(\mathscr{A}) = \{u \in L^2(0,T;H),\ u(t) \in D(A) \text{ a.e. on } (0,T), \\ \exists v \in L^2(0,T;H) \text{ such that } v(t) \in Au(t) \text{ a.e. on } (0,T)\}, \tag{2.3}$$

$$\mathscr{A}u = \{v \in L^2(0,T;H),\ v(t) \in Au(t) \text{ a.e. on } (0,T)\}. \tag{2.4}$$

If A is maximal monotone in H, then $\mathscr{A}$ is maximal monotone in $L^2(0,T;H)$. If A_λ and $\mathscr{A}_\lambda$ are Yosida approximations of A and $\mathscr{A}$, respectively, then $(\mathscr{A}_\lambda u)(t) = A_\lambda u(t)$ for all $\lambda > 0$, a.e. $t \in [0,T]$, for $u \in L^2(0,T;H)$.

Definition 2.1. A sequence $\{A^n\}$ of maximal monotone operators in H is said to be *convergent to A in the sense of the resolvent* if

$$(I+\lambda A^n)^{-1}\xi \longrightarrow (I+\lambda A)^{-1}\xi \quad \text{as } n \longrightarrow \infty,\ \forall \xi \in H,\ \forall \lambda > 0. \tag{2.5}$$

The following characterization of the convergence in the sense of resolvent is true even in reflexive Banach spaces (see [8, page 365]).

Theorem 2.2. *If $A : D(A) \subseteq H \to H$ and $A^n : D(A^n) \subseteq H \to H$ are maximal monotone operators in the Hilbert space H, then A^n converges to A in the sense of the resolvent if and only if A^n is graph-convergent to A (denoted $A^n \overset{G}{\to} A$ as $n \to \infty$), that is, for all $x \in D(A)$, and for all $y \in Ax$, there exist $x_n \in D(A^n)$, $y_n \in A^n x_n$ such that $x_n \to x$, $y_n \to y$ in H.*

We recall now the definition of the Mosco convergence of a sequence of functions and a result concerning the equivalence between the Mosco convergence of the functions (φ^n) and the convergence in the sense of the resolvent of the subdifferential operators $(\partial\varphi^n)$ (see [8, Theorem 3.66, page 373]).

Definition 2.3. If $\varphi, \varphi^n : H \to (-\infty, \infty]$ is a sequence of convex, lower-semicontinuous, proper functions, then φ^n is *convergent to φ in the sense of Mosco* (see [7, 8]) if

(a) there exists $\{x_n\} \subset H$, $x_n \to x$, such that $\varphi^n(x_n) \to \varphi(x)$;
(b) for all $\{x_n\} \subset H$ with $x_n \rightharpoonup x$, $\liminf_{n\to\infty} \varphi^n(x_n) \geq \varphi(x)$.

Theorem 2.4. *If $\varphi^n, \varphi : H \to (-\infty, \infty]$ are convex, lower-semicontinuous, proper functions, then the following statements are equivalent:*

(a) *$\varphi^n \to \varphi$ in the sense of Mosco;*
(b) *$(I + \lambda\partial\varphi^n)^{-1}\xi \to (I + \lambda\partial\varphi)^{-1}\xi$, for all $\lambda > 0$ and for all $\xi \in H$ and there exist $(u, v) \in \partial\varphi$, and $(u_n, v_n) \in \partial\varphi^n$ such that $u_n \to u$, $v_n \to v$ in H and $\varphi^n(u_n) \to \varphi(u)$.*

3. The main result

Consider the boundary value problems

$$pu'' + ru' \in Au + f, \quad \text{a.e. } t \in (0, T), \tag{3.1}$$

$$u(0) = a, \qquad u(T) = b, \tag{3.2}$$

$$pu_n'' + ru_n' \in A^n u_n + f_n, \quad \text{a.e. } t \in (0, T), \tag{3.3}$$

$$u_n(0) = a_n, \qquad u_n(T) = b_n. \tag{3.4}$$

We now state our basic assumptions:

($H1$) A, A^n are nonlinear (possibly multivalued) maximal monotone operators in the real Hilbert space H, with the domains $D(A)$, $D(A^n)$ and $0 \in D(A) \cap D(A^n)$ for all $n \in \mathbb{N}$, $n \geq 1$;
($H2$) $a, b \in D(A)$, a_n, $b_n \in D(A^n)$ are given elements;
($H3$) $f, f_n \in L^2(0, T; H)$;
($H4$) $p, r : [0, T] \to \mathbb{R}$ are continuous functions, $p(t) \geq c > 0$ for all $t \in [0, T]$.

These hypotheses assure the existence and the uniqueness in $W^{2,2}(0, T; H)$ of the solutions to problems (3.1)-(3.2) and (3.3)-(3.4), respectively. In addition, suppose that

($H5$) if $(A^n)^0$ is the minimal section of A^n, then $(A^n)^0 a_n$ and $(A^n)^0 b_n$ are bounded in H;
($H6$) $a_n \to a$, $b_n \to b$ in H and $f_n \to f$ in $L^2(0, T; H)$;

$(H7)$ (A^n) converges to A in the sense of the resolvent, that is,

$$(I+\lambda A^n)^{-1}\xi \longrightarrow (I+\lambda A)^{-1}\xi, \quad n \longrightarrow \infty,\ \forall \xi \in H,\ \forall \lambda > 0. \tag{3.5}$$

The continuous dependence on data result for the problem (3.1)-(3.2) may now be stated.

THEOREM 3.1. *If hypotheses $(H1)$–$(H7)$ hold and u, u_n are the solutions of problems (3.1)-(3.2) and (3.3)-(3.4), respectively, then $u_n(t) \to u(t)$ uniformly on $[0,T]$ and $u_n' \to u'$ in $L^2(0,T;H)$ as $n \to \infty$.*

The proof of this theorem is the purpose of the next section.

Remark 3.2. In [7, Theorem 3.2, page 62 and Proposition 3.7, page 64], the author establishes some conditions when the sum $A^n + B^n$ converges to $A+B$ in the sense of the resolvent. Here A^n and B^n are supposed to be maximal monotone operators convergent to A and B, respectively, in the sense of the resolvent.

Theorem 3.1 above is not a consequence of these general perturbation results. Indeed, problems (3.1)-(3.2) and (3.3)-(3.4) can be written in $L^2(0,T;H)$ as

$$-f \in Bu + \mathscr{A}u, \qquad -f_n \in B^n u_n + \mathscr{A}^n u_n, \tag{3.6}$$

respectively, where $\mathscr{A}$, $\mathscr{A}^n$ are the realizations of A, A^n in $L^2(0,T;H)$, B is given in (1.14), and B^n is analogous to B, but with a_n, b_n instead of a, b. It is known that B, B^n are maximal monotone in $L^2(0,T;H)$ (see [1]). Moreover, $B = \partial\varphi$, $B^n = \partial\varphi^n$, where $\varphi, \varphi^n : L^2(0,T;H) \to (-\infty,\infty]$ are defined by

$$\varphi(u) = \begin{cases} \dfrac{1}{2}\displaystyle\int_0^T \tilde{r}(t)\|u'(t)\|^2 dt, & u \in W^{1,2}(0,T;H),\ u(0)=a,\ u(T)=b, \\ +\infty, & \text{otherwise}, \end{cases}$$
$$\varphi^n(u) = \begin{cases} \dfrac{1}{2}\displaystyle\int_0^T \tilde{r}(t)\|u'(t)\|^2 dt, & u \in W^{1,2}(0,T;H),\ u(0)=a_n,\ u(T)=b_n, \\ +\infty, & \text{otherwise}. \end{cases} \tag{3.7}$$

We show that φ^n is not Mosco convergent to φ. To do this, consider $u \in W^{1,2}(0,T;H)$ with $u(T) \neq b$ and

$$u_n(t) = \begin{cases} a_n, & t=0, \\ b_n, & t=T, \\ u(t) + \dfrac{C}{n}, & 0<t<T, \end{cases} \tag{3.8}$$

where C is a constant in H. It is clear that $u_n \to u$ in $L^2(0,T;H)$ and $\liminf_{n\to\infty}\varphi^n(u_n) < \varphi(u) = +\infty$. Thus condition (b) in Definition 2.3 is not satisfied. Then φ^n does not converge to φ in the sense of Mosco. Theorem 2.4 implies that $\partial\varphi^n$ is not convergent to $\partial\varphi$ in the sense of the resolvent. So Attouch's results for the convergence of the sum $\mathscr{A}^n + B^n$ are not applicable here, even if $\mathscr{A}^n \to \mathscr{A}$ in $L^2(0,T;H)$ in the sense of the resolvent.

A particular case of Theorem 3.1 is obtained assuming that A and A^n are subdifferential mappings and replacing $(H7)$ by the condition

$(H7)'$ $\varphi^n \to \varphi$ in the sense of Mosco.

In this case, we find (in view of Theorem 2.4) the following consequence of Theorem 3.1.

Corollary 3.3. *If $A = \partial\varphi$ and $A^n = \partial\varphi^n$, where $\varphi^n, \varphi : H \to (-\infty, \infty]$ are convex, lower-semicontinuous, proper functions with $0 \in D(\partial\varphi)$, $0 \in D(\partial\varphi^n)$, then under hypotheses $(H2)$–$(H6)$, $(H7)'$, the convergences $u_n(t) \to u(t)$ uniformly on $[0,T]$ and $u_n' \to u'$ in $L^2(0,T;H)$ as $n \to \infty$ are obtained.*

4. The proof of Theorem 3.1

The proof of the main result combines some ideas from [1, 2, 5]. For every given $\lambda > 0$, we put

$$\begin{aligned} y_\lambda &= \left(I + \sqrt{\lambda}A\right)^{-1} a, \qquad z_\lambda = \left(I + \sqrt{\lambda}A\right)^{-1} b, \\ y_{n\lambda} &= \left(I + \sqrt{\lambda}A^n\right)^{-1} a, \qquad z_{n\lambda} = \left(I + \sqrt{\lambda}A^n\right)^{-1} b. \end{aligned} \tag{4.1}$$

By hypothesis (3.5), it follows that

$$y_{n\lambda} \longrightarrow y_\lambda, \quad z_{n\lambda} \longrightarrow z_\lambda, \quad A_\lambda^n \xi \longrightarrow A_\lambda \xi \quad \text{as } n \longrightarrow \infty, \tag{4.2}$$

for all $\lambda > 0$, and for all $\xi \in H$, where A_λ^n is the Yosida approximation of A^n.

Let w_λ, v_λ, $w_{n\lambda}$, $v_{n\lambda}$ be the solutions of the auxiliary boundary value problems

$$\begin{gathered} p w_\lambda'' + r w_\lambda' \in A w_\lambda + f, \quad \text{a.e. } t \in (0,T), \\ w_\lambda(0) = y_\lambda, \qquad w_\lambda(T) = z_\lambda, \end{gathered} \tag{4.3}$$

$$\begin{gathered} p v_\lambda'' + r v_\lambda' = A_\lambda v_\lambda + f, \quad \text{a.e. } t \in (0,T), \\ v_\lambda(0) = y_\lambda, \qquad v_\lambda(T) = z_\lambda, \end{gathered} \tag{4.4}$$

$$\begin{gathered} p w_{n\lambda}'' + r w_{n\lambda}' \in A^n w_{n\lambda} + f_n, \quad \text{a.e. } t \in (0,T), \\ w_{n\lambda}(0) = y_{n\lambda}, \qquad w_{n\lambda}(T) = z_{n\lambda}, \end{gathered} \tag{4.5}$$

$$\begin{gathered} p v_{n\lambda}'' + r v_{n\lambda}' = A_\lambda^n v_{n\lambda} + f_n, \quad \text{a.e. } t \in (0,T), \\ v_{n\lambda}(0) = y_{n\lambda}, \qquad v_{n\lambda}(T) = z_{n\lambda}, \end{gathered} \tag{4.6}$$

respectively. From the general theory recalled in Section 1, we know that each of these problems has a unique solution in $W^{2,2}(0,T;H)$.

For every $t \in [0,T]$, $\lambda > 0$, and $n \in \mathbb{N}$, we can write

$$\begin{aligned} \|u_n(t) - u(t)\| \le{} & \|u_n(t) - w_{n\lambda}(t)\| + \|w_{n\lambda}(t) - v_{n\lambda}(t)\| \\ & + \|v_{n\lambda}(t) - v_\lambda(t)\| + \|v_\lambda(t) - w_\lambda(t)\| \\ & + \|w_\lambda(t) - u(t)\|, \end{aligned} \tag{4.7}$$

$$|u_n' - u'| \le |u_n' - w_{n\lambda}'| + |w_{n\lambda}' - v_{\nu\lambda}'| + |v_{n\lambda}' - v_\lambda'| + |v_\lambda' - w_\lambda'| + |w_\lambda' - u'|. \tag{4.8}$$

Recall that $|\cdot|$ denotes the norm in $L^2(0,T;H)$. We intend to take the superior limit as $n\to\infty$ and then the limit as $\lambda\to 0$ in both (4.7) and (4.8). In order to do this, we estimate each term in (4.7) and (4.8). One begins with some boundedness results.

Lemma 4.1. *Under the hypotheses of Theorem 3.1, if $w_{n\lambda}$ is the solution of problem (4.5), then for every fixed $\lambda>0$,*

$$\begin{gathered}\limsup_{n\to\infty}\|w'_{n\lambda}(0)\|\le c_1\left(\|A_{\sqrt{\lambda}}a\|+\|A_{\sqrt{\lambda}}b\|+\|y_\lambda\|+\|z_\lambda\|+1\right),\\ \limsup_{n\to\infty}\|w'_{n\lambda}(T)\|\le c_2\left(\|A_{\sqrt{\lambda}}a\|+\|A_{\sqrt{\lambda}}b\|+\|y_\lambda\|+\|z_\lambda\|+1\right),\end{gathered}\tag{4.9}$$

$$\limsup_{n\to\infty}|w'_{n\lambda}|\le c_3\left(\|A_{\sqrt{\lambda}}a\|+\|A_{\sqrt{\lambda}}b\|+\|y_\lambda\|+\|z_\lambda\|+1\right),\tag{4.10}$$

$$\limsup_{n\to\infty}|w''_{n\lambda}|\le c_4\left(\|A_{\sqrt{\lambda}}a\|+\|A_{\sqrt{\lambda}}b\|+\|y_\lambda\|+\|z_\lambda\|+1\right),\tag{4.11}$$

$$\limsup_{n\to\infty}|w_{n\lambda}|_C\le c_5\left(\|A_{\sqrt{\lambda}}a\|+\|A_{\sqrt{\lambda}}b\|+\|y_\lambda\|+\|z_\lambda\|+1\right),\tag{4.12}$$

where $|\cdot|_C$ is the norm in $C([0,T];H)$, that is, $|u|_C=\sup_{t\in[0,T]}\|u(t)\|$.

Proof. One approximates (4.5) by

$$\begin{gathered}pw''_{n\lambda\mu}+rw'_{n\lambda\mu}=A^n_\mu w_{n\lambda\mu}+\mu w_{n\lambda\mu}+f_n,\quad \text{a.e. } t\in(0,T),\\ w_{n\lambda\mu}(0)=y_{n\lambda},\qquad w_{n\lambda\mu}(T)=z_{n\lambda},\end{gathered}\tag{4.13}$$

where $A^n_\mu=(1/\mu)[I-(I+\mu A^n)^{-1}]$. We show that $w_{n\lambda\mu}$ is bounded in $C([0,T];H)$ with respect to μ and $w_{n\lambda\mu}\to w_{n\lambda}$ in $C([0,T];H)$ as $\mu\to 0$.

One writes the equation in (4.13) as in (1.15), multiply it by $(\tilde r/p)w_{n\lambda\mu}$ and integrate over $[0,T]$, to obtain

$$\begin{aligned}&\int_0^T\left((\tilde r w'_{n\lambda\mu})',w_{n\lambda\mu}\right)dt\\ &\quad=\int_0^T\frac{\tilde r}{p}(A^n_\mu w_{n\lambda\mu},w_{n\lambda\mu})dt+\mu\int_0^T\frac{\tilde r}{p}\|w_{n\lambda\mu}\|^2dt+\int_0^T\frac{\tilde r}{p}(f_n,w_{n\lambda\mu})dt.\end{aligned}\tag{4.14}$$

Here and everywhere below, we omit the variable t at the functions under integrals.

Without loss of generality, suppose that $0\in A^n0$. Otherwise, we replace A^nu by $\tilde A^nu=A^nu-(A^n)^00$ and f_n by $\tilde f_n=f_n+(A^n)^00$. Since A^n_μ is monotone, by the above equality and $(H4)$, we obtain

$$c|w'_{n\lambda\mu}|^2+\mu|w_{n\lambda\mu}|^2\le\tilde r(T)\|w'_{n\lambda\mu}(T)\|\cdot\|z_{n\lambda}\|+\|w'_{n\lambda\mu}(0)\|\cdot\|y_{n\lambda}\|+|f_n|\cdot|w_{n\lambda\mu}|.\tag{4.15}$$

Since

$$w_{n\lambda\mu}(t)=y_{n\lambda}+\int_0^t w'_{n\lambda\mu}(s)ds,\quad t\in[0,T],\tag{4.16}$$

we deduce that

$$\|w_{n\lambda\mu}(t)\| \le \|y_{n\lambda}\| + \sqrt{T}\left(\int_0^T \|w'_{n\lambda\mu}\|^2 ds\right)^{1/2}, \quad t \in [0,T], \tag{4.17}$$

so

$$|\, w_{n\lambda\mu}\,| \le C_1\left(\|y_{n\lambda}\| + |\, w'_{n\lambda\mu}\,|\right). \tag{4.18}$$

The constant C_1 and all the constants below are positive and independent of n, λ, and μ. Using (4.18) in (4.15), we find

$$\begin{aligned} c\,|\, w'_{n\lambda\mu}\,|^2 \le \tilde{r}(T)\|w'_{n\lambda\mu}(T)\| \cdot \|z_{n\lambda}\| + \|w'_{n\lambda\mu}(0)\| \cdot \|y_{n\lambda}\| \\ + C_1\,|\, f_n\,|\left(\|y_{n\lambda}\| + |\, w'_{n\lambda\mu}\,|\right), \end{aligned} \tag{4.19}$$

hence, by virtue of boundedness of f_n,

$$|\, w'_{n\lambda\mu}\,| \le C_2\left(\|y_{n\lambda}\|^{1/2} \cdot \|w'_{n\lambda\mu}(0)\|^{1/2} + \|z_{n\lambda}\|^{1/2} \cdot \|w'_{n\lambda\mu}(T)\|^{1/2} + \|y_{n\lambda}\|^{1/2} + 1\right). \tag{4.20}$$

We now multiply (4.13) by $w''_{n\lambda\mu}$ and integrate from $t = 0$ to $t = T$:

$$\begin{aligned} &\int_0^T p\|w''_{n\lambda\mu}\|^2 dt + \int_0^T r\left(w'_{n\lambda\mu}, w''_{n\lambda\mu}\right) dt \\ &\quad = \int_0^T \left(A_\mu^n w_{n\lambda\mu}, w''_{n\lambda\mu}\right) dt + \mu \int_0^T \left(w_{n\lambda\mu}, w''_{n\lambda\mu}\right) dt + \int_0^T \left(f_n, w''_{n\lambda\mu}\right) dt. \end{aligned} \tag{4.21}$$

The functions $x \mapsto A_\mu^n x$ and $t \mapsto w_{n\lambda\mu}(t)$ are Lipschitz continuous, so the application $t \mapsto A_\mu^n w_{n\lambda\mu}(t)$ is differentiable a.e. on $[0,T]$. Since $x \mapsto A_\mu^n x$ is monotone, we also have $((A_\mu^n w_{n\lambda\mu}(t))', w'_{n\lambda\mu}(t)) \ge 0$ a.e. $t \in (0,T)$. Then,

$$\left(A_\mu^n w_{n\lambda\mu}, w''_{n\lambda\mu}\right) \le \left(A_\mu^n w_{n\lambda\mu}, w'_{n\lambda\mu}\right)'. \tag{4.22}$$

On the other hand,

$$\left(w_{n\lambda\mu}, w''_{n\lambda\mu}\right) = \left(w_{n\lambda\mu}, w'_{n\lambda\mu}\right)' - \|w'_{n\lambda\mu}\|^2 \le \left(w_{n\lambda\mu}, w'_{n\lambda\mu}\right)'. \tag{4.23}$$

Now (4.22), (4.23), and (4.21) yield

$$\begin{aligned} \int_0^T p\|w''_{n\lambda\mu}\|^2 dt \le{}& \left(A_\mu^n z_{n\lambda}, w'_{n\lambda\mu}(T)\right) - \left(A_\mu^n y_{n\lambda}, w'_{n\lambda\mu}(0)\right) \\ &+ \mu\left(z_{n\lambda}, w'_{n\lambda\mu}(T)\right) - \mu\left(y_{n\lambda}, w'_{n\lambda\mu}(0)\right) \\ &+ \left[\left(\int_0^T \|f_n\|^2 dt\right)^{1/2} + \left(\int_0^T r^2\|w'_{n\lambda\mu}\|^2 dt\right)^{1/2}\right]\left(\int_0^T \|w''_{n\lambda\mu}\|^2 dt\right)^{1/2}, \end{aligned} \tag{4.24}$$

therefore via $(H4)$,

$$
\begin{aligned}
c\left(\int_0^T \|w''_{n\lambda\mu}\|^2 dt\right)^{1/2} \le{}& \left(\int_0^T \|f_n\|^2 dt\right)^{1/2} + \left(\int_0^T r^2\|w'_{n\lambda\mu}\|^2 dt\right)^{1/2} \\
&+ \sqrt{c}\left(\|A_\mu^n y_{n\lambda}\|^{1/2} + \sqrt{\mu}\|y_{n\lambda}\|^{1/2}\right)\|w'_{n\lambda\mu}(0)\|^{1/2} \\
&+ \sqrt{c}\left(\|A_\mu^n z_{n\lambda}\|^{1/2} + \sqrt{\mu}\|z_{n\lambda}\|^{1/2}\right)\|w'_{n\lambda\mu}(T)\|^{1/2}.
\end{aligned}
\tag{4.25}
$$

Inequality (4.20) leads to

$$
\begin{aligned}
\left(\int_0^T \|w''_{n\lambda\mu}\|^2 dt\right)^{1/2} \le{}& C_3 + C_4\left(\|(A^n)^0 y_{n\lambda}\|^{1/2} + \|y_{n\lambda}\|^{1/2}\right)\|w'_{n\lambda\mu}(0)\|^{1/2} \\
&+ C_5\left(\|(A^n)^0 z_{n\lambda}\|^{1/2} + \|z_{n\lambda}\|^{1/2}\right)\|w'_{n\lambda\mu}(T)\|^{1/2} + C_6\|y_{n\lambda}\|^{1/2}
\end{aligned}
\tag{4.26}
$$

for small $\mu > 0$. Observe that

$$
\frac{a - y_{n\lambda}}{\sqrt{\lambda}} = A^n_{\sqrt{\lambda}} a \in A^n y_{n\lambda}, \qquad \frac{b - z_{n\lambda}}{\sqrt{\lambda}} = A^n_{\sqrt{\lambda}} b \in A^n z_{n\lambda}.
\tag{4.27}
$$

Therefore, we get

$$
\left\|(A^n)^0 y_{n\lambda}\right\| \le \left\|A^n_{\sqrt{\lambda}} a\right\|, \qquad \left\|(A^n)^0 z_{n\lambda}\right\| \le \left\|A^n_{\sqrt{\lambda}} b\right\|.
\tag{4.28}
$$

So, (4.26) gives

$$
\begin{aligned}
\left(\int_0^T \|w''_{n\lambda\mu}\|^2 dt\right)^{1/2} \le{}& C_3 + C_4\left(\|A^n_{\sqrt{\lambda}} a\|^{1/2} + \|y_{n\lambda}\|^{1/2}\right)\|w'_{n\lambda\mu}(0)\|^{1/2} \\
&+ C_5\left(\|A^n_{\sqrt{\lambda}} b\|^{1/2} + \|z_{n\lambda}\|^{1/2}\right)\|w'_{n\lambda\mu}(T)\|^{1/2} + C_6\|y_{n\lambda}\|^{1/2}.
\end{aligned}
\tag{4.29}
$$

To estimate $w'_{n\lambda\mu}(0)$ and $w'_{n\lambda\mu}(T)$, we write

$$
\begin{aligned}
w'_{n\lambda\mu}(0) &= \frac{1}{T}\left[z_{n\lambda} - y_{n\lambda} - \int_0^T \left(\int_0^t w''_{n\lambda\mu}(s)ds\right) dt\right], \\
w'_{n\lambda\mu}(T) &= \frac{1}{T}\left[z_{n\lambda} - y_{n\lambda} + \int_0^T \left(\int_t^T w''_{n\lambda\mu}(s)ds\right) dt\right].
\end{aligned}
\tag{4.30}
$$

Now (4.29) and (4.30) imply

$$
\begin{aligned}
\|w'_{n\lambda\mu}(0)\| \le{}& C_7(\|y_{n\lambda}\| + \|z_{n\lambda}\|) + C_8\left(\|A^n_{\sqrt{\lambda}} a\|^{1/2} + \|y_{n\lambda}\|^{1/2}\right)\|w'_{n\lambda\mu}(0)\|^{1/2} \\
&+ C_9\left(\|A^n_{\sqrt{\lambda}} b\|^{1/2} + \|z_{n\lambda}\|^{1/2}\right)\|w'_{n\lambda\mu}(T)\|^{1/2} + C_{10},
\end{aligned}
\tag{4.31}
$$

and analogously for $\|w'_{n\lambda\mu}(T)\|$. Hence,

$$\begin{aligned} \|w'_{n\lambda\mu}(0)\|^{1/2} &\le C_{11}\left(\|A^n_{\sqrt{\lambda}}a\|^{1/2}+\|A^n_{\sqrt{\lambda}}b\|^{1/2}+\|y_{n\lambda}\|^{1/2}+\|z_{n\lambda}\|^{1/2}+1\right),\\ \|w'_{n\lambda\mu}(T)\|^{1/2} &\le C_{12}\left(\|A^n_{\sqrt{\lambda}}a\|^{1/2}+\|A^n_{\sqrt{\lambda}}b\|^{1/2}+\|y_{n\lambda}\|^{1/2}+\|z_{n\lambda}\|^{1/2}+1\right). \end{aligned} \tag{4.32}$$

Using (4.32) in (4.29) and (4.20), respectively, we infer that

$$\begin{aligned} |w''_{n\lambda\mu}| &\le C_{13}\left(\|A^n_{\sqrt{\lambda}}a\|+\|A^n_{\sqrt{\lambda}}b\|+\|y_{n\lambda}\|+\|z_{n\lambda}\|+1\right),\\ |w'_{n\lambda\mu}| &\le C_{14}\left(\|A^n_{\sqrt{\lambda}}a\|+\|A^n_{\sqrt{\lambda}}b\|+\|y_{n\lambda}\|+\|z_{n\lambda}\|+1\right). \end{aligned} \tag{4.33}$$

The last inequality, together with (4.17), leads to

$$|w_{n\lambda\mu}|_C \le C_{15}\left(\|A^n_{\sqrt{\lambda}}a\|+\|A^n_{\sqrt{\lambda}}b\|+\|y_{n\lambda}\|+\|z_{n\lambda}\|+1\right). \tag{4.34}$$

Inequalities (4.32)–(4.34) show that $w'_{n\lambda\mu}(0)$ and $w'_{n\lambda\mu}(T)$ are bounded in H, $w''_{n\lambda\mu}$, $w'_{n\lambda\mu}$ are bounded in $L^2(0,T;H)$, and $w_{n\lambda\mu}$ is bounded in $C([0,T];H)$, all of them with respect to μ. By (4.13) we have also the boundedness of $A^n_\mu w_{n\lambda\mu}$ in $L^2(0,T;H)$ (in μ).

We prove now that $(w'_{n\lambda\mu})_\mu$ is strongly convergent in $C([0,T];H)$. To do this, we write (4.13) for μ and ν, subtract them, multiply by $(\widetilde{r}/p)(w_{n\lambda\mu}-w_{n\lambda\nu})$, and integrate over $[0,T]$. With the aid of

$$J^n_\mu w_{n\lambda\mu}+\mu A^n_\mu w_{n\lambda\mu}=w_{n\lambda\mu}, \tag{4.35}$$

one arrives at

$$\begin{aligned} -\int_0^T \widetilde{r}\|w'_{n\lambda\mu}-w'_{n\lambda\nu}\|^2dt &= \int_0^T \frac{\widetilde{r}}{p}\left(A^n_\mu w_{n\lambda\mu}-A^n_\nu w_{n\lambda\nu}, J^n_\mu w_{n\lambda\mu}-J^n_\nu w_{n\lambda\nu}\right)dt\\ &\quad+\int_0^T \frac{\widetilde{r}}{p}\left(A^n_\mu w_{n\lambda\mu}-A^n_\nu w_{n\lambda\nu}, \mu A^n_\mu w_{n\lambda\mu}-\nu A^n_\nu w_{n\lambda\nu}\right)dt\\ &\quad+\int_0^T \frac{\widetilde{r}}{p}\left(\mu w_{n\lambda\mu}-\nu w_{n\lambda\nu}, w_{n\lambda\mu}-w_{n\lambda\nu}\right)dt. \end{aligned} \tag{4.36}$$

Since $A^n_\mu w_{n\lambda\mu}\in A^n(J^n_\mu w_{n\lambda\mu})$ and A^n is monotone, this implies that

$$\begin{aligned} &\int_0^T \widetilde{r}\|w'_{n\lambda\mu}-w'_{n\lambda\nu}\|^2dt\\ &\qquad\le(\mu+\nu)\left[\int_0^T \frac{\widetilde{r}}{p}\left(A^n_\mu w_{n\lambda\mu}, A^n_\nu w_{n\lambda\nu}\right)dt+\int_0^T \frac{\widetilde{r}}{p}\left(w_{n\lambda\mu}, w_{n\lambda\nu}\right)dt\right]. \end{aligned} \tag{4.37}$$

The boundedness in $L^2(0,T;H)$ of $w_{n\lambda\mu}$ and $A^n_\mu w_{n\lambda\mu}$ with respect to μ shows that

$$|w'_{n\lambda\mu}-w'_{n\lambda\nu}|^2\le K_1^{n\lambda}(\mu+\nu), \tag{4.38}$$

where $K_1^{n\lambda}$ depends linearly on $\|A^n_{\sqrt{\lambda}}a\|$, $\|A^n_{\sqrt{\lambda}}b\|$, $\|y_{n\lambda}\|$, $\|z_{n\lambda}\|$ and is independent of μ and ν. Consequently, $(w'_{n\lambda\mu})_\mu$ strongly converges in $L^2(0,T;H)$, say

$$w'_{n\lambda\mu} \longrightarrow g_{n\lambda} \quad \text{in } L^2(0,T;H) \text{ as } \mu \longrightarrow 0. \tag{4.39}$$

Now we have the estimate

$$\begin{aligned} &\|w'_{n\lambda\mu}(t) - g_{n\lambda}(t)\|^2 - \|w'_{n\lambda\mu}(t_0) - g_{n\lambda}(t_0)\|^2 \\ &\quad \le 2\left(\int_0^T \|w'_{n\lambda\mu} - g_{n\lambda}\|^2 dt\right)^{1/2} \left(\int_0^T \|w''_{n\lambda\mu} - g'_{n\lambda}\|^2 dt\right)^{1/2} \end{aligned} \tag{4.40}$$

with $t_0 \in [0,T]$ such as $\|w'_{n\lambda\mu}(t_0) - g_{n\lambda}(t_0)\| \to 0$ as $\mu \to 0$. Since $(w''_{n\lambda\mu})_\mu$ is bounded in $L^2(0,T;H)$ with respect to μ, one deduces the convergence

$$w'_{n\lambda\mu} \longrightarrow g_{n\lambda} \quad \text{in } C([0,T];H) \text{ as } \mu \longrightarrow 0. \tag{4.41}$$

From (4.16) and (4.41), we obtain that

$$w_{n\lambda\mu}(t) \longrightarrow y_{n\lambda} + \int_0^t g_{n\lambda}(s)ds \overset{\text{not.}}{=} h_{n\lambda}(t), \quad t \in [0,T]. \tag{4.42}$$

It follows that $h_{n\lambda}(0) = y_{n\lambda}$ and $h_{n\lambda}$ is differentiable on $[0,T]$ with $h'_{n\lambda} = g_{n\lambda}$. Hence, $w'_{n\lambda\mu} \to h'_{n\lambda}$ in $C([0,T];H)$, $w_{n\lambda\mu} \rightharpoonup h_{n\lambda}$, $w''_{n\lambda\mu} \rightharpoonup h''_{n\lambda}$ in $L^2(0,T;H)$, and $h_{n\lambda}(T) = z_{n\lambda}$. By (4.35) we get

$$J^n_\mu w_{n\lambda\mu} \rightharpoonup h_{n\lambda} \quad \text{as } \mu \longrightarrow 0 \text{ in } L^2(0,T;H). \tag{4.43}$$

Denoting by B_1 the operator

$$\begin{aligned} B_1 u &= -pu'' - ru' = -\frac{p}{\tilde{r}}(\tilde{r}u')', \\ D(B_1) &= \{u \in W^{2,2}(0,T;H),\ u(0) = y_{n\lambda},\ u(T) = z_{n\lambda}\}, \end{aligned} \tag{4.44}$$

we may write (4.13) under the form

$$-B_1 w_{n\lambda\mu} - \mu w_{n\lambda\mu} - f_n \in \mathcal{A}^n(\mathcal{J}^n_\mu w_{n\lambda\mu}), \tag{4.45}$$

where $\mathcal{J}^n_\mu$ is the realization of J^n_μ in $L^2(0,T;H)$, $(\mathcal{J}^n_\mu u)(t) = J^n_\mu u(t)$ a.e. $t \in [0,T]$ for $u \in L^2(0,T;H)$.

Observe that $-B_1 w_{n\lambda\mu} - \mu w_{n\lambda\mu} - f_n \rightharpoonup -B_1 h_{n\lambda} - f_n$. Taking into account the maximal monotonicity of $\mathcal{A}^n$ in $L^2(0,T;H)$ and (4.43), in order to take the limit in (4.45), it is enough to prove that

$$\langle -B_1 w_{n\lambda\mu} - \mu w_{n\lambda\mu} - f_n, \mathcal{J}^n_\mu w_{n\lambda\mu}\rangle \longrightarrow \langle -B_1 h_{n\lambda} - f_n, h_{n\lambda}\rangle. \tag{4.46}$$

Using (4.35) and the boundedness of $B_1 w_{n\lambda\mu}$, $\mathcal{A}^n w_{n\lambda\mu}$, $w_{n\lambda\mu}$, and $\mathcal{J}^n_\mu w_{n\lambda\mu}$, it suffices to show the convergence

$$\langle -B_1 w_{n\lambda\mu} - f_n, w_{n\lambda\mu}\rangle \longrightarrow \langle -B_1 h_{n\lambda} - f_n, h_{n\lambda}\rangle. \tag{4.47}$$

But $\langle f_n, w_{n\lambda\mu}\rangle \to \langle f_n, h_{n\lambda}\rangle$ and by virtue of (4.41) we get

$$\begin{aligned}\langle -B_1 w_{n\lambda\mu}, w_{n\lambda\mu}\rangle &= \tilde{r}(T)\big(w'_{n\lambda\mu}(T), z_{n\lambda}\big) - \big(w'_{n\lambda\mu}(0), y_{n\lambda}\big) - \int_0^T \tilde{r}\|w'_{n\lambda\mu}\|^2 dt \\ &\longrightarrow \tilde{r}(T)\big(h'_{n\lambda}(T), z_{n\lambda}\big) - \big(h'_{n\lambda}(0), y_{n\lambda}\big) - \int_0^T \tilde{r}\|h'_{n\lambda}\|^2 dt \\ &= \langle -B_1 h_{n\lambda}, h_{n\lambda}\rangle.\end{aligned} \tag{4.48}$$

Thus (4.46) is proved. Now we may pass to the limit as $\mu \to 0$ in (4.45) and find that $h_{n\lambda} \in D(A^n)$ and $-B_1 h_{n\lambda} - f_n \in \mathcal{A}^n h_{n\lambda}$. Since $w_{n\lambda}$ verifies the same equation, by the uniqueness one deduces $h_{n\lambda} = w_{n\lambda}$. Therefore, $w_{n\lambda\mu} \rightharpoonup w_{n\lambda}$, $w''_{n\lambda\mu} \rightharpoonup w''_{n\lambda}$ in $L^2(0,T;H)$, $w'_{n\lambda\mu} \to w'_{n\lambda}$ in $C([0,T];H)$, $w_{n\lambda\mu}(t) \to w_{n\lambda}(t)$ for all $t \in [0,T]$. Now (4.9)–(4.12) follow from (4.32)–(4.34). The proof is finished. □

We now give a boundedness result for the solution (u_n) of (3.4)-(3.5).

Lemma 4.2. *If the hypotheses of Theorem 3.1 are satisfied, then* $\{u'_n(0)\}$ *and* $\{u'_n(T)\}$ *are bounded in* H, $\{u'_n\}$, $\{u''_n\}$ *are bounded in* $L^2(0,T;H)$, *and* $\{u_n\}$ *is bounded in* $C([0,T];H)$.

Proof. Consider the boundary value problem

$$\begin{gathered} p u''_{n\mu} + r u'_{n\mu} = A^n_\mu u_{n\mu} + \mu u_{n\mu} + f_n, \quad \text{a.e. } t \in (0,T), \\ u_{n\mu}(0) = a_n, \qquad u_{n\mu}(T) = b_n. \end{gathered} \tag{4.49}$$

Following the computation from the proof of Lemma 4.1, we get an estimate of the form (4.20) with a_n, b_n instead of $y_{n\lambda}$, $z_{n\lambda}$. Since (a_n), (b_n) are bounded, this can be written as

$$|u'_{n\mu}| \le k_1\|u'_{n\mu}(0)\|^{1/2} + k_2\|u'_{n\mu}(T)\|^{1/2} + k_3, \tag{4.50}$$

where $k_1, k_2, k_3 > 0$ are independent of n and μ.

Similarly, one obtains an inequality of the form (4.26), namely,

$$\begin{aligned}|u''_{n\mu}| \le\ & k_4 + k_5\Big(\|(A^n)^0 a_n\|^{1/2} + \|a_n\|^{1/2}\Big)\|u'_{n\mu}(0)\|^{1/2} \\ & + k_6\Big(\|(A^n)^0 b_n\|^{1/2} + \|b_n\|^{1/2}\Big)\|u'_{n\mu}(T)\|^{1/2} + k_7\|a_n\|^{1/2}.\end{aligned} \tag{4.51}$$

Hypotheses $(H5)$ and $(H6)$ imply the existence of some constants $k_8, k_9, k_{10} > 0$ (independent of n and μ) such that

$$|u''_{n\mu}| \le k_8\|u'_{n\mu}(0)\|^{1/2} + k_9\|u'_{n\mu}(T)\|^{1/2} + k_{10}. \tag{4.52}$$

Next, as in (4.31), one arrives at

$$\|u'_{n\mu}(0)\| \le k_{11}\|u'_{n\mu}(0)\|^{1/2} + k_{12}\|u'_{n\mu}(T)\|^{1/2} + k_{13}, \tag{4.53}$$

and an analogous inequality for $\|u'_{n\mu}(T)\|$, with all constants independent of n and μ. This provides upper bounds for $\|u'_{n\mu}(0)\|$, $\|u'_{n\mu}(T)\|$ and via (4.50), (4.52), for $|u'_{n\mu}|$, $|u''_{n\mu}|$. By $u_{n\mu}(t) = a_n + \int_0^T u'_{n\mu}(s)ds$, $t \in [0,T]$, we find an upper bound for $u_{n\mu}$ in $C([0,T];H)$.

Now, as in the proof of the previous lemma, one shows that $u_{n\mu} \rightharpoonup u_n$, $u''_{n\mu} \rightharpoonup u''_n$ in $L^2(0,T;H)$, $u'_{n\mu} \to u'_n$ in $C([0,T];H)$, and $u_{n\mu}(t) \to u_n(t)$ for $t \in [0,T]$ (as $\mu \to 0$).

Since $\|u'_n(0)\| \le \liminf_{\mu\to 0} \|u'_{n\mu}(0)\|$ and $\|u'_n(T)\| \le \liminf_{\mu\to 0} \|u'_{n\mu}(T)\|$, one deduces that $u'_n(0)$ and $u'_n(T)$ are bounded in H. Analogously, u'_n, u''_n are bounded in $L^2(0,T;H)$ and u_n in $C([0,T];H)$, as claimed. □

Using the same method we can state that the solution $(v_{n\lambda})$ of (4.6) is bounded with respect to n for any fixed $\lambda > 0$. Since (4.6) already contains the Yosida approximation A^n_λ of A^n, we avoid the new parameter μ and work directly with (4.6). One obtains estimates similar to (4.20), (4.29), and (4.32), where $\|y_{n\lambda}\|$, $\|z_{n\lambda}\|$, $\|A^n_{\sqrt{\lambda}}a\|$, $\|A^n_{\sqrt{\lambda}}b\|$ are bounded with respect to n, for every given $\lambda > 0$. Indeed, by (4.2) we have the convergences $y_{n\lambda} \to y_\lambda$, $z_{n\lambda} \to z_\lambda$, $A^n_{\sqrt{\lambda}}a \to A_{\sqrt{\lambda}}a$, $A^n_{\sqrt{\lambda}}b \to A_{\sqrt{\lambda}}b$ as $n \to \infty$ for every $\lambda > 0$, therefore

$$\begin{aligned} \sup_{n\in\mathbb{N}} \|y_{n\lambda}\| &= B_\lambda < \infty, & \sup_{n\in\mathbb{N}} \|z_{n\lambda}\| &= C_\lambda < \infty, \\ \sup_{n\in\mathbb{N}} \|A^n_{\sqrt{\lambda}}a\| &= D_\lambda < \infty, & \sup_{n\in\mathbb{N}} \|A^n_{\sqrt{\lambda}}b\| &= E_\lambda < \infty. \end{aligned} \tag{4.54}$$

These lead to the following result.

Lemma 4.3. *For every fixed $\lambda > 0$, $v'_{n\lambda}(0)$, $v'_{n\lambda}(T)$ are bounded in H with respect to n, $\{v'_{n\lambda}\}$, $\{v''_{n\lambda}\}$ are bounded in $L^2(0,T;H)$ and $\{v_{n\lambda}\}$ is bounded in $C([0,T];H)$.*

Repeating the proof of Lemma 4.1 for problem (4.3), we get the following.

Lemma 4.4. *The solution w_λ of (4.3) is bounded in $C([0,T];H)$, w'_λ, w''_λ are bounded in $L^2(0,T;H)$ and $w'_\lambda(0)$, $w'_\lambda(T)$ are bounded in H.*

Now we are going to estimate each term in (4.7) and (4.8). We begin with the following lemma.

Lemma 4.5. *Under the hypotheses of Theorem 3.1, for every given $\lambda > 0$,*

$$\begin{aligned} \limsup_{n\to\infty} |u'_n - w'_{n\lambda}| \le c_6\Big(&\|A_{\sqrt{\lambda}}a\|^{1/2} + \|A_{\sqrt{\lambda}}b\|^{1/2} + \|y_\lambda\|^{1/2} \\ &+ \|z_\lambda\|^{1/2} + 1\Big)\Big(\|a - y_\lambda\|^{1/2} + \|b - z_\lambda\|^{1/2}\Big), \end{aligned} \tag{4.55}$$

$$\begin{aligned} \limsup_{n\to\infty} |u_n - w_{n\lambda}|_C \le c_7\Big(&\|A_{\sqrt{\lambda}}a\|^{1/2} + \|A_{\sqrt{\lambda}}b\|^{1/2} + \|y_\lambda\|^{1/2} \\ &+ \|z_\lambda\|^{1/2} + 1\Big)\Big(\|a - y_\lambda\|^{1/2} + \|b - z_\lambda\|^{1/2}\Big) + \|a - y_\lambda\|. \end{aligned} \tag{4.56}$$

Proof. Subtracting (3.3) and the equation from (4.5), multiplying by $(\tilde{r}/p)(u_n - w_{n\lambda})$, and integrating over $[0,T]$ by parts, we get via the monotonicity of A^n,

$$\tilde{r}(T)\big(u'_n(T) - w'_{n\lambda}(T), b_n - z_{n\lambda}\big) - \big(u'_n(0) - w'_{n\lambda}(0), a_n - y_{n\lambda}\big) \ge \int_0^T \tilde{r}\|u'_n - w'_{n\lambda}\|^2 dt, \tag{4.57}$$

so

$$c\,|\,u_n' - w_{n\lambda}'\,|^2 \le \widetilde{r}(T)||u_n'(T) - w_{n\lambda}'(T)|| \cdot ||b_n - z_{n\lambda}|| + ||u_n'(0) - w_{n\lambda}'(0)|| \cdot ||a_n - y_{n\lambda}||. \tag{4.58}$$

According to (4.9) and Lemma 4.2, this yields (4.55). Next, from

$$u_n(t) - w_{n\lambda}(t) = a_n - y_{n\lambda} + \int_0^T (u_n' - w_{n\lambda}')(s)ds, \quad t \in [0,T], \tag{4.59}$$

and (4.55), we derive (4.56). □

For the second terms in (4.7) and (4.8), we can also find upper bounds with the aid of y_λ, z_λ, $A_{\sqrt{\lambda}}a$, $A_{\sqrt{\lambda}}b$.

LEMMA 4.6. *Suppose that the above hypotheses hold and let $w_{n\lambda}$, $v_{n\lambda}$ be the solutions of boundary value problems (4.5) and (4.6), respectively. Then*

$$\limsup_{n\to\infty} |\,w_{n\lambda}' - v_{n\lambda}'\,| \le c_8\sqrt{\lambda}\Big(||A_{\sqrt{\lambda}}a|| + ||A_{\sqrt{\lambda}}b|| + ||y_\lambda|| + ||z_\lambda|| + 1\Big), \tag{4.60}$$

$$\limsup_{n\to\infty} |\,w_{n\lambda} - v_{n\lambda}\,|_C \le c_8\sqrt{T\lambda}\Big(||A_{\sqrt{\lambda}}a|| + ||A_{\sqrt{\lambda}}b|| + ||y_\lambda|| + ||z_\lambda|| + 1\Big). \tag{4.61}$$

Proof. One subtracts (4.5) and (4.6), multiplies by $(\widetilde{r}/p)(w_{n\lambda} - v_{n\lambda})$, and integrates from $t = 0$ to $t = T$ to obtain

$$\int_0^T \Big(\big(\widetilde{r}w_{n\lambda}' - \widetilde{r}v_{n\lambda}'\big)', w_{n\lambda} - v_{n\lambda}\Big)dt = \int_0^T \frac{\widetilde{r}}{p}(\alpha_{n\lambda} - A_\lambda^n v_{n\lambda}, w_{n\lambda} - v_{n\lambda})dt. \tag{4.62}$$

Here we have denoted for simplicity by $\alpha_{n\lambda}$ the element $pw_{n\lambda}'' + rw_{n\lambda}' - f_n \in A^n w_{n\lambda}$. Integrating by parts and writing $v_{n\lambda}$ in the right-hand side as $v_{n\lambda} = J_\lambda^n v_{n\lambda} + \lambda A_\lambda^n v_{n\lambda}$, we obtain via the monotonicity of A^n,

$$\int_0^T \widetilde{r}||w_{n\lambda}' - v_{n\lambda}'||^2 dt \le \lambda \int_0^T \frac{\widetilde{r}}{p}(\alpha_{n\lambda} - A_\lambda^n v_{n\lambda}, A_\lambda^n w_{n\lambda})dt. \tag{4.63}$$

This, together with $(\alpha_{n\lambda} - A_\lambda^n v_{n\lambda}, A_\lambda^n v_{n\lambda}) \le (1/2)\|\alpha_{n\lambda}\|^2$, implies

$$\int_0^T \widetilde{r}||w_{n\lambda}' - v_{n\lambda}'||^2 dt \le C\lambda\Big(1 + |\,w_{n\lambda}''\,|^2 + |\,w_{n\lambda}'\,|^2\Big), \tag{4.64}$$

and in view of (4.10) and (4.11), we arrive at (4.60). With the aid of the equality $w_{n\lambda}(t) - v_{n\lambda}(t) = \int_0^t (w_{n\lambda}' - v_{n\lambda}')(s)ds$, $t \in [0,T]$, we can see that (4.61) is also verified. □

Analogously with Lemmas 4.5 and 4.6, we derive the following results.

Lemma 4.7. *If u and w_λ are the solutions of the boundary value problems (3.1)-(3.2) and (4.3), then for every $\lambda > 0$,*

$$|u' - w'_\lambda| \leq c_9\left(\|a - y_\lambda\|^{1/2} + \|b - z_\lambda\|^{1/2}\right),$$
$$|u - w_\lambda|_C \leq \|a - y_\lambda\| + c_{10}\left(\|a - y_\lambda\|^{1/2} + \|b - z_\lambda\|^{1/2}\right) \tag{4.65}$$

with c_9, c_{10} being positive constants.

Lemma 4.8. *If w_λ and v_λ are the solutions of (4.3) and (4.4), then*

$$|w'_\lambda - v'_\lambda| \leq c_{11}\sqrt{\lambda}, \qquad |w_\lambda - v_\lambda|_C \leq c_{12}\sqrt{\lambda}. \tag{4.66}$$

Finally, it will be established that $v'_{n\lambda} - v'_\lambda$ and $v_{n\lambda} - v_\lambda$ tend to 0 as $n \to \infty$, for all $\lambda > 0$, in $L^2(0,T;H)$ and in $C([0,T];H)$, respectively.

Lemma 4.9. *Suppose the assumptions of Theorem 3.1 are satisfied. Then, for every $\lambda > 0$,*

$$\lim_{n\to\infty} |v'_{n\lambda} - v'_\lambda| = 0, \qquad \lim_{n\to\infty} |v_{n\lambda} - v_\lambda|_C = 0. \tag{4.67}$$

Proof. Subtract (4.6) and (4.4), multiply by $(\tilde{r}/p)(v_{n\lambda} - v_\lambda)$, and integrate over $[0,T]$, deducing thus the equality

$$\begin{aligned}
&\tilde{r}(T)\left(v'_{n\lambda}(T) - v'_\lambda(T), z_{n\lambda} - z_\lambda\right) - \left(v'_{n\lambda}(0) - v'_\lambda(0), y_{n\lambda} - y_\lambda\right) - \int_0^T \tilde{r}\|v'_{n\lambda} - v'_\lambda\|^2 dt \\
&\quad = \int_0^T \frac{\tilde{r}}{p}\left(A^n_\lambda v_{n\lambda} - A^n_\lambda v_\lambda, v_{n\lambda} - v_\lambda\right)dt + \int_0^T \frac{\tilde{r}}{p}\left(A^n_\lambda v_\lambda - A_\lambda v_\lambda, v_{n\lambda} - v_\lambda\right)dt \\
&\qquad + \int_0^T \frac{\tilde{r}}{p}\left(f_n - f, v_{n\lambda} - v_\lambda\right)dt
\end{aligned} \tag{4.68}$$

or, in view of the monotonicity of A^n_λ,

$$\begin{aligned}
c|v'_{n\lambda} - v'_\lambda|^2 &\leq \tilde{r}(T)\|v'_{n\lambda}(T) - v'_\lambda(T)\| \cdot \|z_{n\lambda} - z_\lambda\| + \|v'_{n\lambda}(0) - v'_\lambda(0)\| \cdot \|y_{n\lambda} - y_\lambda\| \\
&\quad + \left[\left(\int_0^T \frac{\tilde{r}}{p}\|A^n_\lambda v_\lambda - A_\lambda v_\lambda\|^2 dt\right)^{1/2} + \left(\int_0^T \frac{\tilde{r}}{p}\|f_n - f\|^2 dt\right)^{1/2}\right] \\
&\quad \times \left(\int_0^T \frac{\tilde{r}}{p}\|v_{n\lambda} - v_\lambda\|^2 dt\right)^{1/2}.
\end{aligned} \tag{4.69}$$

According to the boundedness from Lemma 4.3, this leads to

$$|v'_{n\lambda} - v'_\lambda|^2 \leq k_\lambda\left(\|y_{n\lambda} - y_\lambda\| + \|z_{n\lambda} - z_\lambda\| + |A^n_\lambda v_\lambda - A_\lambda v_\lambda| + |f_n - f|\right) \tag{4.70}$$

for all $\lambda > 0$, $n \in \mathbb{N}$, where k_λ is independent of n. By (4.2) we infer that

$$A^n_\lambda v_\lambda(t) \longrightarrow A_\lambda v_\lambda(t) \quad \text{uniformly on } [0,T], \text{ as } n \longrightarrow \infty. \tag{4.71}$$

Using this, together with $(H6)$ and the other convergences from (4.2) into (4.70), we find the first part of (4.67). The second limit is immediate.

The end of the proof of Theorem 3.1. We come back to (4.7) and (4.8) and apply Lemmas 4.5–4.9. Therefore, for small $\lambda > 0$,

$$\begin{aligned}\limsup_{n\to\infty} |u_n - u|_C \\ &\leq c_{13}\left(||A_{\sqrt{\lambda}}a||^{1/2} + ||A_{\sqrt{\lambda}}b||^{1/2} + ||y_\lambda||^{1/2} + ||z_\lambda||^{1/2} + 1\right)\left(||a - y_\lambda||^{1/2} + ||b - z_\lambda||^{1/2}\right) \\ &\quad + 2||a - y_\lambda|| + c_8\sqrt{T\lambda}(||A_{\sqrt{\lambda}}a|| + ||A_{\sqrt{\lambda}}b|| + ||y_\lambda|| + ||z_\lambda|| + 1) + c_{12}\sqrt{\lambda},\end{aligned} \tag{4.72}$$

where $c_{13} > 0$ is independent of λ. A similar inequality is available for $\limsup_{n\to\infty} |u_n' - u'|$.

Taking into account the boundedness of $A_{\sqrt{\lambda}}a$ and $A_{\sqrt{\lambda}}b$ and the convergences $y_\lambda \to a$, $z_\lambda \to b$, we may pass to the limit as $\lambda \to 0$ in the above inequality and conclude that $u_n(t) \to u(t)$ as $n \to \infty$, uniformly on $[0,T]$. Analogously, $u_n' \to u'$ in $L^2(0,T;H)$ and the proof is complete. □

5. Internal approximations

In this section, we give a numerical approximation of the solution u of the problem

$$\begin{gathered} pu''(t) + ru'(t) = Au(t), \quad 0 < t < T, \\ u(0) = a, \qquad u(T) = b, \end{gathered} \tag{5.1}$$

by the solution u_N of an internal scheme of approximation.

Suppose that H is a separable real Hilbert space, provided with the scalar product $(\cdot,\cdot)$ and the corresponding norm $\|\cdot\|$ and

$$p, r : [0,T] \longrightarrow \mathbb{R} \quad \text{are continuous}, \qquad p(t) \geq c > 0 \quad \forall t \in [0,T]. \tag{5.2}$$

Consider the univoque operator $A : H \to H$ satisfying the following assumption:

$(H8)$ A is monotone, hemicontinuous, and everywhere defined on H.

Then A is maximal monotone in H (see [13, page 40]), and therefore for all $a, b \in H$, problem (5.1) has a unique solution $u \in W^{2,2}(0,T;H)$ (see [1]).

Let $\{e_i\}_{i=1}^{\infty}$ be an orthonormal basis in H. For any fixed positive integer N, denote by P_N the orthogonal projector given by $P_N x = \sum_{i=1}^{N}(x,e_i)e_i$ for all $x \in H$ and let $H_N = P_N H$. It is known that $P_N^2 = P_N$ and P_N is selfadjoint, that is, $(P_N x, y) = (x, P_N y)$ for all $x, y \in H$ (see, e.g., [17]).

One defines the operator $A_N : D(A_N) = H_N \subset H \to H$, $A_N = P_N A$. So, for every $\widetilde{x}_N = P_N x \in H_N$ (with $x \in H$), we have $A_N \widetilde{x}_N \in H_N$ and

$$A_N \widetilde{x}_N = P_N A P_N x = P_N A\left(\sum_{i=1}^{N}(x,e_i)e_i\right). \tag{5.3}$$

It is easy to check that, in view of $(H8)$, the operator A_N is monotone, hemicontinuous, univoque, and everywhere defined on H_N. Consequently, it is maximal monotone in H_N.

Consider now the approximating problem

$$\begin{aligned} pu_N''(t) + ru_N'(t) &= A_N u_N(t), \quad 0 < t < T, \\ u_N(0) = P_N a, &\qquad u_N(T) = P_N b. \end{aligned} \tag{5.4}$$

It is clear that (5.4) has a unique solution $u_N \in W^{2,2}(0,T;H_N)$.

Assume in addition that $A0 = 0$ and A is bounded, that is, it maps bounded sets onto bounded sets.

We now show that

$$(I + \lambda A_N)^{-1} P_N x \longrightarrow (I + \lambda A)^{-1} x \quad (N \longrightarrow \infty),\ \forall \lambda > 0,\ \forall x \in H. \tag{5.5}$$

To do this, we put $y_N = (I + \lambda A_N)^{-1} P_N x$ and $y = (I + \lambda A)^{-1} x$. Therefore, we get $y_N \in H_N$ and

$$y_N - P_N y_N + \lambda (P_N A y_N - P_N A y) = 0. \tag{5.6}$$

Multiplying by $y_N - P_N y$ in H, we obtain $\| y_N - P_N y\|^2 + \lambda(P_N A y_N - P_N A y, y_N - P_N y) = 0$. Since P_N is selfadjoint, $P_N^2 = P_N$, and $P_N y_N = y_N$, one deduces that

$$\| y_N - P_N y \|^2 + \lambda (A y_N - A y, y_N - y) + \lambda (A y_N - A y, y - P_N y) = 0. \tag{5.7}$$

The sequence $\{y_N\}$ is bounded in H for every fixed $\lambda > 0$. Indeed, since $A0 = 0$ and $(I + \lambda A_N)^{-1}$ is a contraction, it follows that

$$\| y_N \| = \left\| (I + \lambda A_N)^{-1} P_N x - (I + \lambda A_N)^{-1} 0 \right\| \le \| P_N x \|. \tag{5.8}$$

Hence, $\{y_N\}$ is bounded in H.

Passing to the superior limit as $N \to \infty$ in (5.7) and using the monotonicity and the boundedness of A, we find that $y_N \to y$ in H as $N \to \infty$, that is, (5.5) holds.

Using again the boundedness of A and that fact that P_N is selfadjoint with $P_N^2 = P_N$, we can easily show that $A_N P_N a$ and $A_N P_N b$ are bounded in H. Thus condition $(H5)$ is verified.

As a consequence of Theorem 3.1, we state the following internal approximating result.

Proposition 5.1. *Assume that (5.2) holds, $A : H \to H$ is a bounded operator satisfying $(H8)$, $A0 = 0$ and $a, b \in H$ are given. Denoting by u and u_N the unique solutions of the boundary values problems (5.1) and (5.4), respectively, where $A_N = P_N A : H_N \to H_N$, the convergences $u_N(t) \to u(t)$ uniformly on $[0,T]$ and $u_N' \to u'$ in $L^2(0,T;H)$ as $N \to \infty$ are obtained.*

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