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Research Article

On the Generalized Hyers-Ulam Stability of a Cauchy-Jensen Functional Equation

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In 2006, W. G. Park and J. H. Bae investigated the Hyers-Ulam stability of a Cauchy-Jensen functional equation. In this paper, we improve their results and obtain better results for a Cauchy-Jensen functional equation. Also, we establish new theorems for the generalized Hyers-Ulam stability of a Cauchy-Jensen functional equation.

1. Introduction

The stability problem of functional equations originated from a question of Ulam [1] concerning the stability of group homomorphisms. Given a group G_1, a metric group (G_2,d), and $\varepsilon > 0$, does there exist a $\delta > 0$ such that if $h: G_1 \to G_2$ satisfies

$$d(h(xy),h(x)h(y)) < \delta \tag{1.1}$$

for all $x, y \in G_1$, then a homomorphism $H: G_1 \to G_2$ exists with

$$d(h(x),H(x)) < \varepsilon \tag{1.2}$$

for all $x \in G_1$? If the answer is affirmative, we would say that the equation of homomorphism $H(xy) = H(x)H(y)$ is stable.

In 1941, Hyers [2] gave a first affirmative answer to the question of Ulam for Banach spaces. In 1951, Bourgin [3] was the second author to treat the Ulam stability problem. In 1978, Gruber [4] remarked that Ulam's problem is of particular interest in probability theory, and in the case of functional equations of different types, and Rassias [5] and then Găvruţa [6] obtained the generalized results of Hyer's theorem which allow the Cauchy difference to be unbounded. The stability problems of several functional equations have been extensively investigated by a number of authors [7–16]. Rassias [17] and Jun et al. [18] established the Hyers-Ulam stability of a Cauchy-Jensen functional equation which has a different type from the one mentioned in this paper.

Throughout this paper, let X and Y be vector spaces. A mapping $g : X \to Y$ is called a Cauchy mapping (resp., a Jensen mapping) if g satisfies the functional equation $g(x+y) = g(x) + g(y)$ (resp., $2g((x+y)/2) = g(x) + g(y)$).

A mapping $f : X \times X \to Y$ is called a Cauchy-Jensen mapping [19] if f satisfies the system of equations

$$\begin{aligned} f(x+y,z) &= f(x,z) + f(y,z), \\ 2f\left(x, \frac{y+z}{2}\right) &= f(x,y) + f(x,z). \end{aligned} \tag{1.3}$$

It is easy to see that a mapping $f : X \times X \to Y$ is a Cauchy-Jensen mapping if and only if the mapping f satisfies the functional equation

$$2f\left(x+y, \frac{z+w}{2}\right) = f(x,z) + f(x,w) + f(y,z) + f(y,w) \tag{1.4}$$

for all $x, y, z, w \in X$. For a mapping $g : X \to Y$, consider the functional equation

$$g(x+y+z) + g(x) + g(y) + g(z) = g(x+y) + g(y+z) + g(z+x). \tag{1.5}$$

In 2002, Rassias via his paper entitled "On the Ulam stability of mixed type mappings on restricted domains" investigated the relation (1.5) and established Ulam stability of mixed type mappings: additive and quadratic. In 2006, Park and Bae [19] investigated the relation between (1.4) and (1.5) and obtained the stability of (1.3) and (1.4). In this paper, we improve their results for the relation between (1.4) and (1.5) and obtain better results by adopting the different method of proof. Also we establish new theorems for the generalized Hyers-Ulam stability of a Cauchy-Jensen mapping.

2. The relation of (1.4) and (1.5)

In this section, we improve the Park and Bae results [19] for the relation of (1.4) and (1.5).

Theorem 2.1. *Let $g : X \to Y$ be a mapping satisfying (1.5) and let $f : X \times X \to Y$ be the mapping given by*

$$f(x,y) := \frac{1}{2}\big(g(x+y) - g(-x) - g(y)\big) \tag{2.1}$$

for all $x, y \in X$. Then f satisfies (1.4) for all $x, y \in X$ and

$$g(x) = f(x,x) \tag{2.2}$$

for all $x \in X$.

Proof. Letting $x = y = z = 0$ in (1.5), $g(0) = 0$. Replacing z by $-x-y$ in (1.5),

$$g(x+y)-g(-x-y)=g(x)+g(y)-g(-x)-g(-y) \tag{2.3}$$

for all $x,y \in X$. From (2.1) and the above equality, we have

$$\begin{aligned}2f(x+y,z)&=g(x+y+z)-g(-x-y)-g(z)\\&=-g(x)-g(y)-2g(z)+g(y+z)+g(z+x)+g(x+y)-g(-x-y)\\&=g(y+z)+g(z+x)-g(-x)-g(-y)-2g(z)\\&=2f(x,z)+2f(y,z)\end{aligned} \tag{2.4}$$

for all $x,y,z \in X$. Setting $z = y$ in (1.5) and letting $y = z$ in (1.5), one can obtain

$$\begin{aligned}2g(x+y)&=g(x)+2g(y)+g(x+2y)-g(2y),\\2g(x+z)&=g(x)+2g(z)+g(x+2z)-g(2z),\end{aligned} \tag{2.5}$$

respectively, for all $x,y,z \in X$. By (1.5) and the above equalities,

$$\begin{aligned}4f(x,y+z)&=2g(x+y+z)-2g(-x)-2g(y+z)\\&=-2g(x)-2g(y)-2g(z)-2g(-x)+2g(x+y)+2g(x+z)\\&=g(x+2z)-g(-x)-g(2z)+g(x+2y)-g(-x)-g(2y)\\&=2f(x,2y)+2f(x,2z)\end{aligned} \tag{2.6}$$

for all $x,y,z \in X$. By (2.4) and (2.6), f satisfies (1.4).

Replacing y and z by x and $-x$, respectively, we get

$$g(2x)=3g(x)+g(-x). \tag{2.7}$$

Setting $y = x$ in (2.1) and using (2.7), we see that the equality (2.2) holds. □

THEOREM 2.2. *Let $f : X\times X\to Y$ be a mapping satisfying (1.4), and let $g : X\to Y$ be the mapping given by (2.2) for all $x \in X$. Then g satisfies (1.5) and*

$$f(x,y)+f(y,x)=g(x+y)-\frac{1}{2}\big(g(-x)+g(-y)+g(x)+g(y)\big) \tag{2.8}$$

for all $x,y \in X$. In particular, if $f(x,y) = f(y,x)$ for all $x,y \in X$, then (2.1) holds.

Proof. In (1.4), we easily get

$$f(0,x)=0,\qquad 2f(x,y)=f(2x,y),\qquad 2f(x,y)=f(x,2y)+f(x,0) \tag{2.9}$$

for all $x,y \in X$. Let $f_1 : X\times X\to Y$ and $f_2 : X\to Y$ be the mappings defined by

$$\begin{aligned}f_1(x,y)&:=f(x,y)-f(x,0),\\f_2(x)&:=f(x,0)\end{aligned} \tag{2.10}$$

for all $x, y \in X$. By the definition of f_1, we have

$$\begin{aligned} f_1(x,0) &= 0, \qquad & f_1(0,x) &= 0, \\ 2f_1(x,y) &= f_1(x,2y), \qquad & 2f_1(x,y) &= f_1(2x,y) \end{aligned} \tag{2.11}$$

for all $x, y \in X$. By (2.9) and the above equalities, we get

$$\begin{aligned} f_1(x,y+z) &= f(x,y+z) - f(x,0) \\ &= \frac{1}{2}f(x,2y) + \frac{1}{2}f(x,2z) - f(x,0) \\ &= f(x,y) - \frac{1}{2}f(x,0) + f(x,z) - \frac{1}{2}f(x,0) - f(x,0) \\ &= f_1(x,y) + f_1(x,z), \\ f_1(x+y,z) &= f_1(x,z) + f_1(y,z) \end{aligned} \tag{2.12}$$

for all $x, y, z \in X$. Also we have

$$f_1(x+y,z+w) = f_1(x,z) + f_1(y,z) + f_1(x,w) + f_1(y,w), \tag{2.13}$$

$$\begin{aligned} &f_1(x+y+z,x+y+z) + f_1(x,x) + f_1(y,y) + f_1(z,z) \\ &\quad = f_1(x+y,x+y) + f_1(y+z,y+z) + f_1(z+x,z+x) \end{aligned} \tag{2.14}$$

for all $x, y, z, w \in X$. By the definition of f_2,

$$f_2(x+y) = f(x+y,0) = f(x,0) + f(y,0) = f_2(x) + f_2(y) \tag{2.15}$$

for all $x, y \in X$, and so

$$f_2(x+y+z) + f_2(x) + f_2(y) + f_2(z) = f_2(x+y) + f_2(y+z) + f_2(z+x) \tag{2.16}$$

for all $x, y, z \in X$. Since

$$g(x) := f(x,x) = f_1(x,x) + f_2(x), \tag{2.17}$$

the equality (1.5) holds by (2.14) and (2.16). By (2.13),

$$\begin{aligned} f_1(x,x) &= f_1(2x,2x) + f_1(2x,-x) + f_1(-x,2x) + f_1(-x,-x) \\ &= 4f_1(x,x) + 2f_1(x,-x) + 2f_1(-x,x) + f_1(-x,-x), \\ f_1(-x,-x) &= 4f_1(-x,-x) + 2f_1(x,-x) + 2f_1(-x,x) + f_1(x,x) \end{aligned} \tag{2.18}$$

for all $x \in X$. Hence

$$f_1(x,x) = f_1(-x,-x) \tag{2.19}$$

for all $x \in X$, and

$$\begin{aligned}
&\frac{1}{2}g(x+y) - \frac{1}{4}g(-x) - \frac{1}{4}g(-y) - \frac{1}{4}g(x) - \frac{1}{4}g(y) \\
&\quad = \frac{1}{2}f_1(x+y,x+y) - \frac{1}{4}\big(f_1(-x,-x) + f_1(x,x) + f_1(y,y) + f_1(-y,-y)\big) \\
&\qquad + \frac{1}{2}f_2(x+y) - \frac{1}{4}\big(f_2(-x) + f_2(x) + f_2(y) + f_2(-y)\big) \\
&\quad = \frac{1}{2}\big(f_1(x,x) + f_1(x,y) + f_1(y,x) + f_1(y,y)\big) \\
&\qquad - \frac{1}{2}\big(f_1(x,x) + f_1(y,y)\big) + \frac{1}{2}f_2(x+y) \\
&\quad = \frac{1}{2}\big(f_1(x,y) + f_1(y,x)\big) + \frac{1}{2}f_2(x) + \frac{1}{2}f_2(y) \\
&\quad = \frac{1}{2}\big(f(x,y) + f(y,x)\big)
\end{aligned} \tag{2.20}$$

for all $x, y \in X$. This completes the proof. □

3. Stability of (1.3) and (1.4)

In this section, let Y be a Banach space. For the given mapping $f : X \times X \to Y$, we define

$$\begin{aligned}
Df(x,y,z,w) &:= 2f\left(x+y, \frac{z+w}{2}\right) - f(x,z) - f(x,w) - f(y,z) - f(y,w), \\
D_1 f(x,y,z) &:= f(x+y,z) - f(x,z) - f(y,z), \\
D_2 f(x,y,z) &:= 2f\left(x, \frac{y+z}{2}\right) - f(x,y) - f(x,z)
\end{aligned} \tag{3.1}$$

for all $x, y, z, w \in X$.

Theorem 3.1. *Let $\varphi : X \times X \times X \to [0,\infty)$ and $\psi : X \times X \times X \to [0,\infty)$ be two functions satisfying*

$$\sum_{j=0}^{\infty} \frac{1}{2^j}\varphi(2^j x, 2^j y, z) < \infty, \qquad \sum_{j=0}^{\infty} \frac{1}{2^j}\psi(2^j x, y, z) < \infty, \tag{3.2}$$

$$\sum_{j=0}^{\infty} \frac{1}{2^j}\varphi(x, y, 2^j z) < \infty, \qquad \sum_{j=0}^{\infty} \frac{1}{2^j}\psi(x, 2^j y, 2^j z) < \infty, \tag{3.3}$$

$$\lim_{i\to\infty} \frac{1}{2^i}\sum_{j=0}^{\infty} \frac{1}{2^j}\varphi(2^j x, 2^j y, 2^i z) = 0 \tag{3.4}$$

for all $x, y, z \in X$. Let $f : X \times X \to Y$ be a mapping such that

$$\|D_1 f(x,y,z)\| \le \varphi(x,y,z), \tag{3.5}$$

$$\|D_2 f(x,y,z)\| \le \psi(x,y,z) \tag{3.6}$$

for all $x,y,z \in X$. Then there exist a unique Cauchy-Jensen mapping $F_1 : X \times X \to Y$ and a unique biadditive mapping $F_2 : X \times X \to Y$ such that

$$\|f(x,y) - F_1(x,y)\| \le \sum_{j=0}^{\infty} \frac{1}{2^{j+1}} \varphi(2^j x, 2^j x, y), \tag{3.7}$$

$$\|f(x,y) - f(x,0) - F_2(x,y)\| \le \sum_{j=0}^{\infty} \frac{1}{2^{j+1}} \psi(x, 2^{j+1} y, 0), \tag{3.8}$$

$$F_1(x,y) - F_1(x,0) = F_2(x,y) \tag{3.9}$$

for all $x,y \in X$. The mappings $F_1, F_2 : X \times X \to Y$ are given by

$$F_1(x,y) := \lim_{j\to\infty} \frac{1}{2^j} f(2^j x, y), \qquad F_2(x,y) := \lim_{j\to\infty} \frac{1}{2^j} f(x, 2^j y) \tag{3.10}$$

for all $x,y \in X$.

Proof. Letting $y = x$ and replacing z by y in (3.5), we get

$$\|f(2x,y) - 2f(x,y)\| \le \varphi(x,x,y) \tag{3.11}$$

for all $x,y \in X$. Thus

$$\left\| \frac{1}{2^j} f(2^j x, y) - \frac{1}{2^{j+1}} f(2^{j+1} x, y) \right\| \le \frac{1}{2^{j+1}} \varphi(2^j x, 2^j x, y) \tag{3.12}$$

for all $x,y \in X$. For given integers l,m $(0 \le l < m)$,

$$\left\| \frac{1}{2^l} f(2^l x, y) - \frac{1}{2^m} f(2^m x, y) \right\| \le \sum_{j=l}^{m-1} \frac{1}{2^{j+1}} \varphi(2^j x, 2^j x, y) \tag{3.13}$$

for all $x,y \in X$. By (3.2), the sequence $\{(1/2^j) f(2^j x, y)\}$ is a Cauchy sequence for all $x,y \in X$. Since Y is complete, the sequence $\{(1/2^j) f(2^j x, y)\}$ converges for all $x,y \in X$. Define $F_1 : X \times X \to Y$ by

$$F_1(x,y) := \lim_{j\to\infty} \frac{1}{2^j} f(2^j x, y) \tag{3.14}$$

for all $x,y \in X$. Putting $l = 0$ and taking $m \to \infty$ in (3.13), one can obtain the inequality (3.7). By (3.5) and (3.6),

$$\begin{gathered} \left\| \frac{1}{2^j} D_1 f(2^j x, 2^j y, z) \right\| \le \frac{1}{2^j} \varphi(2^j x, 2^j y, z), \\ \left\| \frac{1}{2^j} D_2 f(2^j x, y, z) \right\| \le \frac{1}{2^j} \psi(2^j x, y, z) \end{gathered} \tag{3.15}$$

for all $x,y,z \in X$ and all j. Letting $j \to \infty$ in the above two inequalities and using (3.2), F_1 is a Cauchy-Jensen mapping. Now let $F_1' : X \times X \to Y$ be another Cauchy-Jensen mapping satisfying (3.7). Then we have

$$\begin{aligned}||F_1(x,y)-F_1'(x,y)|| &\le \frac{1}{2^n}||f(2^nx,y)-F_1(2^nx,y)||+\frac{1}{2^n}||f(2^nx,y)-F_1'(2^nx,y)||\\ &\le \sum_{j=n}^{\infty}\frac{1}{2^{j+1}}\varphi(2^jx,2^jx,y)+\sum_{j=n}^{\infty}\frac{1}{2^{j+1}}\varphi(2^jx,2^jx,y)\end{aligned} \tag{3.16}$$

for all $n\in N$ and $x,y\in X$. As $n\to\infty$, we may conclude that $F_1(x,y)=F_1'(x,y)$ for all $x,y\in X$. Thus such a Cauchy-Jensen mapping $F_1 : X\times X\to Y$ is unique.

Next, replacing y by $2y$ and z by 0 in (3.6), one can obtain

$$\left\|\left(f(x,y)-f(x,0)\right)-\frac{1}{2}\left(f(x,2y)-f(x,0)\right)\right\|\le\frac{1}{2}\psi(x,2y,0) \tag{3.17}$$

for all $x,y\in X$. By the same method as above, F_2 is a unique biadditive mapping which satisfies (3.8), where $F_2(x,y):=\lim_{j\to\infty}(1/2^j)f(x,2^jy)$ for all $x,y\in X$. From (3.17) and the definitions of F_1 and F_2, the equalities

$$\begin{aligned}F_1(x,y)-F_1(x,0)&=\frac{1}{2}\left(F_1(x,2y)-F_1(x,0)\right),\\ F_1(x,y)-F_1(x,0)&=\frac{1}{2^n}F_1(x,2^ny)-\frac{1}{2^n}F_1(x,0),\\ F_2(x,y)&=\frac{1}{2^n}F_2(x,2^ny)\end{aligned} \tag{3.18}$$

hold for all $n\in N$ and for all $x,y\in X$. Hence the inequality

$$\begin{aligned}&||F_1(x,y)-F_1(x,0)-F_2(x,y)||\\ &\quad=\left\|\frac{1}{2^n}F_1(x,2^ny)-\frac{1}{2^n}F_1(x,0)-\frac{1}{2^n}F_2(x,2^ny)\right\|\\ &\quad\le\frac{1}{2^n}||f(x,2^ny)-F_1(x,2^ny)||+\frac{1}{2^n}||f(x,0)-F_1(x,0)||\\ &\qquad+\frac{1}{2^n}||f(x,2^ny)-f(x,2^n0)-F_2(x,2^ny)||\\ &\quad\le\frac{1}{2^n}\sum_{j=0}^{\infty}\frac{1}{2^{j+1}}\left(\varphi(2^jx,2^jx,2^ny)+\varphi(2^jx,2^jx,0)\right)+\sum_{j=n}^{\infty}\frac{1}{2^{j+1}}\psi(x,2^{j+1}y,0)\end{aligned} \tag{3.19}$$

holds for all $n\in N$ and for all $x,y\in X$. Taking $n\to\infty$ and using (3.3) and (3.4), we have (3.9). □

Theorem 3.2. *Let $\varphi,\psi : X\times X\times X\to[0,\infty)$ be two functions satisfying*

$$\sum_{j=0}^{\infty}2^j\varphi\left(\frac{x}{2^j},\frac{y}{2^j},z\right)<\infty,\qquad \sum_{j=0}^{\infty}2^j\psi\left(\frac{x}{2^j},y,z\right)<\infty,\tag{3.20}$$

$$\sum_{j=0}^{\infty}2^j\varphi\left(x,y,\frac{z}{2^j}\right)<\infty,\qquad \sum_{j=0}^{\infty}2^j\psi\left(x,\frac{y}{2^j},\frac{z}{2^j}\right)<\infty,\tag{3.21}$$

$$\lim_{i\to\infty}2^i\sum_{j=0}^{\infty}2^j\varphi\left(\frac{x}{2^j},\frac{y}{2^j},\frac{z}{2^i}\right)=0\tag{3.22}$$

for all $x,y,z\in X$. Let $f : X\times X\to Y$ be a mapping satisfying (3.5) and (3.6) for all $x,y,z\in X$. Then there exist a unique Cauchy-Jensen mapping $F_1 : X\times X\to Y$ and a unique biadditive mapping $F_2 : X\times X\to Y$ such that

$$\|f(x,y)-F_1(x,y)\|\le\sum_{j=0}^{\infty}2^j\varphi\left(\frac{x}{2^{j+1}},\frac{x}{2^{j+1}},y\right),\tag{3.23}$$

$$\|f(x,y)-f(x,0)-F_2(x,y)\|\le\sum_{j=0}^{\infty}2^j\psi\left(x,\frac{y}{2^j},0\right),\tag{3.24}$$

$$F_1(x,y)-F_1(x,0)=F_2(x,y)\tag{3.25}$$

for all $x,y\in X$. The mappings $F_1,F_2 : X\times X\to Y$ are given by

$$F_1(x,y):=\lim_{j\to\infty}2^jf\left(\frac{x}{2^j},y\right),\qquad F_2(x,y):=\lim_{j\to\infty}2^j\left(f\left(x,\frac{y}{2^j}\right)-f(x,0)\right)\tag{3.26}$$

for all $x,y\in X$.

Proof. Replacing x, y, z by $x/2$, $x/2$, y in (3.5), respectively, we have

$$\left\|f(x,y)-2f\left(\frac{x}{2},y\right)\right\|\le\varphi\left(\frac{x}{2},\frac{x}{2},y\right)\tag{3.27}$$

for all $x,y\in X$. Thus

$$\left\|2^jf\left(\frac{x}{2^j},y\right)-2^{j+1}f\left(\frac{x}{2^{j+1}},y\right)\right\|\le 2^j\varphi\left(\frac{x}{2^{j+1}},\frac{x}{2^{j+1}},y\right)\tag{3.28}$$

for all $x,y\in X$. For given integers l, m $(0\le l<m)$,

$$\left\|2^lf\left(\frac{x}{2^l},y\right)-2^mf\left(\frac{x}{2^m},y\right)\right\|\le\sum_{j=l}^{m-1}2^j\varphi\left(\frac{x}{2^{j+1}},\frac{x}{2^{j+1}},y\right)\tag{3.29}$$

for all $x,y\in X$. By (3.20), the sequence $\{2^jf(x/2^j,y)\}$ is a Cauchy sequence for all $x,y\in X$. Since Y is complete, the sequence $\{2^jf(x/2^j,y)\}$ converges for all $x,y\in X$. Define

$F_1 : X \times X \to Y$ by

$$F_1(x, y) := \lim_{j\to\infty} 2^j f\left(\frac{x}{2^j}, y\right) \tag{3.30}$$

for all $x, y \in X$. Putting $l = 0$ and taking $m \to \infty$ in (3.29), one can obtain the inequality (3.23). By (3.5) and (3.6),

$$\begin{gathered} \left\|2^j D_1 f\left(\frac{x}{2^j}, \frac{y}{2^j}, z\right)\right\| \le 2^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}, z\right), \\ \left\|2^j D_2 f\left(\frac{x}{2^j}, y, z\right)\right\| \le 2^j \psi\left(\frac{x}{2^j}, y, z\right) \end{gathered} \tag{3.31}$$

for all $x, y, z \in X$ and all j. Letting $j \to \infty$ in the above two inequalities and using (3.20), F_1 is a Cauchy-Jensen mapping. Now let $F_1' : X \times X \to Y$ be another Cauchy-Jensen mapping satisfying (3.23). Then we have

$$\begin{aligned} \|F_1(x, y) - F_1'(x, y)\| &\le 2^n \left\|f\left(\frac{x}{2^n}, y\right) - F_1\left(\frac{x}{2^n}, y\right)\right\| + 2^n \left\|f\left(\frac{x}{2^n}, y\right) - F_1'\left(\frac{x}{2^n}, y\right)\right\| \\ &\le \sum_{j=n}^{\infty} 2^{j+1} \varphi\left(\frac{x}{2^{j+1}}, \frac{x}{2^{j+1}}, y\right) \end{aligned} \tag{3.32}$$

for all $n \in N$ and $x, y \in X$. As $n \to \infty$, we may conclude that $F_1(x, y) = F_1'(x, y)$ for all $x, y \in X$. Thus such a Cauchy-Jensen mapping $F_1 : X \times X \to Y$ is unique.

Next, replacing z by 0 in (3.6), one can obtain

$$\left\|\left(f(x, y) - f(x, 0)\right) - 2\left(f\left(x, \frac{y}{2}\right) - f(x, 0)\right)\right\| \le \psi(x, y, 0) \tag{3.33}$$

for all $x, y \in X$. By the same method as above, F_2 is a unique biadditive mapping which satisfies (3.24), where $F_2(x, y) := \lim_{j\to\infty} 2^j (f(x, y/2^j) - f(x, 0))$ for all $x, y \in X$. From (3.33) and the definitions of F_1 and F_2, the equalities

$$\begin{gathered} F_1(x, y) - F_1(x, 0) = 2\left(F_1\left(x, \frac{y}{2}\right) - F_1(x, 0)\right), \\ F_1(x, y) - F_1(x, 0) = 2^n F_1\left(x, \frac{y}{2^n}\right) - 2^n F_1(x, 0), \\ F_2(x, y) = 2^n F_2\left(x, \frac{y}{2^n}\right) 2 \end{gathered} \tag{3.34}$$

hold for all $n \in N$ and for all $x, y \in X$. Hence, by (3.23), (3.24), and (3.34), the inequality

$$\begin{aligned}
&||F_1(x,y) - F_1(x,0) - F_2(x,y)|| \\
&\quad = \left\| 2^n F_1\left(x, \frac{y}{2^n}\right) - 2^n F_1(x,0) - 2^n F_2\left(x, \frac{y}{2^n}\right) \right\| \\
&\quad \le 2^n \left\| f\left(x, \frac{y}{2^n}\right) - F_1\left(x, \frac{y}{2^n}\right) \right\| + 2^n ||f(x,0) - F_1(x,0)|| \\
&\qquad + 2^n \left\| f\left(x, \frac{y}{2^n}\right) - f(x,0) - F_2\left(x, \frac{y}{2^n}\right) \right\| \\
&\quad \le 2^n \sum_{j=0}^{\infty} 2^j \left(\varphi\left(\frac{x}{2^{j+1}}, \frac{x}{2^{j+1}}, \frac{y}{2^n}\right) + \varphi\left(\frac{x}{2^{j+1}}, \frac{x}{2^{j+1}}, 0\right) \right) + \sum_{j=n}^{\infty} 2^j \psi\left(x, \frac{y}{2^j}, 0\right)
\end{aligned} \tag{3.35}$$

holds for all $n \in N$ and for all $x, y \in X$. Taking $n \to \infty$ and using (3.21) and (3.22), we have (3.25). □

Theorem 3.3. *Let $\varphi, \psi : X \times X \times X \to [0, \infty)$ be two functions satisfying (3.2), (3.21), and*

$$\lim_{i \to \infty} 2^i \sum_{j=0}^{\infty} \frac{1}{2^j} \varphi\left(2^j x, 2^j y, \frac{z}{2^i}\right) = 0 \tag{3.36}$$

for all $x, y, z \in X$. Let $f : X \times X \to Y$ be a mapping satisfying (3.5) and (3.6) for all $x, y, z \in X$. Then there exist a unique Cauchy-Jensen mapping $F_1 : X \times X \to Y$ and a unique biadditive mapping $F_2 : X \times X \to Y$ satisfying (3.7), (3.9), and (3.24) for all $x, y \in X$. The mappings $F_1, F_2 : X \times X \to Y$ are given by

$$F_1(x,y) = \lim_{j \to \infty} \frac{1}{2^j} f(2^j x, y), \qquad F_2(x,y) = \lim_{j \to \infty} 2^j \left(f\left(x, \frac{y}{2^j}\right) - f(x,0) \right) \tag{3.37}$$

for all $x, y \in X$.

Proof. By the proofs of Theorems 3.1 and 3.2, there exist a unique Cauchy-Jensen mapping $F_1 : X \times X \to Y$ and a unique biadditive mapping $F_2 : X \times X \to Y$ satisfying (3.7) and (3.24) for all $x, y \in X$. The mappings $F_1, F_2 : X \times X \to Y$ are given by (3.37). From (3.33) and the definitions of F_1 and F_2, the equalities in (3.34) hold for all $n \in N$ and for all $x, y \in X$. Hence the inequality

$$\begin{aligned}
&||F_1(x,y) - F_1(x,0) - F_2(x,y)|| \\
&\quad = \left\| 2^n F_1\left(x, \frac{y}{2^n}\right) - 2^n F_1(x,0) - 2^n F_2\left(x, \frac{y}{2^n}\right) \right\| \\
&\quad \le 2^n \left\| f\left(x, \frac{y}{2^n}\right) - F_1\left(x, \frac{y}{2^n}\right) \right\| + 2^n ||f(x,0) - F_1(x,0)|| \\
&\qquad + 2^n \left\| f\left(x, \frac{y}{2^n}\right) - f(x,0) - F_2\left(x, \frac{y}{2^n}\right) \right\| \\
&\quad \le 2^n \sum_{j=0}^{\infty} \frac{1}{2^{j+1}} \left(\varphi\left(2^j x, 2^j x, \frac{y}{2^n}\right) + \varphi(2^j x, 2^j x, 0) \right) + \sum_{j=n}^{\infty} 2^j \psi\left(x, \frac{y}{2^j}, 0\right)
\end{aligned} \tag{3.38}$$

holds for all $n \in N$ and for all $x, y \in X$. Taking $n \to \infty$ and using (3.21) and (3.36), we have (3.9). □

Theorem 3.4. *Let $\varphi, \psi : X \times X \times X \to [0, \infty)$ be two functions satisfying (3.3), (3.20), and*

$$\lim_{i \to \infty} \frac{1}{2^i} \sum_{j=0}^{\infty} 2^j \varphi\left(\frac{x}{2^i}, \frac{y}{2^i}, 2^j z\right) = 0 \tag{3.39}$$

for all $x, y, z \in X$. Let $f : X \times X \to Y$ be a mapping satisfying (3.5) and (3.6) for all $x, y, z \in X$. Then there exist a unique Cauchy-Jensen mapping $F_1 : X \times X \to Y$ and a unique biadditive mapping $F_2 : X \times X \to Y$ satisfying (3.8), (3.9), and (3.23) for all $x, y \in X$. The mappings $F_1, F_2 : X \times X \to Y$ are given by

$$F_1(x, y) = \lim_{j \to \infty} 2^j f\left(\frac{x}{2^j}, y\right), \qquad F_2(x, y) = \lim_{j \to \infty} \frac{1}{2^j} f\left(x, 2^j y\right) \tag{3.40}$$

for all $x, y \in X$.

Proof. By the proofs of Theorems 3.1 and 3.2, there exist a unique Cauchy-Jensen mapping $F_1 : X \times X \to Y$ and a unique biadditive mapping $F_2 : X \times X \to Y$ satisfying (3.8) and (3.23) for all $x, y \in X$. The mappings $F_1, F_2 : X \times X \to Y$ are given by (3.40). From (3.17) and the definitions of F_1 and F_2, the equalities in (3.18) hold for all $n \in N$ and for all $x, y \in X$. Hence we have the inequality

$$\begin{aligned}
&||F_1(x, y) - F_1(x, 0) - F_2(x, y)|| \\
&\quad = \left|\left|\frac{1}{2^n} F_1\left(x, 2^n y\right) - \frac{1}{2^n} F_1(x, 0) - \frac{1}{2^n} F_2\left(x, 2^n y\right)\right|\right| \\
&\quad = \left|\left|\frac{1}{2^n} f\left(x, 2^n y\right) - \frac{1}{2^n} F_1\left(x, 2^n y\right)\right|\right| + \frac{1}{2^n} ||F_1(x, 0)|| \\
&\qquad + \frac{1}{2^n} ||f\left(x, 2^n y\right) - f\left(x, 2^n 0\right) - F_2\left(x, 2^n y\right)|| + \frac{1}{2^n} ||f(x, 0)|| \\
&\quad \le \frac{1}{2^n} \sum_{j=0}^{\infty} 2^j \varphi\left(\frac{x}{2^{j+1}}, \frac{x}{2^{j+1}}, 2^n y\right) + \sum_{j=n}^{\infty} \frac{1}{2^{j+1}} \psi\left(x, 2^{j+1} y, 0\right) \\
&\qquad + \frac{1}{2^n} ||F_1(x, 0)|| + \frac{1}{2^n} ||f(x, 0)||.
\end{aligned} \tag{3.41}$$

Taking $n \to \infty$ and using (3.3) and (3.39), we have (3.9).

From Theorems 3.1 and 3.3, we obtain the stability of the functional equation (1.4) in the following corollary. □

Corollary 3.5. *Let $\varphi_1, \varphi_2 : X \times X \to [0, \infty)$ be two functions satisfying*

$$\sum_{j=0}^{\infty} \frac{1}{2^j} \varphi_1(2^j x, 2^j y) < \infty, \qquad \sum_{j=0}^{\infty} \frac{1}{2^j} \varphi_2(2^j x, 2^j y) < \infty, \tag{3.42}$$

or

$$\sum_{j=0}^{\infty} \frac{1}{2^j} \varphi_1(2^j x, 2^j y) < \infty, \qquad \sum_{j=0}^{\infty} 2^j \varphi_2\left(\frac{x}{2^j}, \frac{y}{2^j}\right) < \infty \tag{3.43}$$

for all $x, y \in X$. Let $f : X \times X \to Y$ be a mapping such that

$$\|Df(x, y, z, w)\| \leq \varphi_1(x, y)\varphi_2(z, w) \tag{3.44}$$

for all $x, y, z, w \in X$. Then there exists a unique Cauchy-Jensen mapping $F : X \times X \to Y$ such that

$$\|f(x, y) - F(x, y)\| \leq \sum_{j=0}^{\infty} \frac{1}{2^{j+2}} \varphi_1(2^j x, 2^j x)\varphi_2(y, y) \tag{3.45}$$

for all $x, y \in X$. The mapping $F : X \times X \to Y$ is given by

$$F(x, y) = \lim_{j \to \infty} \frac{1}{2^j} f(2^j x, y) \tag{3.46}$$

for all $x, y \in X$.

Proof. From (3.44), we have

$$\begin{aligned} \|D_1 f(x, y, z)\| &= \left\|\frac{1}{2} Df(x, y, z, z)\right\| \leq \frac{1}{2}\varphi_1(x, y)\varphi_2(z, z), \\ \|D_2 f(x, y, z)\| &= \left\|Df\left(\frac{x}{2}, \frac{x}{2}, y, z\right) - \frac{1}{2}Df\left(\frac{x}{2}, \frac{x}{2}, y, y\right) - \frac{1}{2}Df\left(\frac{x}{2}, \frac{x}{2}, z, z\right)\right\| \\ &\leq \varphi_1\left(\frac{x}{2}, \frac{x}{2}\right)\left(\varphi_2(y, z) + \frac{1}{2}\varphi_2(y, y) + \frac{1}{2}\varphi_2(z, z)\right) \end{aligned} \tag{3.47}$$

for all $x, y, z \in X$. Let $\varphi, \psi : X \times X \times X \to [0, \infty)$ be the maps defined by

$$\begin{aligned} \varphi(x, y, z) &:= \frac{1}{2}\varphi_1(x, y)\varphi_2(z, z), \\ \psi(x, y, z) &:= \varphi_1\left(\frac{x}{2}, \frac{x}{2}\right)\left(\varphi_2(y, z) + \frac{1}{2}\varphi_2(y, y) + \frac{1}{2}\varphi_2(z, z)\right) \end{aligned} \tag{3.48}$$

for all $x, y, z \in X$. Then we can apply Theorem 3.1 for the case φ_1 and φ_2 satisfying (3.42), and Theorem 3.3 for the case φ_1 and φ_2 satisfying (3.43); we get the desired results.

From Theorems 3.2 and 3.4, we obtain another stability of the functional equation (1.4) in the following corollary. □

Corollary 3.6. *Let $\varphi_1, \varphi_2 : X \times X \to [0, \infty)$ be two functions satisfying*

$$\sum_{j=0}^{\infty} 2^j \varphi_1\left(\frac{x}{2^j}, \frac{y}{2^j}\right) < \infty, \qquad \sum_{j=0}^{\infty} 2^j \varphi_2\left(\frac{x}{2^j}, \frac{y}{2^j}\right) < \infty \tag{3.49}$$

for all $x, y \in X$, or satisfying

$$\sum_{j=0}^{\infty} 2^j \varphi_1\left(\frac{x}{2^j}, \frac{y}{2^j}\right) < \infty, \qquad \sum_{j=0}^{\infty} \frac{1}{2^j} \varphi_2(2^j x, 2^j y) < \infty \tag{3.50}$$

for all $x, y \in X$. Let $f : X \times X \to Y$ be a mapping such that

$$||Df(x, y, z, w)|| \leq \varphi_1(x, y)\varphi_2(z, w) \tag{3.51}$$

for all $x, y, z, w \in X$. Then there exists a unique Cauchy-Jensen mapping $F : X \times X \to Y$ such that

$$||f(x, y) - F(x, y)|| \leq \sum_{j=0}^{\infty} 2^{j-1} \varphi_1\left(\frac{x}{2^{j+1}}, \frac{x}{2^{j+1}}\right) \varphi_2(y, y) \tag{3.52}$$

for all $x, y \in X$. The mapping $F : X \times X \to Y$ is given by

$$F(x, y) := \lim_{j \to \infty} 2^j f\left(\frac{x}{2^j}, y\right) \tag{3.53}$$

for all $x, y \in X$.

Now we obtain the stability of the Cauchy-Jensen functional equation (1.4) on a normed space in the following corollary.

Corollary 3.7. *Let $p, q \neq 1$, θ be nonnegative real numbers and X a normed space. Let $f : X \times X \to Y$ be a mapping such that*

$$||Df(x, y, z, w)|| \leq \theta(\|x\|^p + \|y\|^p)(\|z\|^q + \|w\|^q) \tag{3.54}$$

for all $x, y, z, w \in X$. Then there exists a unique Cauchy-Jensen mapping $F : X \times X \to Y$ such that

$$||f(x, y) - F(x, y)|| \leq \frac{2\theta}{|2 - 2^p|} \|x\|^p \|y\|^q. \tag{3.55}$$

Proof. Let $\varphi_1, \varphi_2 : X \times X \to [0, \infty)$ be the maps defined by

$$\varphi_1(x, y) := \theta(\|x\|^p + \|y\|^p), \qquad \varphi_2(x, y) := \|x\|^q + \|y\|^q \tag{3.56}$$

for all $x, y \in X$. For the cases $0 \leq p < 1$ and $1 < p$, we can apply Corollaries 3.5 and 3.6, respectively. □

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