ON A SINGULAR SUBLINEAR POLYHARMONIC PROBLEM

SONIA BEN OTHMAN

This paper deals with a class of singular nonlinear polyharmonic equations on the unit ball B in $\mathbb{R}^n$ ($n \geq 2$) where the combined effects of a singular and a sublinear term allow us by using the Schauder fixed point theorem to establish an existence result for the following problem: $(-\Delta)^m u = \varphi(\cdot, u) + \psi(\cdot, u)$ in B (in the sense of distributions), $u > 0$, $\lim_{|x|\to 1} u(x)/(1-|x|)^{m-1} = 0$. Our approach is based on estimates for the polyharmonic Green function on B with zero Dirichlet boundary conditions.

1. Introduction

In this paper, we study the existence and the asymptotic behaviour of a positive solution for the following higher-order singular problem involving sublinear term

$$(-\Delta)^m u = \varphi(\cdot, u) + \psi(\cdot, u) \text{ in } B \quad \text{(in the sense of distributions)},$$

$$u > 0,$$

$$\lim_{|x|\to 1} \frac{u(x)}{\left(1 - |x|\right)^{m-1}} = 0, \quad\quad (P)$$

where B is the unit ball in $\mathbb{R}^n$ ($n \geq 2$) and m is a positive integer.

Our motivation to study the problem (P) comes from [11], where Shi and Yao investigated the existence of nonnegative solutions for the following elliptic problem:

$$\Delta u + k(x)u^{-\gamma} + \lambda u^\alpha = 0 \quad \text{in } \Omega,$$

$$u(x) > 0, \quad x \in \Omega, \quad\quad (1.1)$$

where Ω is a bounded $C^{1,1}$ domain in $\mathbb{R}^n$ ($n \geq 2$), γ, α are two constants in $(0,1)$, λ is

a real parameter, and k is a Hölder continuous function in $\overline{\Omega}$. The pure singular elliptic equation

$$\Delta u + p(x)u^{-\gamma} = 0, \quad \gamma > 0, \ x \in D \subseteq \mathbb{R}^n, \tag{1.2}$$

has been extensively studied for both bounded and unbounded domain D in $\mathbb{R}^n$ ($n \geq 2$). We refer to ([3–5, 7–9], and references therein) for various existence and uniqueness results related to solutions for above equation. On the other hand, the problem (P) with a sublinear term $\psi(\cdot, u)$ and a singular term $\varphi(\cdot, u) = 0$, has been studied in the unit ball by Mâagli et al. in [10]. Thus a natural question to ask is, for more general singular and sublinear terms combined in the nonlinearity, whether or not the problem (P) has a solution, which we aim to study in this paper.

Our tools are based essentially on some inequalities satisfied by the Green function $G_{m,n}$ of $(-\Delta)^m$ in B with Dirichlet boundary conditions $(\partial/\partial \nu)^j u = 0; 0 \leq j \leq m - 1$. This allows us to state some properties of functions in the Kato class $K_{m,n}(B) = K_{m,n}$ which was introduced in [1] (see Definition 1.1 below).

In [2], Boggio showed that $G_{m,n}$ is positive and it is given by

$$G_{m,n}(x, y) = k_{m,n}|x - y|^{2m-n} \int_1^{[x,y]/|x-y|} \frac{(v^2 - 1)^{m-1}}{v^{n-1}} dv, \tag{1.3}$$

where $k_{m,n}$ is a positive constant and $[x,y]^2 = |x - y|^2 + (1 - |x|^2)(1 - |y|^2)$ for $x, y \in B$. Note that the positivity does not hold for the Green function of the m-polyharmonic operator in arbitrary bounded domain (see, e.g., [6]).

Only for $m = 1$, we do not have this restriction.

Definition 1.1. A Borel measurable function q in B belongs to the class $K_{m,n}$ if q satisfies the following condition:

$$\lim_{\alpha \to 0} \left(\sup_{x \in B} \int_{B \cap B(x,\alpha)} \left(\frac{\delta(y)}{\delta(x)} \right)^m G_{m,n}(x, y) \, |q(y)| \, dy \right) = 0, \tag{1.4}$$

where $\delta(x) = 1 - |x|$ is the Euclidian distance between x and ∂B.

We refer in this paper to the potential of a measurable nonnegative function f, defined on B by

$$Vf(x) = \int_B G_{m,n}(x, y) f(y) dy. \tag{1.5}$$

Our plan in this paper is as follows. Section 2 is devoted to collect some preliminary results about the Green function $G_{m,n}(x, y)$ and the class $K_{m,n}$. In Section 3, we suppose that φ and ψ satisfy the following hypotheses.

(H_1) φ is a nonnegative Borel measurable function on $B \times (0, \infty)$, continuous and nonincreasing with respect to the second variable.

(H_2) For all $c > 0$, $x \mapsto \varphi(x, c(\delta(x))^m)/(\delta(x))^{m-1} \in K_{m,n}$.

(H_3) ψ is a nonnegative Borel measurable function on $B \times (0, \infty)$, continuous with respect to the second variable such that there exist a nontrivial nonnegative function p

and a nonnegative function q on B satisfying for $x \in B$ and $t > 0$,

$$x \longmapsto \frac{q(x)}{(\delta(x))^{m-1}} \in K_{m,n}, \tag{1.6}$$

$$p(x)f(t) \leq \psi(x,t) \leq q(x)g(t) \quad \text{for } (x,t) \in B \times (0,\infty), \tag{1.7}$$

where f is a measurable nondecreasing function on $[0,\infty)$ satisfying

$$\lim_{t \to 0^+} \frac{f(t)}{t} = +\infty, \tag{1.8}$$

and g is a nonnegative measurable function locally bounded on $[0,\infty)$ satisfying

$$\limsup_{t \to \infty} \frac{g(t)}{t} < \|Vq\|_\infty. \tag{1.9}$$

Using a fixed point argument, we will state the following existence result.

THEOREM 1.2. *Under hypotheses (H_1)–(H_3), problem (P) has a positive continuous solution u satisfying for each $x \in B$,*

$$a(\delta(x))^m \leq u(x) \leq V(\varphi(\cdot, a\delta(\cdot)^m))(x) + bVq(x), \tag{1.10}$$

where a, b are positive constants.

Typical examples of nonlinearities satisfying (H_1)–(H_3) are

$$\varphi(x,t) = p(x)(\delta(x))^{m\gamma+m-1} t^{-\gamma} \tag{1.11}$$

for $\gamma \geq 0$, and

$$\psi(x,t) = q(x)(\delta(x))^{m-1} t^\alpha \mathrm{Log}(1 + t^\beta) \tag{1.12}$$

for $\alpha, \beta \geq 0$ such that $\alpha + \beta < 1$, where p and q are nonnegative functions in $K_{m,n}$.

Notations. In order to simplify our statements, we define some convenient notation.
  (i) $B := \{x \in \mathbb{R}^n; |x| < 1\}$ with $n \geq 2$.
  (ii) We denote $s \wedge t = \min(s,t)$ and $s \vee t = \max(s,t)$ for $s,t \in \mathbb{R}$.
  (iii) For $(x,y) \in B^2$, $[x,y]^2 = |x-y|^2 + (1-|x|^2)(1-|y|^2)$.
  (iv) $\mathcal{B}(B)$ denotes the set of Borel measurable functions in B and $\mathcal{B}^+(B)$ is the set of nonnegative ones.
  (v) $C_0(B) := \{\omega$ continuous on B and $\lim_{|x| \to 1} \omega(x) = 0\}$. We recall that $C_0(B)$ is a Banach space with the uniform norm $\|\omega\|_\infty = \sup_{x \in B} |\omega(x)|$.
  (vi) Let f and g be two positive functions on a set S. We call, $f \sim g$, if there is $c > 0$ such that

$$\frac{1}{c}g(x) \leq f(x) \leq cg(x) \quad \forall x \in S, \tag{1.13}$$

$f \preceq g$, if there is $c > 0$ such that

$$f(x) \leq cg(x) \quad \forall x \in S. \tag{1.14}$$

(vii) For any $f \in \mathcal{B}(B)$, we put

$$\|\varphi\|_B := \sup_{x \in B} \int_B \left(\frac{\delta(y)}{\delta(x)}\right)^m G_{m,n}(x,y) \, |f(y)| \, dy. \tag{1.15}$$

2. Properties of the Green function and the Kato class $K_{m,n}$

The existence result that we are going to prove suggests to collect in this section some estimates on the Green function and some properties of functions belonging to the Kato class $K_{m,n}$.

PROPOSITION 2.1. *For each $m \geq 1$, the following estimates are satisfied on B^2:*

$$G_{m,n}(x,y) \preceq (\delta(x)\delta(y))^{m-1} G_{1,n}(x,y), \tag{2.1}$$

$$(\delta(x)\delta(y))^m \preceq G_{m,n}(x,y). \tag{2.2}$$

For $n > 2m$,

$$G_{m,n}(x,y) \preceq \frac{1}{|x-y|^{n-2m}}. \tag{2.3}$$

For $n < 2m$,

$$G_{m,n}(x,y) \preceq \frac{(\delta(x)\delta(y))^m}{[x,y]^n}. \tag{2.4}$$

For $n = 2m$,

$$G_{m,n}(x,y) \preceq \frac{(\delta(x))^{m-1}(\delta(y))^{m-1}}{[x,y]^{2(m-1)}} \operatorname{Log}\left(1 + \frac{\delta(x)\delta(y)}{|x-y|^2}\right). \tag{2.5}$$

Proof. The proof of assertions (2.1)–(2.4) can be found in [1]. If $n = 2m$, it follows from (1.3) that for (x,y) in B^2, we have

$$G_{m,n}(x,y) \preceq \int_1^{[x,y]/|x-y|} \frac{1}{v}\left(1 - \frac{1}{v^2}\right)^{m-1} dv$$

$$\preceq \left(1 - \frac{|x-y|^2}{[x,y]^2}\right)^{m-1} \operatorname{Log}\left(\frac{[x,y]}{|x-y|}\right) \tag{2.6}$$

$$\preceq \frac{(\delta(x))^{m-1}(\delta(y))^{m-1}}{[x,y]^{2(m-1)}} \operatorname{Log}\left(1 + \frac{\delta(x)\delta(y)}{|x-y|^2}\right),$$

which gives (2.5). $\qquad \square$

PROPOSITION 2.2 (see [10]). *Let q be a nonnegative function in $K_{m,n}$. Then the following assertions hold:*

(i) $\|q\|_B < \infty$,

(ii) *the function $x \mapsto (\delta(x))^{2m-1} q \in L^1(B)$. In the sequel, we will give some estimates on the radial potential*

$$x \longmapsto \int_{S^{n-1}} G_{m,n}(x, r\omega) d\sigma(\omega), \tag{2.7}$$

where σ is the normalized measure on the unit sphere S^{n-1} of $\mathbb{R}^n$.

PROPOSITION 2.3. *For $x \in B$, and $r \in (0, 1)$,*

$$\int_{S^{n-1}} G_{m,n}(x, r\omega) d\sigma(\omega) \preceq (\delta(x))^{m-1} (1-r)^{m-1} \rho(|x| \vee r), \tag{2.8}$$

where for $t \in (0, 1)$,

$$\rho(t) = \begin{cases} \dfrac{1-t}{t^{(n-2m)^+}} & \text{if } n \neq 2m, \\[4mm] \operatorname{Log} \dfrac{1}{t} & \text{if } n = 2m, \end{cases} \tag{2.9}$$

and $(n - 2m)^+ = \max(0, n - 2m)$.

Proof. To show the claim, we use inequalities in Proposition 2.1.

For $x \in B$ and $r \in (0, 1)$, we distinguish three cases.

(i) If $n > 2m$, we obtain by (2.1) and elementary calculus that

$$\int_{S^{n-1}} G_{m,n}(x, r\omega) d\sigma(\omega) \preceq (\delta(x))^{m-1}(1-r)^{m-1} \int_{S^{n-1}} G_{1,n}(x, r\omega) d\sigma(\omega)$$

$$\preceq (\delta(x))^{m-1}(1-r)^{m-1} \frac{1 - |x| \vee r}{(|x| \vee r)^{n-2}}. \tag{2.10}$$

On the other hand, by (2.3) we have

$$\int_{S^{n-1}} G_{m,n}(x, r\omega) d\sigma(\omega) \preceq \int_{S^{n-1}} \frac{d\sigma(\omega)}{|x - r\omega|^{n-2m}}$$

$$\preceq \int_0^\pi \frac{(\sin\theta)^{n-2}}{(|x|^2 + r^2 - 2r|x|\cos\theta)^{(n-2m)/2}} d\theta$$

$$\preceq \int_0^\pi \frac{(\sin\theta)^{n-2}}{[((|x| \wedge r) - (|x| \vee r)\cos\theta)^2 + (|x| \vee r)^2 (\sin\theta)^2]^{(n-2m)/2}} d\theta$$

$$\preceq \frac{1}{(|x| \vee r)^{n-2m}}. \tag{2.11}$$

Using the fact that for each $a, b > 0$, we have $\min(a, b) \sim ab/(a+b)$, we obtain

$$\int_{S^{n-1}} G_{m,n}(x, r\omega) d\sigma(\omega)$$

$$\preceq \min\left(\frac{1}{(|x| \vee r)^{n-2m}}, (\delta(x))^{m-1}(1-r)^{m-1}\frac{1-|x| \vee r}{(|x| \vee r)^{(n-2)}}\right)$$

$$\preceq \frac{(1-|x| \vee r)^m (1-|x| \wedge r)^{m-1}}{(|x| \vee r)^{n-2m}}\left(\frac{1}{(|x| \vee r)^{2m-1} + (1-|x| \vee r)^{2m-1}}\right) \qquad (2.12)$$

$$\preceq \frac{(1-|x| \vee r)^m (1-|x| \wedge r)^{m-1}}{(|x| \vee r)^{n-2m}}.$$

(ii) If $n = 2m$, then we have from (2.5)

$$\int_{S^{n-1}} G_{m,n}(x, r\omega) d\sigma(\omega)$$

$$\preceq (\delta(x))^{m-1}(1-r)^{m-1}\int_0^\pi \frac{(\sin\theta)^{2m-2}}{(|x|^2 r^2 + 1 - 2r|x|\cos\theta)^{m-1}}$$

$$\times \mathrm{Log}\left(1 + \frac{(1-|x|^2)(1-r^2)}{(|x|^2 + r^2 - 2r|x|\cos\theta)}\right) d\theta$$

$$\preceq (\delta(x))^{m-1}(1-r)^{m-1}\int_0^\pi \frac{(\sin\theta)^{2(m-1)}}{\left((|x|r - \cos\theta)^2 + (\sin\theta)^2\right)^{m-1}} \qquad (2.13)$$

$$\times \mathrm{Log}\left(1 + \frac{(1-|x|^2)(1-r^2)}{(|x|^2 + r^2 - 2r|x|\cos\theta)}\right) d\theta$$

$$\preceq (\delta(x))^{m-1}(1-r)^{m-1}\int_0^\pi \mathrm{Log}\left(1 + \frac{(1-|x|^2)(1-r^2)}{(|x|^2 + r^2 - 2r|x|\cos\theta)}\right) d\theta$$

$$\preceq (\delta(x))^{m-1}(1-r)^{m-1}\mathrm{Log}\left(\frac{1}{|x| \vee r}\right).$$

(iii) If $n < 2m$, then applying (2.4) we have by elementary calculus

$$\int_{S^{n-1}} G_{m,n}(x, r\omega) d\sigma(\omega)$$

$$\preceq (\delta(x))^m (1-r)^m \int_0^\pi \frac{(\sin\theta)^{n-2}}{(|x|^2 r^2 + 1 - 2r|x|\cos\theta)^{n/2}}$$

$$\preceq (\delta(x))^m (1-r)^m \int_0^\pi \frac{(\sin\theta)^{n-2}}{\left((|x|r - \cos\theta)^2 + (\sin\theta)^2\right)^{(n-2)/2}} \frac{1}{(|x|^2 r^2 + 1 - 2r|x|\cos\theta)} d\theta$$

$$\leq (\delta(x))^m (1-r)^m \int_0^\pi \frac{1}{(|x|^2 r^2 + 1 - 2r|x|\cos\theta)} d\theta$$

$$\leq (\delta(x))^m (1-r)^m \frac{1}{1-r|x|} \leq (1-|x| \vee r)^m (1-|x| \wedge r)^{m-1}.$$

(2.14)

This completes the proof. □

We characterize in the following the radial functions in the class $K_{m,n}$.

PROPOSITION 2.4. *Let a be a radial function in B, then the following assertions are equivalent:*

(1) $a \in K_{m,n}$,
(2) $\int_0^1 r^{n-1}(1-r)^{2(m-1)}\rho(r)|a(r)|dr < \infty$,
where ρ is the function defined by (2.9).

Proof. See [1, Proposition 3.8]. □

Example 2.5. Let q be the function defined in B by

$$q(x) = \frac{1}{(\delta(x))^\lambda (\mathrm{Log}\, 2/\delta(x))^\mu}.$$

(2.15)

By Proposition 2.4, $q \in K_{m,n}$ if and only if $\lambda < 2m$ and $\mu \in \mathbb{R}$ or $\lambda = 2m$ and $\mu > 1$.

Example 2.6. Let $p > \max(n/2m, 1)$, let $q \geq 1$ such that $1/p + 1/q = 1$, and let a be a radial function in $L^p(B)$. Then for $\lambda < 2m - 1/p$ and $\mu \in \mathbb{R}$ or $\lambda = 2m - 1/p$ and $\mu > 1/q$, the function

$$x \longmapsto \frac{a(x)}{\delta(x)^\lambda (\mathrm{Log}\, 2/\delta(x))^\mu}$$

(2.16)

is in $K_{m,n}$.

Indeed, using the Hölder inequality we obtain

$$\int_0^1 r^{n-1} \frac{(1-r)^{2(m-1)-\lambda}}{(\mathrm{Log}\,(2/(1-r)))^\mu} \rho(r) |a(r)| dr$$

$$\leq \|a\|_p \left(\int_0^1 r^{n-1} (\rho(r))^q \frac{(1-r)^{[2(m-1)-\lambda]q}}{(\mathrm{Log}\,(2/(1-r)))^{\mu q}} dr \right)^{1/q}$$

$$= \|a\|_p \left[\left(\int_0^{1/2} r^{n-1} (\rho(r))^q \frac{(1-r)^{[2(m-1)-\lambda]q}}{(\mathrm{Log}\,(2/(1-r)))^{\mu q}} dr \right)^{1/q} \right.$$

(2.17)

$$\left. + \left(\int_{1/2}^1 r^{n-1} (\rho(r))^q \frac{(1-r)^{[2(m-1)-\lambda]q}}{(\mathrm{Log}\,(2/(1-r)))^{\mu q}} dr \right)^{1/q} \right]$$

$$= \|a\|_p (I_1 + I_2).$$

Since for $r \in (0, 1/2]$, we have $1 - r \sim 1$, then we deduce from (2.9) that $I_1 < \infty$.

On the other hand, using that $\text{Log}(1/r) \sim 1 - r$, for $r \in [1/2, 1)$, then we obtain by (2.9),

$$I_2 \preceq \left(\int_4^\infty \frac{dr}{t^{(2m-\lambda-1)q+2}(\text{Log}\,t)^{\mu q}} \right)^{1/q}. \tag{2.18}$$

Hence, we have obviously $I_2 < \infty$, for $\lambda < 2m - 1/p$ and $\mu \in \mathbb{R}$ or $\lambda = 2m - 1/p$ and $\mu > 1/q$. So, the result follows from Proposition 2.4.

Remark 2.7. For $x \in B$, put $\varphi_{\lambda,\mu}(x) = 1/\delta(x)^\lambda(\text{Log}(2/\delta(x)))^\mu$. Then it follows from (2.2) that there exists a constant $c > 0$ such that for each $a \in \mathcal{B}^+(B)$,

$$c\delta(x)^m \int_B (\delta(y))^m a(y)\varphi_{\lambda,\mu}(x)dy \leq V(a\varphi_{\lambda,\mu})(x). \tag{2.19}$$

In the next proposition, we will give upper estimates on $V(a\varphi_{\lambda,\mu})$.

PROPOSITION 2.8. *Let $p > \max(n/2m, 1)$ and let $q \geq 1$ such that $1/p + 1/q = 1$. Then for $\lambda < m + 1 - 1/p$ and $\mu \in \mathbb{R}$ or for $\lambda = m + 1 - 1/p$ and $\mu > 1/q$, there exists a constant $c > 0$, such that for each nonnegative radial function $a \in L^p(B)$ and $x \in B$, the following estimates hold:*

$$V(a\varphi_{\lambda,\mu})(x) \leq c\|a\|_p \times \begin{cases} (\delta(x))^{2m-\lambda-(1/p)} \left(\text{Log} \dfrac{2}{\delta(x)} \right)^{-\mu} & \text{if } m - \dfrac{1}{p} < \lambda < m+1 - \dfrac{1}{p}, \mu \in \mathbb{R}, \\[2ex] (\delta(x))^m & \text{if } \lambda < m - \dfrac{1}{p}, \mu \in \mathbb{R}, \\[2ex] (\delta(x))^m \left(\text{Log}\,\text{Log} \dfrac{4}{\delta(x)} \right)^{1/q} & \text{if } \lambda = m - \dfrac{1}{p}, \mu = \dfrac{1}{q}, \\[2ex] (\delta(x))^m \left(\text{Log} \dfrac{2}{\delta(x)} \right)^{1/q-\mu} & \text{if } \lambda = m - \dfrac{1}{p}, \mu < \dfrac{1}{q}, \\[2ex] (\delta(x))^m & \text{if } \lambda = m - \dfrac{1}{p}, \mu > \dfrac{1}{q}, \\[2ex] (\delta(x))^{m-1} \left(\text{Log} \dfrac{2}{\delta(x)} \right)^{1/q-\mu} & \text{if } \lambda = m+1 - \dfrac{1}{p}, \mu > \dfrac{1}{q}. \end{cases}$$

$$\tag{2.20}$$

Proof. Let $p > \max(n/2m, 1)$, let $q \geq 1$ such that $1/p + 1/q = 1$ and let $a(x) = a(|x|)$ be a nonnegative function in $L^p(B)$. Using Proposition 2.3 and the Hölder inequality, we

obtain that for each $x \in B$,

$$V(a\varphi_{\lambda,\mu})(x) \le c \int_0^1 r^{n-1} \frac{a(r)}{(1-r)^\lambda (\text{Log}(2/(1-r)))^\mu} \left(\int_{S^{n-1}} G_{m,n}(x, r\omega) d\sigma(\omega) \right) dr$$

$$\le c(\delta(x))^{m-1} \int_0^1 r^{n-1} \frac{(1-r)^{m-\lambda-1}}{(\text{Log}(2/(1-r)))^\mu} \rho(|x| \vee r) a(r) dr$$

$$\le c(\delta(x))^{m-1} \|a\|_p \left(\int_0^1 r^{n-1} \frac{(1-r)^{(m-\lambda-1)q}}{(\text{Log}(2/(1-r)))^{\mu q}} (\rho(|x| \vee r))^q dr \right)^{1/q}.$$

$$(2.21)$$

First, if $|x| \in (0, 1/2]$, then we deduce by (2.9) that

$$\left(\int_0^1 r^{n-1} \frac{(1-r)^{(m-\lambda-1)q}}{(\text{Log}(2/(1-r)))^{\mu q}} (\rho(|x| \vee r))^q dr \right)^{1/q}$$

$$\le \left(\int_0^1 r^{n-1} \frac{(1-r)^{(m-\lambda-1)q}}{(\text{Log}(2/(1-r)))^{\mu q}} (\rho(r))^q dr \right)^{1/q} \le 1.$$

$$(2.22)$$

Now, if $|x| \in [1/2, 1)$, we have $\rho(x) \le (1 - |x|)$ and

$$\left(\int_0^1 r^{n-1} \frac{(1-r)^{(m-\lambda-1)q}}{(\text{Log}(2/(1-r)))^{\mu q}} (\rho(|x| \vee r))^q dr \right)^{1/q}$$

$$\le \delta(x) \left(\int_0^{|x|} r^{n-1} \frac{(1-r)^{(m-\lambda-1)q}}{(\text{Log}(2/(1-r)))^{\mu q}} dr \right)^{1/q} + \left(\int_{|x|}^1 r^{n-1} \frac{(1-r)^{(m-\lambda)q}}{(\text{Log}(2/(1-r)))^{\mu q}} dr \right)^{1/q}$$

$$= I(x) + J(x).$$

$$(2.23)$$

By elementary calculus, we obtain that for $|x| \in [1/2, 1)$,

$$I(x) \sim \delta(x) \left(\int_0^{1/2} r^{n-1} dr + \int_{1/2}^{|x|} (1-r)^{(m-\lambda-1)q} \left(\text{Log}\left(\frac{2}{1-r} \right) \right)^{-\mu q} dr \right)^{1/q}$$

$$\sim \delta(x) \left[1 + \left(\int_4^{2/(1-|x|)} t^{(\lambda+1-m)q-2} (\text{Log}\, t)^{-\mu q} dr \right)^{1/q} \right].$$

$$(2.24)$$

Hence, we distinguish the following cases.
(i) If $\lambda > m - 1/p$, then we have

$$I(x) \sim (\delta(x))^{m-\lambda+1/q} \left(\text{Log}\, \frac{2}{\delta(x)} \right)^{-\mu}.$$

$$(2.25)$$

(ii) If $\lambda < m - 1/p$, then we have

$$I(x) \sim \delta(x).$$

$$(2.26)$$

(iii) If $\lambda = m - 1/p$, then we have

$$
I(x) \sim \begin{cases}
\delta(x)\left(\text{Log}\,\text{Log}\,\dfrac{2}{1-|x|}\right)^{1/q} & \text{if } \mu = \dfrac{1}{q}, \\[3mm]
\delta(x)\left(\text{Log}\,\dfrac{2}{1-|x|}\right)^{1/q-\mu} & \text{if } \mu < \dfrac{1}{q}, \\[3mm]
\delta(x) & \text{if } \mu > \dfrac{1}{q}.
\end{cases}
\tag{2.27}
$$

On the other hand, we have

$$
J(x) \sim \left(\int_{2/(1-|x|)}^{\infty} t^{(\lambda-m)q-2}(\text{Log}\,t)^{-\mu q}\right)^{1/q}.
\tag{2.28}
$$

So, by elementary calculus, we obtain

$$
J(x) \sim \begin{cases}
(\delta(x))^{m-\lambda+1/q}\left(\text{Log}\,\dfrac{2}{\delta(x)}\right)^{-\mu} & \text{if } \lambda < m+1-\dfrac{1}{p},\ \mu \in \mathbb{R}, \\[3mm]
\left(\text{Log}\,\dfrac{2}{\delta(x)}\right)^{1/q-\mu} & \text{if } \lambda = m+1-\dfrac{1}{p},\ \mu > \dfrac{1}{q}.
\end{cases}
\tag{2.29}
$$

Hence, for $|x| \in [1/2, 1)$, we have

$$
\left(\int_0^1 r^{n-1}\frac{(1-r)^{(m-\lambda-1)q}}{(\text{Log}\,(2/(1-r)))^{\mu q}}\,(\rho(|x| \vee r))^q\,dr\right)^{1/q}
$$

$$
\preccurlyeq \begin{cases}
(\delta(x))^{m-\lambda+1/q}\left(\text{Log}\,\dfrac{2}{\delta(x)}\right)^{-\mu} & \text{if } m-\dfrac{1}{p} < \lambda < m+1-\dfrac{1}{p},\ \mu \in \mathbb{R}, \\[3mm]
\delta(x)\left(\text{Log}\,\dfrac{2}{\delta(x)}\right)^{1/q-\mu} & \text{if } \lambda = m-\dfrac{1}{p},\ \mu < \dfrac{1}{q}, \\[3mm]
\delta(x)\left(\text{Log}\,\text{Log}\,\dfrac{2}{\delta(x)}\right)^{1/q} & \text{if } \lambda = m-\dfrac{1}{p},\ \mu = \dfrac{1}{q}, \\[3mm]
\delta(x) & \text{if } \lambda = m-\dfrac{1}{p},\ \mu > \dfrac{1}{q}, \\[3mm]
\delta(x) & \text{if } \lambda < m-\dfrac{1}{p},\ \mu \in \mathbb{R}, \\[3mm]
\left(\text{Log}\,\dfrac{2}{\delta(x)}\right)^{1/q-\mu} & \lambda = m+1-\dfrac{1}{p},\ \mu > \dfrac{1}{q}.
\end{cases}
\tag{2.30}
$$

Thus, by combination of the two cases, we obtain the result.   □

3. Proof of the main existence result

In this section, we aim at proving Theorem 1.2, so we need the following result (see [10, Proposition 4]).

PROPOSITION 3.1. *For a fixed nonnegative function q in $K_{m,n}$,*

$$\mathcal{M}_q := \{f \in \mathcal{B}^+(B);\ f \le q\}, \tag{3.1}$$

then the family of functions

$$\left\{\frac{1}{(\delta(x))^{m-1}} \int_B (\delta(y))^{m-1} G_{m,n}(x,y) f(y) dy;\ f \in \mathcal{M}_q\right\} \tag{3.2}$$

is relatively compact in $C_0(B)$.

Proof of Theorem 1.2. Assuming H_1–H_3, we will use a fixed point argument to construct a solution of the problem (P).

By (1.6), the function $\theta(x) = q(x)/(\delta(x))^{m-1} \in K_{m,n}$, then using Proposition 3.1, we obtain

$$\left\|V\left((\delta(\cdot))^{m-1}\theta\right)\right\|_\infty = \|Vq\|_\infty < \infty. \tag{3.3}$$

Consequently, we deduce by (1.7) and (1.8), that p is a nontrivial nonnegative function in $L^1_{\mathrm{loc}}(B)$, then there exists a compact K in B such that we have

$$0 < \alpha := \int_K (\delta(y))^m p(y) dy < \infty. \tag{3.4}$$

Put $\beta := \min\{(\delta(x))^m;\ x \in K\}$, then from (1.8), we deduce that there exists a constant $a > 0$ such that

$$c\alpha f(a\beta) \ge a. \tag{3.5}$$

Take $0 < \eta < 1/\|Vq\|_\infty$, then we deduce by (1.9) that there exists $\rho > 0$ such that for $t \ge \rho$, we have $g(t) \le \eta t$. Hence, put $\gamma = \sup_{0 \le t \le \rho} g(t)$, then we obtain

$$0 \le g(t) \le \eta t + \gamma \quad \forall t \ge 0. \tag{3.6}$$

Finally, it follows from (2.2) and (1.6) that there exists a constant $c_1 > 0$ such that for each $x \in B$,

$$c_1(\delta(x))^m \le Vq(x), \tag{3.7}$$

and using further the hypothesis (H_2) and the Proposition 3.1, we deduce that $\|V(\varphi(\cdot, a(\delta(\cdot))^m))\|_\infty < \infty$. Hence, let $b = \max\{a/c_1, (\eta\|V(\varphi(\cdot, a(\delta(\cdot))^m))\|_\infty + \gamma)/(1 - \eta\|Vq\|_\infty)\}$ and consider the closed convex set

$$\Lambda := \{u \in C_0(B) : a(\delta(\cdot))^m \le u \le V(\varphi(\cdot, a(\delta(\cdot))^m)) + bVq\}. \tag{3.8}$$

Obviously, by (3.7) we have that the set Λ is nonempty. Define the integral operator T on Λ by

$$Tu(x) := \int_B G_{m,n}(x,y)[\varphi(y,u(y)) + \psi(y,u(y))]dy \quad \forall x \in B. \tag{3.9}$$

Let us prove that $T\Lambda \subset \Lambda$. Let $u \in \Lambda$ and $x \in B$, then by (1.7) and (3.6) we have

$$\begin{aligned}
Tu(x) &\leq V(\varphi(\cdot, a(\delta(\cdot))^m))(x) + \int_B G_{m,n}(x,y)q(y)g(u(y))dy \\
&\leq V(\varphi(\cdot, a(\delta(\cdot))^m))(x) + \int_B G_{m,n}(x,y)q(y)[\eta u(y) + \gamma]dy \\
&\leq V(\varphi(\cdot, a(\delta(\cdot))^m))(x) \\
&\quad + \int_B G_{m,n}(x,y)q(y)[\eta(\|V(\varphi(\cdot, a(\delta(\cdot))^m))\|_\infty + b\|Vq\|_\infty) + \gamma]dy.
\end{aligned} \tag{3.10}$$

Since $\eta(\|V(\varphi(\cdot, a(\delta(\cdot))^m))\|_\infty) + \gamma \leq b(1 - \eta\|Vq\|_\infty)$, then

$$\begin{aligned}
\int_B G_{m,n}(x,y)&q(y)[\eta(\|V(\varphi(\cdot, a(\delta(\cdot))^m))\|_\infty + b\|Vq\|_\infty) + \gamma]dy \\
&\leq b\int_B G_{m,n}(x,y)q(y)dy = bVq(x), \\
Tu(x) &\leq V(\varphi(\cdot, a(\delta(\cdot))^m))(x) + bVq(x).
\end{aligned} \tag{3.11}$$

Moreover, from the monotonicity of f and using the relations (1.7), (2.2), and (3.5), there exists a constant $c > 0$ such that for each $x \in B$, we have

$$\begin{aligned}
Tu(x) &\geq \int_B G_{m,n}(x,y)\psi(y,u(y))dy \geq c(\delta(x))^m \int_B (\delta(y))^m p(y)f(u(y))dy \\
&\geq cf(a\beta)\left(\int_K (\delta(y))^m p(y)dy\right)(\delta(x))^m \geq a(\delta(x))^m.
\end{aligned} \tag{3.12}$$

On the other hand, we have that for each $u \in \Lambda, \varphi(\cdot,u) \leq \varphi(\cdot, a(\delta(\cdot))^m)$ and $\psi(\cdot,u) \leq g(u)q \leq bq$, which implies that

$$\varphi(\cdot,u) + \psi(\cdot,u) \in \mathcal{M}_{(\varphi(\cdot, a(\delta(\cdot))^m) + bq)}. \tag{3.13}$$

So we deduce by (H$_2$), (1.6), and Proposition 3.1 that $T\Lambda$ is relatively compact in $C_0(B)$. In particular, for all $u \in \Lambda$, $Tu \in C_0(B)$ and we deduce that $T\Lambda \subset \Lambda$.

Next, we prove the continuity of T in Λ. Let $(u_k)_k$ be a sequence in Λ which converges uniformly to a function u in Λ. Then since φ and ψ are continuous with respect to the second variable, we deduce by the dominated convergence theorem that

$$Tu_k(x) \longrightarrow Tu(x) \quad \text{as } x \longrightarrow \infty. \tag{3.14}$$

Now, since $T\Lambda$ is relatively compact in $C_0(B)$, then we obtain the uniform convergence. Hence T is a compact operator mapping from Λ to itself. So by the Schäuder fixed point

theorem, there exists a function $u \in \Lambda$ such that

$$u(x) = \int_B G_{m,n}(x,y)[\varphi(y,u(y)) + \psi(y,u(y))]dy \quad \forall x \in B. \tag{3.15}$$

Finally, we need to verify that u is a positive continuous solution for the problem (P). Indeed, since the functions $\theta(x) = q/(\delta(\cdot))^{m-1}$ and $x \mapsto \varphi(\cdot, a(\delta(\cdot x))^m)/(\delta(x \cdot))^{m-1}$ are in $K_{m,n}$, we deduce by (3.13) and Proposition 2.2 that the function

$$\varphi(\cdot, u) + \psi(\cdot, u) \in L^1_{\mathrm{loc}}(B) \tag{3.16}$$

holds. Moreover, since $u \in C_0(B)$, we deduce from (3.15) that $V(\varphi(\cdot, u) + \psi(\cdot, u))$ is in $L^1_{\mathrm{loc}}(B)$. Hence u satisfies (in the sense of distributions) the equation

$$(-\Delta)^m u = \varphi(\cdot, u) + \psi(\cdot, u) \quad \text{in } B. \tag{3.17}$$

Moreover, since u satisfies

$$a\delta(x) \le \frac{u(x)}{(\delta(x))^{m-1}} \le \frac{1}{(\delta(x))^{m-1}}(V(\varphi(\cdot, a\delta^m))(x) + bVq(x)), \tag{3.18}$$

then it follows from hypotheses (H_2), (1.6), and Proposition 3.1 that $\lim_{|x| \to 1}(u(x)/(\delta(x))^{m-1}) = 0$. This completes the proof. $\qquad\square$

Example 3.2. Let $\alpha, \beta \ge 0$ such that $0 \le \alpha + \beta < 1$, $\max(\sigma, \lambda) < m$, and $\gamma > 0$. Then the problem

$$(-\Delta)^m u = (\delta(x))^{m\gamma - \sigma}(u(x))^{-\gamma} + (\delta(x))^{-\lambda}(u(x))^{\alpha} \mathrm{Log}\left(1 + (u(x))^{\beta}\right) \quad \text{in } B,$$

$$u > 0,$$

$$\lim_{|x| \to 1} \frac{u(x)}{(\delta(x))^{m-1}} = 0 \tag{3.19}$$

has a positive solution $u \in C_0(B)$ satisfying

$$a(\delta(x))^m \le u(x) \le b(\delta(x))^m, \tag{3.20}$$

where a, b are positive constants.

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Sonia Ben Othman: Département de Mathematiques, Faculté des Sciences de Tunis,
Campus Universitaire, 1060 Tunis, Tunisia
E-mail address: sonia.benothman@fsb.rnu.tn