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Research Article

Some New Bounds for Mathieu's Series

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Two upper and lower bounds for Mathieu's series are established, which refine to a certain extent a sharp double inequality obtained by Alzer-Brenner-Ruehr in 1998. Moreover, the very closer lower and upper bounds for $\zeta(3)$ are deduced.

1. Introduction

In 1890, Mathieu in [1] defined $S(r)$ for $r > 0$ by

$$S(r) = \sum_{n=1}^{\infty} \frac{2n}{(n^2 + r^2)^2} \tag{1.1}$$

and conjectured that $S(r) < 1/r^2$. We call formula (1.1) Mathieu's series.

There has been a lot of literature about the estimations of $S(r)$ for more than 100 years till 1998, for example, [2–14] and the references therein. In [9], Makai proved that

$$\frac{1}{r^2 + 1/2} < S(r) < \frac{1}{r^2}. \tag{1.2}$$

In 1998, Alzer et al. presented in [2] that

$$\frac{1}{r^2 + 1/2\zeta(3)} < S(r) < \frac{1}{r^2 + 1/6}, \tag{1.3}$$

where ζ denotes the zeta function and the constants $1/2\zeta(3)$ and $1/6$ in (1.3) are the best possible.

After 2000, among other things, several open problems on the estimations and integral representations of generalized Mathieu's series were posed in [15–17] by Guo and Qi. Stimulated by or originated from these open problems, a lot of articles such as [18–37] have been published in variant reputable journals by many mathematicians all over the world.

In this article, by utilizing the well-known telescope technique ever used in [9, 38], we would like to improve or refine the sharp double inequality (1.3) and to establish a very closer double inequality for $\zeta(3)$.

Our main results are the following four theorems.

Theorem 1.1. *For* $r > 0$,

$$S(r) > \frac{1}{r^2 + 1/6 + (r^2+6)/3(9r^2+8)} = \frac{1}{r^2 + 1/2 - 2(4r^2+1)/3(9r^2+8)}. \tag{1.4}$$

Remark 1.2. By standard argument, it is showed readily that inequality (1.4) is better than the left-hand side inequality in (1.3) when $r > 2\sqrt{(5\zeta(3)-6)/(27-11\zeta(3))} = 0.05\ldots$.

Theorem 1.3. *For* $r > 0$,

$$\begin{aligned} \frac{1}{r^2 + 1/6 + 5/6(2r^2+3)} = \frac{1}{r^2 + 1/2 - (4r^2+1)/6(2r^2+3)} &< S(r) \\ &< \frac{1}{r^2 + 1/2 - (4r^2+1)/2\left(2r^2 - 3 + 4\sqrt{r^4+2r^2+5}\right)}. \end{aligned} \tag{1.5}$$

Remark 1.4. It is not difficult to verify that the left-hand side inequality in (1.5) is better than the left-hand side inequality in (1.3) when $r > \sqrt{(8\zeta(3)-9)/2[3-\zeta(3)]} = 0.41\ldots$ and that the right-hand side inequality in (1.5) is better than the right-hand side inequality in (1.3) when $r < \sqrt{239/16} = 3.86\ldots$.

It is important to remark that inequality (1.4) and the left-hand side inequality in (1.5) do not include each other, which can be proved straightforwardly.

Theorem 1.5. *For* $r > 0$,

$$S(r) < \frac{1}{\sqrt{r^4+2r^2+2} - 1}. \tag{1.6}$$

Remark 1.6. It is easy to deduce that inequality (1.6) is better than the right-hand side inequality in (1.3) when $0 < r < \sqrt{23/12} = 1.38\ldots$.

Theorem 1.7. *For* $m \in \mathbb{N}$, *let* $S_3(m) = \sum_{n=1}^{m}(1/n^3)$. *Then*

$$\frac{1}{2m^2 + 2m + 1 - 1/6(m^2+m+3/2)} < \zeta(3) - S_3(m) < \frac{1}{2m^2 + 2m + 1 - 1/6(m^2+m+1)}. \tag{1.7}$$

Remark 1.8. Calculation by Mathematica 5.2 shows that

$$\zeta(3) = 1.202056903159594285399\ldots. \tag{1.8}$$

If taking m from 1 to 9, the sums of the right side term in (1.7) and $S_3(m)$ are

$$\begin{array}{lll} 1.202247191011235955, & 1.202064220183486239, & 1.202057560382342322, \\ 1.202057003155139651, & 1.202056924652726768, & 1.202056909039779896, \\ 1.202056905080018071, & 1.202056903877571143, & 1.202056903458154800. \end{array} \tag{1.9}$$

If taking m from 1 to 9, the sums of the left side term in (1.7) and $S_3(m)$ are

$$\begin{array}{lll} 1.201923076923076923, & 1.202054794520547945, & 1.202056799882886839, \\ 1.202056893315403149, & 1.202056901714344462, & 1.202056902872941459, \\ 1.202056903088695828, & 1.202056903138840387, & 1.202056903152657143. \end{array} \tag{1.10}$$

These numerical computations by mathematic 5.2 reveals that inequalities in (1.7) give much accurate approximations from left and right.

Corollary 1.9. *If* $1 \le \delta < 3/2$ *and* $m \ge \sqrt{(3\delta^2 - \delta + 1/12)/(6 - 4\delta)} - 1$, *then*

$$\zeta(3) < S_3(m) + \frac{1}{2m^2 + 2m + 1 - 1/6(m^2 + m + \delta)}. \tag{1.11}$$

Remark 1.10. In [39, 40], the number $\zeta(3)$ was estimated by using Jordan's inequality and its refinements. In [41, 42], some more general conclusions were obtained.

Remark 1.11. Finally, an open problem is posed: find the best possible constants a and b such that

$$\frac{1}{r^2 + 1/2 - (4r^2 + 1)/12(r^2 + a)} < S(r) < \frac{1}{r^2 + 1/2 - (4r^2 + 1)/12(r^2 + b)} \tag{1.12}$$

holds true for all $r > 0$.

It is clear that $a \le 3/2$ and $b \ge 1/4$.

2. Proofs of theorems and corollary

Now we are in a position to prove our theorems and corollary.

Proof of Theorem 1.1. For $n \in \mathbb{N}$, let

$$w_n(r) = n(n-1) + r^2 + \frac{1}{2} - \frac{\theta}{n^2 + \gamma}, \tag{2.1}$$

where $\theta = (1/3)(r^2 + 1/4)$ and γ is a possible and undetermined positive function of r such that

$$\frac{1}{w_n(r)} - \frac{1}{w_{n+1}(r)} \le \frac{2n}{(n^2 + r^2)^2}. \tag{2.2}$$

Straightforward computation yields that

$$\frac{1}{w_n(r)}-\frac{1}{w_{n+1}(r)}=\frac{2n\{1+\theta(1+1/2n)/(n^2+\gamma)[(n+1)^2+\gamma]\}}{(n^2+r^2)^2+\theta Q(n,r,\gamma)/(n^2+\gamma)[(n+1)^2+\gamma]},\tag{2.3}$$

where

$$\begin{aligned}Q(n,r,\gamma)&=n^4+4n^3+(4\gamma-2r^2-1)n^2\\&\quad+(6\gamma-2r^2-2)n+3\gamma^2+2(1-r^2)\gamma-\frac{2r^2}{3}-\frac{5}{12}.\end{aligned}\tag{2.4}$$

It is easy to see that if

$$\frac{1+1/2n}{Q(n,r,\gamma)}\le\frac{1}{(n^2+r^2)^2},\tag{2.5}$$

then inequality (2.2) holds. Further, inequality (2.5) is equivalent to

$$\begin{aligned}&n^4+4n^3+(4\gamma-2r^2-1)n^2+(6\gamma-2r^2-2)n+3\gamma^2\\&\quad+2(1-r^2)\gamma-\frac{2r^2}{3}-\frac{5}{12}\ge\left(1+\frac{1}{2n}\right)(n^2+r^2)^2,\end{aligned}\tag{2.6}$$

which can be rewritten as

$$\begin{aligned}&7n^3+(8\gamma-8r^2-2)n^2+(12\gamma-6r^2-4)n\\&\quad+6\gamma^2+4(1-r^2)\gamma-2r^4-\frac{4r^2}{3}-\frac{5}{6}-\frac{r^4}{n}\ge 0,\end{aligned}\tag{2.7}$$

which can be further rearranged as

$$\begin{aligned}f(n,\gamma)&\triangleq(n-1)\left[7n^2+(8\gamma-8r^2+5)n+20\gamma-14r^2+1+\frac{r^4}{n}\right]\\&\quad+6\gamma^2+4(6-r^2)\gamma-3r^4-\frac{46}{3}r^2+\frac{1}{6}\ge 0.\end{aligned}\tag{2.8}$$

Direct computation reveals that

$$f\left(n,\frac{9r^2}{8}\right)=(n-1)\left[7n^2+(r^2+5)n+\frac{17}{2}r^2+1+\frac{r^4}{n}\right]+\frac{3}{32}r^4+\frac{35}{3}r^2+\frac{1}{6}>0,\tag{2.9}$$

but

$$f(n,r^2)=(n-1)\left(7n^2+5n+6r^2+\frac{r^4}{n}\right)-r^4+\frac{26}{3}r^2+\frac{1}{6}\tag{2.10}$$

is negative if r is large enough. Consequently, if taking $\gamma=9r^2/8$, then inequality (2.2) is valid. Summing up on both sides of (2.2), with respect to $n=1,2,\ldots$, leads to (1.4). The proof of Theorem 1.1 is finished. □

Proof of Theorem 1.3. Now let us consider the sequence

$$\nu_n(r) = n(n-1) + r^2 + \frac{1}{2} - \frac{\theta}{n(n-1)+\beta} \tag{2.11}$$

for $n \in \mathbb{N}$, where θ and β are two undetermined functions of r, in order that

$$\frac{1}{\nu_n(r)} - \frac{1}{\nu_{n+1}(r)} < \frac{2n}{(n^2+r^2)^2}. \tag{2.12}$$

Direct calculation yields that

$$\frac{1}{\nu_n(r)} - \frac{1}{\nu_{n+1}(r)} = \frac{2n + 2\theta n/(n^2-n+\beta)(n^2+n+\beta)}{(n^2+r^2)^2 + P(n,r,\theta,\beta)/(n^2-n+\beta)(n^2+n+\beta)}, \tag{2.13}$$

where

$$\begin{aligned} P(n,r,\theta,\beta) = {} & \left(r^2 + \frac{1}{4} - 2\theta\right)n^4 + \left(r^2+\frac{1}{4}\right)\beta^2 - \theta\beta(2r^2+1) + \theta^2 \\ & + \left[\left(r^2+\frac{1}{4}\right)(2\beta-1) - \theta(2\beta+2r^2+3)\right]n^2. \end{aligned} \tag{2.14}$$

Letting $r^2 + 1/4 - 2\theta = \theta$ and

$$\left(r^2+\frac{1}{4}\right)(2\beta-1) - \theta(2\beta+2r^2+3) = 2\theta r^2 \tag{2.15}$$

give

$$\theta = \frac{1}{3}\left(r^2+\frac{1}{4}\right), \qquad \beta = r^2 + \frac{3}{2}. \tag{2.16}$$

Consequently,

$$\begin{aligned} P(n,r,\theta,\beta) &= \theta n^4 + 2\theta r^2 n^2 + 3\theta\beta^2 - \theta\beta(2r^2+1) + \theta^2 \\ &= \theta(n^2+r^2)^2 + \theta[3\beta^2 - \beta(2r^2+1) + \theta - r^4] \\ &= \theta(n^2+r^2)^2 + \frac{16}{3}\theta(r^2+1). \end{aligned} \tag{2.17}$$

As a result,

$$\begin{aligned} \frac{1}{\nu_2(r)} - \frac{1}{\nu_{n+1}(r)} &= \frac{2n + 2\theta n/(n^2-n+\beta)(n^2+n+\beta)}{(n^2+r^2)^2 + (\theta(n^2+r^2)^2 + 16\theta(r^2+1)/3)/(n^2-n+\beta)(n^2+n+\beta)} \\ &< \frac{2n + 2\theta n/(n^2-n+\beta)(n^2+n+\beta)}{(n^2+r^2)^2 + \theta(n^2+r^2)^2/(n^2-n+\beta)(n^2+n+\beta)} = \frac{2n}{(n^2+r^2)^2}. \end{aligned} \tag{2.18}$$

Summing up on both sides of the above inequality with respect to $n \in \mathbb{N}$ leads to

$$S(r) > \frac{1}{\nu_1} = \frac{1}{r^2 + 1/2 - \theta/\beta} = \frac{1}{r^2 + 1/2 - (4r^2+1)/(12r^2+18)}. \tag{2.19}$$

As mentiond above, taking $\theta = (1/3)(r^2 + 1/4)$ and simplifying yield that

$$\begin{aligned} P(n,r,\theta,\beta) &= \theta\left(n^2+r^2\right)^2 - \theta\left[\left(4r^2+6-4\beta\right)n^2 - 3\beta^2 + \left(2r^2+1\right)\beta + r^4 - \theta\right] \\ &= \theta\left(n^2+r^2\right)^2 - \theta\left(4r^2+6-4\beta\right)\left(n^2-1\right) + \theta R, \end{aligned} \tag{2.20}$$

where

$$R = 3\beta^2 - \left(2r^2-3\right)\beta - r^4 - \frac{11}{3}r^2 - \frac{71}{12}. \tag{2.21}$$

Now choosing $\beta > 0$ such that $R = 0$ gives

$$\beta = \frac{2r^2 - 3 + 4\sqrt{r^4+2r^2+5}}{6}. \tag{2.22}$$

It is observed that

$$4r^2 + 6 - 4\beta = \frac{8}{3}\left(r^2 + 3 - \sqrt{r^4+2r^2+5}\right) > 0 \tag{2.23}$$

and, for $n \in \mathbb{N}$,

$$P(n,r,\theta,\beta) = \theta\left(n^2+r^2\right)^2 - \frac{8\theta}{3}\left(r^2+3-\sqrt{r^4+2r^2+5}\right)\left(n^2-1\right) \geq \theta\left(n^2+r^2\right)^2. \tag{2.24}$$

Therefore,

$$\frac{1}{\nu_n(r)} - \frac{1}{\nu_{n+1}(r)} > \frac{2n}{\left(n^2+r^2\right)^2}. \tag{2.25}$$

Summing up on both sides from $n = 1$ to ∞ gives

$$\frac{1}{\nu_1(r)} = \frac{1}{r^2 + 1/2 - \theta/\beta} = \frac{1}{r^2 + 1/2 - (4r^2+1)/2\left(2r^2 - 3 + 4\sqrt{r^4+2r^2+5}\right)} > S(r). \tag{2.26}$$

The proof of Theorem 1.3 is complete. □

Proof of Theorem 1.5. Let $u_n(r) = n(n-1) + r^2 + \mu(r)$ for $n \in \mathbb{N}$, where

$$\mu(r) = \sqrt{\left(r^2+1\right)^2 + 1} - \left(r^2+1\right) > 0. \tag{2.27}$$

Then,

$$\frac{1}{u_n(r)} - \frac{1}{u_{n+1}(r)} = \frac{2n}{\left(n^2+r^2\right)^2 - [1-2\mu(r)]n^2 + \mu^2(r) + 2r^2\mu(r)}. \tag{2.28}$$

From (2.27), it is deduced that $\mu^2(r)+2r^2\mu(r)=1-2\mu(r)>0$. Hence,

$$\frac{1}{u_n(r)}-\frac{1}{u_{n+1}(r)}=\frac{2n}{\left(n^2+r^2\right)^2-[1-2\mu(r)](n^2-1)}\ge\frac{2n}{\left(n^2+r^2\right)^2},\tag{2.29}$$

and then

$$\sum_{n=1}^{\infty}\frac{2n}{\left(n^2+r^2\right)^2}<\frac{1}{u_1}=\frac{1}{r^2+\mu(r)}=\frac{1}{\sqrt{r^4+2r^2+2}-1}.\tag{2.30}$$

The proof of Theorem 1.5 is complete. □

Proof of Theorem 1.7. Let

$$t_n=2n^2-2n+1-\frac{1}{6(n^2-n+\delta)},\tag{2.31}$$

where δ is a fixed positive number and $n\in\mathbb{N}$. Direct computation gives

$$\frac{1}{t_n}-\frac{1}{t_{n+1}}=\frac{4n+2n/6(n^2-n+\delta)(n^2+n+\delta)}{4n^4+(2n^4+(8\delta-12)n^2+6\delta^2-2\delta+1/6)/6(n^2-n+\delta)(n^2+n+\delta)}.\tag{2.32}$$

If $\delta=3/2$, then $8\delta-12=0$ and

$$\frac{1}{t_n}-\frac{1}{t_{n+1}}=\frac{4n+2n/6(n^2-n+3/2)(n^2+n+3/2)}{4n^4+(2n^4+32/3)/6(n^2-n+3/2)(n^2+n+3/2)}<\frac{1}{n^3}.\tag{2.33}$$

Summing up on both sides of the above inequality for n from $m+1$ to infinity produces

$$\frac{1}{t_{m+1}}=\frac{1}{2m^2+1-1/6(m^2+m+3/2)}<\sum_{n=m+1}^{\infty}\frac{1}{n^3}.\tag{2.34}$$

Adding $S_3(m)$ on both sides of the above inequality leads to the left-hand side inequality in (1.7).

If $\delta=1$ and $n>1$, then

$$\frac{1}{t_n}-\frac{1}{t_{n+1}}=\frac{4n+2n/6(n^2-n+1)(n^2+n+1)}{4n^4+(2n^4-[4(n^2-1)-1/6])/6(n^2-n+1)(n^2+n+1)}>\frac{1}{n^3}.\tag{2.35}$$

Summing up on both sides of the above inequality for n from $m+1$to infinity yields that

$$\frac{1}{2m^2+2m+1-1/2(m^2+m+1)}>\sum_{n=m+1}^{\infty}\frac{1}{n^3}.\tag{2.36}$$

This is equivalent to the right-hand side inequality in (1.7). Theorem 1.7 is proved. □

Proof of Corollary 1.9. It is easy to see that

$$2n^4+(8\delta-12)n^2+6\delta^2-2\delta+\frac{1}{6}=2n^4-(12-8\delta)\left(n^2-\frac{3\delta^2-\delta+1/12}{6-4\delta}\right).\tag{2.37}$$

If $1 \leq \delta < 3/2$ and $n \geq \sqrt{(3\delta^2 - \delta + 1/12)/(6 - 4\delta)}$, from (2.32), it is deduced that

$$\frac{1}{t_n} - \frac{1}{t_{n+1}} \geq \frac{1}{n^3}. \tag{2.38}$$

By the same argument as mentiond above, when $m \geq \sqrt{(3\delta^2 - \delta + 1/12)/(6 - 4\delta)} - 1$, inequality

$$\frac{1}{t_{m+1}} = \frac{1}{2m^2 + 2m + 1 - 1/6(m^2 + m + \delta)} > \sum_{n=m+1}^{\infty} \frac{1}{n^3} \tag{2.39}$$

is obtained, which is equivalent to (1.11). The proof of Corollary 1.9 is complete. □

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