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Research Article

Meromorphic Functions Sharing a Small Function

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We will study meromorphic functions that share a small function, and prove the following result: let $f(z)$ and $g(z)$ be two transcendental meromorphic functions in the complex plane and let $n \geq 11$ be a positive integer. Assume that $a(z)(\not\equiv 0)$ is a common small function with respect to $f(z)$ and $g(z)$. If $f^n f'$ and $g^n g'$ share $a(z)$ CM, then either $f^n(z)f'(z)g^n(z)g'(z) \equiv a^2(z)$, or $f(z) \equiv tg(z)$ for a constant satisfying $t^{n+1} = 1$. As applications, we give several examples.

1. Introduction and main result

In this paper, a meromorphic function always means a function which is meromorphic in the whole complex plane. Let $f(z)$ be a nonconstant meromorphic function. We use the following standard notations of value distribution theory:

$$\begin{aligned} &T(r,f), m(r,f), N(r,f), \overline{N}(r,f), N\left(r,\frac{1}{f}\right), \overline{N}\left(r,\frac{1}{f}\right), \Theta(a,f) \\ &\quad = 1 - \overline{\lim_{r\to\infty}}\frac{\overline{N}(r,1/(f-a))}{T(r,f)}, \ldots \end{aligned} \tag{1.1}$$

(see [1–3]). We use $S(r,f)$ to denote any function satisfying

$$S(r,f) = o\{T(r,f)\}, \tag{1.2}$$

as $r \to \infty$, possibly outside of a set E with finite measure.

Let $a(z)$ be a meromorphic function in the complex plane. If $T(r,a) = S(r,f)$, then $a(z)$ is called a small function related to $f(z)$.

Let b be a finite complex number. We denote by $N_{2)}(r,1/(f-b))$ the counting function for zeros of $f(z)-b$ (or poles of $1/(f(z)-b)$) with multiplicity at most 2, and by $\overline{N}_{2)}(r,1/(f-b))$ the corresponding one for which multiplicity is not counted. Let $N_{(2}(r,1/(f-b))$ be the counting function for zeros of $f(z)-b$ with multiplicity at least 2 and $\overline{N}_{(2}(r,1/(f-b))$ the corresponding one for which multiplicity is not counted. Set

$$N_2\left(r,\frac{1}{f-b}\right) = \overline{N}\left(r,\frac{1}{f-b}\right) + \overline{N}_{(2}\left(r,\frac{1}{f-b}\right). \tag{1.3}$$

Suppose that $f(z)$ and $g(z)$ are two meromorphic functions, and $a(z)$ is a small function related to both of them. We say that $f(z)$ and $g(z)$ share the small function $a(z)$ CM, if $f(z)-a(z)$ and $g(z)-a(z)$ assume the same zeros with the same multiplicities. We say that $f(z)$ and $g(z)$ share the value a CM if $a(z) \equiv a(\in \overline{C})$ is constant.

In the 1920's, Nevanlinna [2] proved his famous four-valued theorem, which is an important result about uniqueness of meromorphic functions. Then many results about meromorphic functions that share more than or equal to two values have been obtained (see [4]). In 1997, Yang and Hua [5] studied meromorphic functions sharing one value.

Theorem 1.1 (see [5]). *Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions and let $n \geq 11$ be a positive integer. If $f^n(z)f'(z)$ and $g^n(z)g'(z)$ share 1 CM, then either $f(z) = c_1e^{cz}$, $g(z) = c_2e^{-cz}$, where c_1, c_2, and c are three constants satisfying $(c_1c_2)^{n+1}c^2 = -1$, or $f(z) \equiv tg(z)$ for a constant such that $t^{n+1} = 1$.*

Fang and Qiu [6] investigated meromorphic functions sharing fixed point later.

Theorem 1.2 (see [6]). *Let $f(z)$ and $g(z)$ be two nonconstant meromorphic(entire) functions and let $n \geq 11$ $(n \geq 6)$ be a positive integer. If $f^n(z)f'(z)$ and $g^n(z)g'(z)$ share z CM, then either $f(z) = c_1e^{cz^2}$, $g(z) = c_2e^{-cz^2}$, where c_1, c_2, and c are three constants satisfying $4(c_1c_2)^{n+1}c^2 = -1$, or $f(z) \equiv tg(z)$ for a constant such that $t^{n+1} = 1$.*

Recently, Banerjee [7] also studied meromorphic functions sharing one value, generating Theorem 1.1. In this paper, we extend the results above as follows.

Theorem 1.3. *Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions, and let $a(z)(\not\equiv 0)$ be a common small function with respect to them, and let $n \geq 11$ be a positive integer. If $f^n(z)f'(z)$ and $g^n(z)g'(z)$ share $a(z)$ CM, then either $f^n(z)f'(z)g^n(z)g'(z) \equiv a^2(z)$, or $f(z) \equiv tg(z)$ for a constant such that $t^{n+1} = 1$.*

2. Lemmas

In order to prove Theorem 1.3, we need the following lemmas.

Lemma 2.1 (see [4]). *Suppose that $f(z)$ is a nonconstant meromorphic function in the complex plane, and k is a positive integer. Then*

$$N\left(r,\frac{1}{f^{(k)}}\right)\le N\left(r,\frac{1}{f}\right)+k\overline{N}(r,f)+S(r,f). \tag{2.1}$$

Lemma 2.2 (see [3, 4]). *Suppose that $f(z)$ is a nonconstant meromorphic function in the complex plane, and $a(\in C\cup\infty)$ is any complex number. Then,*

$$\sum_{a}\Theta(a,f)\le 2. \tag{2.2}$$

Lemma 2.3 (see [3, 4, 8]). *Let $f(z)$ and $g(z)$ be two meromorphic functions in the complex plane. If f and g share 1 CM, then one of the following cases must occur:*

(i) $T(r,f)+T(r,g)\le 2\{N_2(r,1/f)+N_2(r,1/g)+N_2(r,f)+N_2(r,g)\}+S(r,f)+S(r,g)$,

(ii) *either $f\equiv g$ or $fg\equiv 1$.*

3. Proof of Theorem 1.3

Let $F(z)=f^n(z)f'(z)/a(z)$ and $G(z)=g^n(z)g'(z)/a(z)$. Then we know that $F(z)$ and $G(z)$ share 1 CM. From Lemma 2.1, we have

$$N\left(r,\frac{1}{f'}\right)\le N\left(r,\frac{1}{f}\right)+\overline{N}(r,f)+S(r,f). \tag{3.1}$$

By $T(r,a)=S(r,f)$ and (3.1), we obtain

$$\begin{aligned}
N_2\left(r,\frac{1}{F}\right) &\le N_2\left(r,\frac{1}{f^nf'}\right)+N_2(r,a)\\
&\le 2\overline{N}\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{f'}\right)+\overline{N}_{(2}\left(r,\frac{1}{f'}\right)+S(r,f)\\
&\le 2\overline{N}\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{f'}\right)+S(r,f)\\
&=\frac{2}{n}\left(n\overline{N}\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{f'}\right)\right)+\left(1-\frac{2}{n}\right)N\left(r,\frac{1}{f'}\right)+S(r,f)\\
&\le\frac{2}{n}N\left(r,\frac{1}{f^nf'}\right)+\left(1-\frac{2}{n}\right)\frac{1}{n+1}\left[nN\left(r,\frac{1}{f}\right)+n\overline{N}(r,f)+N\left(r,\frac{1}{f'}\right)\right]+S(r,f)\\
&=\frac{3}{n+1}N\left(r,\frac{1}{f^nf'}\right)+\frac{n-2}{n+1}\overline{N}(r,f)+S(r,f).
\end{aligned} \tag{3.2}$$

Thus we have, by elementary Nevanlinna theory,

$$\begin{aligned}
N_2\left(r,\frac{1}{F}\right)+N_2(r,F) &\le \frac{3}{n+1}N\left(r,\frac{1}{f^nf'}\right)+\frac{n-2}{n+1}\overline{N}(r,f)+N_2(r,f^nf')+S(r,f)\\
&= \frac{3}{n+1}N\left(r,\frac{1}{f^nf'}\right)+\frac{n-2}{n+1}\overline{N}(r,f)+2\overline{N}(r,f)+S(r,f)\\
&= \frac{3}{n+1}N\left(r,\frac{1}{Fa}\right)+\frac{3n}{n+1}\overline{N}(r,f)+S(r,f)\\
&\le \frac{3}{n+1}N\left(r,\frac{1}{F}\right)+\frac{3n}{(n+1)(n+2)}N(r,f^nf')+S(r,f)\\
&\le \frac{3}{n+1}N\left(r,\frac{1}{F}\right)+\frac{3n}{(n+1)(n+2)}N(r,F)+S(r,f)\\
&\le \left(\frac{3}{n+1}+\frac{3n}{(n+1)(n+2)}\right)T(r,F)+S(r,f)\\
&= \frac{6}{n+2}T(r,F)+S(r,f).
\end{aligned} \tag{3.3}$$

On the other hand, since

$$nT(r,f)=T\left(r,\frac{f^nf'}{a}\frac{a}{f'}\right)+S(r,f)\le T(r,F)+2T(r,f)+S(r,f), \tag{3.4}$$

we have $S(r,f)=S(r,F)$, and therefore,

$$N_2\left(r,\frac{1}{F}\right)+N_2(r,F)\le\frac{6}{n+2}T(r,F)+S(r,F). \tag{3.5}$$

Similarly, we have

$$N_2\left(r,\frac{1}{G}\right)+N_2(r,G)\le\frac{6}{n+2}T(r,G)+S(r,G). \tag{3.6}$$

Combining with Lemma 2.3, suppose first that

$$T(r,F)+T(r,G)\le 2\left\{N_2\left(r,\frac{1}{F}\right)+N_2\left(r,\frac{1}{G}\right)+N_2(r,F)+N_2(r,G)\right\}+S(r,F)+S(r,G). \tag{3.7}$$

By (3.5)–(3.7), we obtain

$$\frac{n-10}{n+2}\{T(r,F)+T(r,G)\}\le S(r,F)+S(r,G). \tag{3.8}$$

Since both F and G are transcendental meromorphic functions and $n\ge 11$, then we deduce a contradiction from (3.8).

Therefore, we deduce that either $F(z)G(z)\equiv 1$ or $F(z)\equiv G(z)$ from Lemma 2.3, that is, either $f^n(z)f'(z)g^n(z)g'(z)\equiv a^2(z)$ or $f^{n+1}(z)\equiv g^{n+1}(z)+c$, where c is a constant.

Suppose that $c \neq 0$, then

$$\begin{aligned}\Theta(\infty, f^{n+1}) + \Theta(0, f^{n+1}) + \Theta(c, f^{n+1}) &= \Theta(\infty, f^{n+1}) + \Theta(0, f^{n+1}) + \Theta(0, g^{n+1}) \\ &\geq 3\left(1 - \frac{1}{n+1}\right) \geq \frac{33}{12},\end{aligned} \tag{3.9}$$

which contradicts Lemma 2.2. Thus, $f(z) \equiv tg(z)$ for a constant such that $t^{n+1} = 1$. The proof is complete.

Remark 3.1. At this time, it is not easy to obtain the representation of $f(z)$ and $g(z)$ like in Theorems 1.1 and 1.2. Suppose that either $a(z)$ is an entire function or all of its poles are simple. Set

$$\frac{f'}{f} = m(z), \qquad \frac{g'}{g} = n(z), \tag{3.10}$$

then we get $f(z) = c_1 e^{\int_{z_0}^{z} m(z)dz}$ and $g(z) = c_2 e^{\int_{z_0}^{z} n(z)dz}$, where c_1 and c_2 are two nonzero constants, and the integral path $[z_0, z](z_0 \neq z)$ does not pass the poles of either $m(z)$ or $n(z)$. Combining with the equality in Theorem 1.3

$$f^n(z) f'(z) g^n(z) g'(z) \equiv a^2(z), \tag{3.11}$$

we have

$$(c_1 c_2)^{n+1} e^{(n+1)(\int_{z_0}^{z} m(z)dz + \int_{z_0}^{z} n(z)dz)} m(z) n(z) \equiv a^2(z). \tag{3.12}$$

In particular,

$$f(z) = c_1 e^{c\int_{z_0}^{z} a(z)dz}, \qquad g(z) = c_2 e^{-c\int_{z_0}^{z} a(z)dz}, \tag{3.13}$$

if $m(z) = -n(z) = ca(z)$, where c is a constant such that $(c_1c_2)^{n+1}c^2 = -1$. Hence (3.13) is one of the representations of $f(z)$ and $g(z)$ which can be obtained from (3.11) under the condition.

Example 3.2. If $a(z) = e^z$, then by Theorem 1.3 and the Remark 3.1, we can obtain two representations of $f(z)$ and $g(z)$: $f(z) \equiv tg(z)$ for a constant such that $t^{n+1} = 1$; $f(z) = c_1 e^{ce^z}, g(z) = c_2 e^{-ce^z}$, where c_1, c_2, and c are three constants satisfying $(c_1c_2)^{n+1}c^2 = -1$.

Example 3.3. Suppose that

$$a(z) = \frac{2z^3 - 2z^2 + 1}{z^2 - z}, \tag{3.14}$$

then by Remark 3.1, we have

$$f(z) = c_1 e^{c\int_{z_0}^{z} a(z)dz} = c_1 \left(\frac{z-1}{z}\right)^c e^{cz^2}, \qquad g(z) = c_2 e^{-c\int_{z_0}^{z} a(z)dz} = c_2 \left(\frac{z}{z-1}\right)^c e^{-cz^2}, \tag{3.15}$$

where c_1, c_2 and c are three constants satisfying $(c_1c_2)^{n+1}c^2 = -1$. This is one of the relations of $f(z)$ and $g(z)$.

Example 3.4. If $a(z)$ has a pole of order $m(>1)$, then we cannot get (3.13). Suppose that

$$a(z)=\frac{1+z^2}{z^2}, \tag{3.16}$$

then $f(z)=c_1e^{-c/z}e^{cz}$ and $g(z)=c_2e^{c/z}e^{-cz}$ are not meromorphic functions in the complex plane.

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