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Research Article

On a Two-Variable p-Adic l_q-Function

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We prove that a two-variable p-adic l_q-function has the series expansion $l_{p,q}(s,t,\chi) = ([2]_q/[2]_F)\sum_{a=1,(p,a)=1}^{F}(-1)^a(\chi(a)q^a/\langle a+pt\rangle^s)\sum_{m=0}^{\infty}\binom{-s}{m}(F/\langle a+pt\rangle)^m E^*_{m,q^F}$ which interpolates the values $l_{p,q}(-n,t,\chi) = E^*_{n,\chi_n,q}(pt) - p^n\chi_n(p)([2]_q/[2]_{q^p})E^*_{n,\chi_n,q^p}(t)$, whenever n is a nonpositive integer. The proof of this original construction is due to Kubota and Leopoldt in 1964, although the method given in this note is due to Washington.

1. Introduction

The ordinary Euler polynomials $E_n(t)$ are defined by the equation

$$\frac{2e^{tx}}{e^x+1} = \sum_{n=0}^{\infty}E_n(t)\frac{x^n}{n!}. \tag{1.1}$$

Setting $t = 1/2$ and normalizing by 2^n gives the ordinary Euler numbers

$$E_n = 2^n E_n\left(\frac{1}{2}\right). \tag{1.2}$$

The ordinary Euler polynomials appear in many classical results (see [1]). In [2], the values of these polynomials at rational arguments were expressed in term of the Hurwitz zeta function. Congruences for Euler numbers have also received much attention from the point of view of p-adic interpolation. In [3], Kim et al. recently defined the natural q-extension of ordinary Euler numbers and polynomials by p-adic integral representation and proved properties generalizing those satisfied by E_n and $E_n(t)$. They also constructed the one-variable p-adic q-l-function $l_{p,q}(s,\chi)$ for Dirichlet characters χ and $s \in \mathbb{C}_p$ with $|s|_p < p^{1-(1/(p-1))}$, with the property that

$$l_{p,q}(-n,\chi) = E^*_{n,\chi\omega^{-n},q} - [2]_q[2]^{-1}_{q^p} p^n \chi\omega^{-n}(p) E^*_{n,\chi\omega^{-n},q^p} \tag{1.3}$$

for $n = 0, 1, \ldots$, where $E^*_{n,\chi\omega^{-n},q}$ is a generalized q-Euler number associated with the Dirichlet characters $\chi\omega^{-n}$ (see Section 2 for definitions).

In the present paper, we will construct a specific two-variable p-adic l_q-function $l_{p,q}(s,t,\chi)$ by means of a method provided in [4–6]. We also prove that $l_{p,q}(s,t,\chi)$ is analytic in s and t for $s \in \mathbb{C}_p$ with $|s|_p < p^{1-(1/(p-1))}$ and $t \in \mathbb{C}_p$ with $|t|_p \le 1$, which interpolates the values

$$l_{p,q}(-n,t,\chi) = E^*_{n,\chi_n,q}(pt) - p^n \chi_n(p) \frac{[2]_q}{[2]_{q^p}} E^*_{n,\chi_n,q^p}(t), \tag{1.4}$$

whenever n is a nonpositive integer. This two-variable function is a generalization of the one-variable p-adic q-l-function, which is the function obtained by putting $t = 0$ in $l_{p,q}(s,t,\chi)$ (cf. [3–11]).

Throughout this paper $\mathbb{Z}$, $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ will denote the ring of integers, the ring of p-adic rational integers, the field of p-adic rational numbers, and the completion of the algebraic closure of $\mathbb{Q}_p$, respectively. We will use $\mathbb{Z}^+$ for the set of nonpositive integers. Let v_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = 1/p$. When one talks of q-extension, q is variously considered as an indeterminate, a complex number $q \in \mathbb{C}$, or a p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}_p$, then we normally assume $|1-q|_p < 1$. If $q \in \mathbb{C}$, then we assume that $|q| < 1$. Also, we use the following notations:

$$[x]_q = \frac{1-q^x}{1-q}, \quad [x]_{-q} = \frac{1-(-q)^x}{1+q}, \quad \text{(cf. [7, 8])}. \tag{1.5}$$

Let d be a fixed integer, and let

$$X = X_d = \lim_{\overleftarrow{N}} (\mathbb{Z}/dp^N\mathbb{Z}), \quad X^* = \bigcup_{\substack{0<a<dp \\ (a,p)=1}} a + dp\mathbb{Z}_p, \quad a + dp^N\mathbb{Z}_p = \left\{x \in X \mid x \equiv a \pmod{dp^N}\right\}, \tag{1.6}$$

where $a \in \mathbb{Z}$ lies in $0 \le a < dp^N$. Let $UD(\mathbb{Z}_p)$ be the space of uniformly differentiable function on $\mathbb{Z}_p$. For $f \in UD(\mathbb{Z}_p)$, the p-adic q-integral is defined by

$$I_q(f) = \int_{\mathbb{Z}_p} f(a)\,d\mu_q(a) = \int_X f(a)\,d\mu_q(a) = \lim_{N\to\infty} \frac{1}{[dp^N]_q} \sum_{a=0}^{dp^N-1} f(a) q^a \quad \text{for } |1-q|_p < 1. \tag{1.7}$$

In [8], the bosonic integral was considered from a more physical point of view to the limit $q \to 1$ as follows:

$$I_1(f) = \lim_{q\to1} I_q(f) = \int_{\mathbb{Z}_p} f(a)\,d\mu_1(a) = \lim_{N\to\infty} \frac{1}{p^N} \sum_{a=0}^{p^N-1} f(a). \tag{1.8}$$

Furthermore, we can consider the fermionic integral in contrast to the conventional "bosonic integral." That is, $I_{-1}(f) = \int_{\mathbb{Z}_p} f(a)d\mu_{-1}(a)$ (see [9]). From this, we derive $I_{-1}(f_1) + I_{-1}(f) = 2f(0)$, where $f_1(a) = f(a+1)$. Also, we have

$$I_{-1}(f_n) + (-1)^{n-1}I_{-1}(f) = 2\sum_{a=0}^{n-1}(-1)^{n-1-a}f(a), \tag{1.9}$$

where $f_n(a) = f(a+n)$ and $n \in \mathbb{Z}^+$ (see [9]). For $|1-q|_p < 1$, we consider fermionic p-adic q-integral on $\mathbb{Z}_p$ which is the q-extension of $I_{-1}(f)$ as follows:

$$I_{-q}(f) = \int_{\mathbb{Z}_p} f(a)d\mu_{-q}(a) = \lim_{N\to\infty}\frac{1}{[dp^N]_{-q}}\sum_{a=0}^{dp^N-1} f(a)(-q)^a \quad \text{(cf. [3])}. \tag{1.10}$$

2. q-Euler numbers and polynomials

In this section, we review some notations and facts in [3].

From (1.10), we can derive the following formula:

$$qI_{-q}(f_1) + I_{-q}(f) = [2]_q f(0), \tag{2.1}$$

where $f_1(a)$ is translation with $f_1(a) = f(a+1)$. If we take $f(a) = e^{ax}$, then we have $f_1(a) = e^{(a+1)x} = e^{ax}e^x$. From (2.1), we derive $(qe^x+1)I_{-q}(e^{ax}) = [2]_q$. Hence, we obtain

$$I_{-q}(e^{ax}) = \int_{\mathbb{Z}_p} e^{ax}d\mu_{-q}(a) = \frac{[2]_q}{qe^x+1}. \tag{2.2}$$

We now set

$$\frac{[2]_q}{qe^x+1} = \sum_{n=0}^{\infty} E^*_{n,q}\frac{x^n}{n!}. \tag{2.3}$$

$E^*_{n,q}$ is called the nth q-Euler number. By (2.2) and (2.3), we see that

$$\int_{\mathbb{Z}_p} a^n d\mu_{-q}(a) = E^*_{n,q}. \tag{2.4}$$

From (2.2), we also note that

$$\int_{\mathbb{Z}_p} e^{(t+a)x}d\mu_{-q}(a) = \frac{[2]_q}{qe^x+1}e^{tx}. \tag{2.5}$$

In view of (2.3) and (2.5), we can consider q-Euler polynomials associated with t as follows:

$$\frac{[2]_q}{qe^x+1}e^{tx} = \sum_{n=0}^{\infty}E^*_{n,q}(t)\frac{x^n}{n!}, \qquad \int_{\mathbb{Z}_p}(t+a)^n d\mu_{-q}(a) = E^*_{n,q}(t). \tag{2.6}$$

Put $\lim_{q\to 1}E^*_{n,q}=E^*_n$ and $\lim_{q\to 1}E^*_{n,q}(t)=E^*_n(t)$. Then, we have $E_n(t)=E^*_n(t)$ and

$$E_n=\sum_{m=0}^{n}2^m\binom{n}{m}E^*_m, \tag{2.7}$$

where E_n and $E_n(t)$ are the ordinary Euler numbers and polynomials. By (2.3) and (2.6), we easily see that $E^*_{n,q}(t)=\sum_{m=0}^{n}\binom{n}{m}t^{n-m}E^*_{m,q}$. For $d\in\mathbb{Z}^+$, let $f_d(a)=f(a+d)$. Then, we have

$$q^dI_{-q}(f_d)+(-1)^{d-1}I_{-q}(f)=[2]_q\sum_{a=0}^{d-1}(-1)^{d-a-1}q^af(a),\quad \text{see [3]}. \tag{2.8}$$

If d is an odd positive integer, we have

$$q^dI_{-q}(f_d)+I_{-q}(f)=[2]_q\sum_{a=0}^{d-1}(-1)^aq^af(a). \tag{2.9}$$

Let χ be a Dirichlet character with conductor $d=d_\chi$ (=odd) $\in\mathbb{Z}^+$. If we take $f(a)=\chi(a)e^{(t+a)x}$, then we have $f_d(a)=f(a+d)=\chi(a)e^{dx}e^{(t+a)x}$. From (1.7) and (2.9), we derive

$$\int_X\chi(a)e^{(t+a)x}d\mu_{-q}(a)=\frac{[2]_q\sum_{a=1}^{d}(-1)^aq^a\chi(a)e^{(t+a)x}}{q^de^{dx}+1}. \tag{2.10}$$

In view of (2.10), we also consider the generalized q-Euler polynomials associated with χ as follows:

$$F_{\chi,q}(x,t)=\frac{[2]_q\sum_{a=1}^{d}(-1)^aq^a\chi(a)e^{(t+a)x}}{q^de^{dx}+1}=\sum_{n=0}^{\infty}E^*_{n,\chi,q}(t)\frac{x^n}{n!}. \tag{2.11}$$

From (2.10) and (2.11), we derive the following equation:

$$\int_X\chi(a)(t+a)^nd\mu_{-q}(a)=E^*_{n,\chi,q}(t) \tag{2.12}$$

for $n\geq 0$. Put $\lim_{q\to 1}E^*_{n,\chi,q}(t)=E^*_{n,\chi}(t)$. On the other hand, the generalized q-Euler polynomials associated with χ are easily expressed as the q-Euler polynomials:

$$E^*_{n,\chi,q}(t)=d^n\frac{[2]_q}{[2]_{q^d}}\sum_{a=1}^{d}(-1)^aq^a\chi(a)E^*_{n,q^d}\left(\frac{a+t}{d}\right),\quad n\geq 0. \tag{2.13}$$

Let χ be a Dirichlet character with conductor $d=d_\chi\in\mathbb{Z}^+$. It is well known (see [11, 12]) that, for positive integers m and n,

$$\sum_{a=1}^{dn}\chi(a)a^m=\frac{1}{m+1}\left(B_{m+1,\chi}(dn)-B_{m+1,\chi}(0)\right), \tag{2.14}$$

where $B_{m+1,\chi}(t)$ are the generalized Bernoulli polynomials. When $d = d_\chi$ (=odd) $\in \mathbb{Z}^+$, note that

$$
\begin{aligned}
&\frac{[2]_q \sum_{a=1}^{d}(-1)^a q^a \chi(a) e^{(t+a)x}\left(1-\left(-q^d e^{dx}\right)^n\right)}{1-\left(-q^d e^{dx}\right)} \\
&\quad = [2]_q \sum_{a=1}^{d}\sum_{l=0}^{n-1}(-1)^{a+dl} q^{a+dl}\chi(a+dl) e^{x(t+a+dl)} = [2]_q \sum_{a=1}^{dn}(-1)^a q^a \chi(a) e^{x(t+a)} \\
&\quad = \sum_{m=0}^{\infty}\left([2]_q \sum_{a=1}^{dn}(-1)^a q^a \chi(a)(t+a)^m\right)\frac{x^m}{m!}.
\end{aligned}
\tag{2.15}
$$

By (2.11), the relation (2.15) can be rewritten as

$$
\frac{[2]_q \sum_{a=1}^{d}(-1)^a q^a \chi(a) e^{(t+a)x}\left(1-\left(-q^d e^{dx}\right)^n\right)}{1-\left(-q^d e^{dx}\right)} = \sum_{m=0}^{\infty}\left(E_{m,\chi,q}^*(t) + (-1)^{n+1} q^{dn} E_{m,\chi,q}^*(t+dn)\right)\frac{x^m}{m!}.
\tag{2.16}
$$

Now, we give the q-analog of (2.14) for the generalized Euler polynomials. From (2.15) and (2.16), it is easy to see that

$$
\sum_{a=1}^{dn}(-1)^a q^a \chi(a)(t+a)^m = \frac{1}{[2]_q}\left(E_{m,\chi,q}^*(t) + (-1)^{n+1} q^{dn} E_{m,\chi,q}^*(t+dn)\right)
\tag{2.17}
$$

for positive integers m and n. In particular, replacing q by 1 in (2.17), if $\chi = \chi^0$, the principal character ($d_\chi = 1$), and $t = 0$, then

$$
\sum_{a=1}^{n-1}(-1)^a a^m = \frac{1}{2}\left(E_m(0) + (-1)^{n+1} E_m(n)\right).
\tag{2.18}
$$

Definition 2.1. Let $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$. Let χ be a primitive Dirichlet character with conductor $d = d_\chi$ (=odd) $\in \mathbb{Z}^+$. One sets

$$
l_q(s,t,\chi) = [2]_q \sum_{n=0}^{\infty}\frac{(-1)^n q^n \chi(n)}{(t+n)^s}, \quad 0 < t \le 1.
\tag{2.19}
$$

Remark 2.2. We assume that $q \in \mathbb{C}$ with $|q| < 1$. Let χ be a primitive Dirichlet character with conductor $d = d_\chi$ (=odd) $\in \mathbb{Z}^+$. From (2.11), we consider the below integral which is known as the Mellin transformation of $F_{\chi,q}(x,t)$ (cf. [13])

$$
\begin{aligned}
\frac{1}{\Gamma(s)}\int_0^\infty x^{s-1} F_{\chi,q}(-x,t)\,dx &= [2]_q \sum_{a=1}^{d}(-1)^a q^a \chi(a)\frac{1}{\Gamma(s)}\int_0^\infty x^{s-1}\frac{e^{-(t+a)x}}{1-\left(-q^d e^{-dx}\right)}dx \\
&= [2]_q \sum_{a=1}^{d}(-1)^a q^a \chi(a+dl)\sum_{l=0}^{\infty}(-1)^{dl}\frac{q^{dl}}{(a+dl+t)^s}.
\end{aligned}
\tag{2.20}
$$

We write $n = a + dl$, where $n = 1, 2, \ldots,$ and obtain

$$
\frac{1}{\Gamma(s)}\int_0^\infty x^{s-1} F_{\chi,q}(-x,t)\,dx = [2]_q \sum_{n=0}^{\infty}\frac{(-1)^n q^n \chi(n)}{(t+n)^s} = l_q(s,t,\chi).
\tag{2.21}
$$

Note that $l_q(s,t,\chi)$ is an analytic function in the whole complex s-plane. By using a geometric series in (2.11), we obtain

$$[2]_q e^{tx}\sum_{n=0}^{\infty}(-1)^n q^n \chi(n) e^{nx} = \sum_{n=0}^{\infty} E^*_{n,\chi,q}(t)\frac{x^n}{n!}. \tag{2.22}$$

We also note that

$$E^*_{n,\chi,q}(t) = \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^k [2]_q e^{tx}\sum_{n=0}^{\infty}(-1)^n q^n \chi(n) e^{nx}\bigg|_{x=0}. \tag{2.23}$$

By Definition 2.1 and (2.23), we obtain the following proposition.

Proposition 2.3. *For $n \in \mathbb{Z}^+$, one has $l_q(-n,t,\chi) = E^*_{n,\chi,q}(t)$.*

The values of $l_q(s,t,\chi)$ at negative integers are algebraic, hence may be regarded as being in an extension of $\mathbb{Q}_p$. We therefore look for a p-adic function which agrees with $l_q(s,t,\chi)$ at the negative integers in Section 3.

3. A two-variable p-adic l_q-function

We will consider the p-adic analog of the l_q-functions which are introduced in the previous section (see Definition 2.1). Throughout this section we assume that p is an odd prime. Note that there exist $\varphi(p)$ distinct solutions, modulo p, to the equation $x^{\varphi(p)}-1=0$, and each solution must be congruent to one of the values $a \in \mathbb{Z}$, where $1 \le a < p$, $(a,p)=1$. Thus, given $a \in \mathbb{Z}$ with $(a,p)=1$, there exists a unique $\omega(a) \in \mathbb{Z}_p$, where $\omega(a)^{\varphi(p)}=1$, such that $\omega(a) \equiv a \pmod{p\mathbb{Z}_p}$. Letting $\omega(a)=0$ for $a \in \mathbb{Z}$, such that $(a,p) \neq 1$, it can be seen that ω is actually a Dirichlet character having conductor $d_\omega = p$, called the Teichmüller character. Let $\langle a\rangle = \omega^{-1}(a)a$. Then, $\langle a\rangle \equiv 1 \pmod{p\mathbb{Z}_p}$. For the context in the sequel, an extension of the definition of the Teichmüller character is needed. We denote a particular subring of $\mathbb{C}_p$ as

$$R = \{a \in \mathbb{C}_p \mid |a|_p \le 1\}. \tag{3.1}$$

If $t \in \mathbb{C}_p$, such that $|t|_p \le 1$, then for any $a \in \mathbb{Z}$, $a+pt \equiv a \pmod{pR}$. Thus, for $t \in \mathbb{C}_p$, $|t|_p \le 1$, $\omega(a+pt)=\omega(a)$. Also, for these values of t, let $\langle a+pt\rangle = \omega^{-1}(a)(a+pt)$. Let χ be the Dirichlet character of conductor $d=d_\chi$. For $n \ge 1$, we define χ_n to be the primitive character associated with the character $\chi_n : (\mathbb{Z}/\mathrm{lcm}(d,p)\mathbb{Z})^\times \to \mathbb{C}^\times$ defined by $\chi_n(a) = \chi(a)\omega^{-n}(a)$.

We define an interpolation function for generalized q-Euler polynomials.

Definition 3.1. Let χ be the Dirichlet character with conductor $d = d_\chi$ (=odd) and let F be a positive integral multiple of p and d. Now, one defines the two-variable p-adic l_q-function as follows:

$$l_{p,q}(s,t,\chi) = \frac{[2]_q}{[2]_{q^F}}\sum_{\substack{a=1\\(p,a)=1}}^{F}(-1)^a\chi(a)q^a\langle a+pt\rangle^{-s}\sum_{m=0}^{\infty}\binom{-s}{m}\left(\frac{F}{\langle a+pt\rangle}\right)^m E^*_{m,q^F}. \tag{3.2}$$

Let $D = \{s \in \mathbb{C}_p \mid |s|_p < p^{1-(1/(p-1))}\}$ and let $a \in \mathbb{Z},\ (a,p) = 1$. For $t \in \mathbb{C}_p,\ |t|_p \leq 1$, the same argument as that given in the proof of the main theorem of [4, 5] can be used to show that the functions $\sum_{m=0}^{\infty}\binom{s}{m}(F/(a+pt))^m E^*_{m,q^F}$ and $\langle a+pt\rangle^s = \sum_{m=0}^{\infty}\binom{s}{m}(\langle a+pt\rangle - 1)^m$ are analytic for $s \in D$. According to this method, we see that the function $\sum_{m=0}^{\infty}\binom{s}{m}(F/(a+pt))^m E^*_{m,q^F}$ is analytic for $t \in \mathbb{C}_p,\ |t|_p \leq 1$, whenever $s \in D$. It readily follows that $\langle a+pt\rangle^s = \langle a\rangle^s \sum_{m=0}^{\infty}\binom{s}{m}(a^{-1}pt)^m$ is analytic for $t \in \mathbb{C}_p,\ |t|_p \leq 1$, when $s \in D$. Therefore,

$$l_{p,q}(s,t,\chi) \text{ is analytic for } t \in \mathbb{C}_p, \quad |t|_p \leq 1, \tag{3.3}$$

provided $s \in D$ (see [5]).

We set

$$h_{p,q}(s,t,a \mid F) = (-1)^a q^a \langle a+pt\rangle^{-s} \frac{[2]_q}{[2]_{q^F}} \sum_{m=0}^{\infty} \binom{-s}{m} \left(\frac{F}{a+pt}\right)^m E^*_{m,q^F}. \tag{3.4}$$

Thus, we note that

$$h_{p,q}(-n,t,a \mid F) = \omega^{-n}(a)(-1)^a q^a F^n \frac{[2]_q}{[2]_{q^F}} E^*_{n,q^F}\left(\frac{a+pt}{F}\right) \tag{3.5}$$

for $n \in \mathbb{Z}^+$. We also consider the two-variable p-adic l_q-functions which interpolate the generalized q-Euler polynomials at negative integers as follows:

$$l_{p,q}(s,t,\chi) = \sum_{\substack{a=1\\(p,a)=1}}^{F} \chi(a) h_{p,q}(s,t,a \mid F). \tag{3.6}$$

We will in the process derive an explicit formula for this function. Before we begin this derivation, we need the following result concerning generalized q-Euler polynomials.

Lemma 3.2. *Let F be a positive integral multiple of p and $d = d_\chi$. Then, for each $n \in \mathbb{Z},\ n \geq 0$,*

$$E^*_{n,\chi,q}(t) = F^n \frac{[2]_q}{[2]_{q^F}} \sum_{a=1}^{F} (-1)^a q^a \chi(a) E^*_{n,q^F}\left(\frac{a+t}{F}\right). \tag{3.7}$$

Proof. By (2.6) and (2.11), we note that

$$F_{\chi,q}(x,t) = \frac{[2]_q}{[2]_{q^d}} \sum_{a=1}^{d} (-1)^a q^a \chi(a) \sum_{n=0}^{\infty} E^*_{n,q^d}\left(\frac{a+t}{d}\right) \frac{(dx)^n}{n!}. \tag{3.8}$$

Then, we have

$$E^*_{n,\chi,q}(t) = d^n \frac{[2]_q}{[2]_{q^d}} \sum_{a=1}^{d} (-1)^a q^a \chi(a) E^*_{n,q^d}\left(\frac{a+t}{d}\right). \tag{3.9}$$

On the other hand, if $F = dp$, then we get

$$
\begin{aligned}
&\frac{[2]_q}{[2]_{q^F}} \sum_{a=1}^{F} (-1)^a q^a \chi(a) \sum_{n=0}^{\infty} E^*_{n,q^F}\left(\frac{a+t}{F}\right) \frac{(Fx)^n}{n!} \\
&\quad = [2]_q \sum_{a=1}^{d} \sum_{b=0}^{p-1} (-1)^{a+bd} q^{a+bd} \chi(a+bd) \frac{e^{(a+bd)x} e^{tx}}{q^F e^{Fx}+1} \\
&\quad = [2]_q \sum_{a=1}^{d} (-1)^a q^a \chi(a) e^{(a+t)x} \sum_{b=0}^{p-1} (-1)^{bd} q^{bd} e^{bdx} \frac{1}{q^F e^{Fx}+1} \\
&\quad = \frac{[2]_q}{[2]_{q^d}} \sum_{a=1}^{d} (-1)^a q^a \chi(a) \sum_{n=0}^{\infty} E^*_{n,q^d}\left(\frac{a+t}{d}\right) \frac{(dx)^n}{n!}.
\end{aligned}
\tag{3.10}
$$

This completes the proof. □

Set $\chi_n = \chi\omega^{-n}$. From (3.5) and (3.6), we obtain

$$
\begin{aligned}
l_{p,q}(-n,t,\chi) &= F^n \frac{[2]_q}{[2]_{q^F}} \sum_{\substack{a=1 \\ (p,a)=1}}^{F} \chi_n(a)(-1)^a q^a E^*_{n,q^F}\left(\frac{a+pt}{F}\right) \\
&= F^n \frac{[2]_q}{[2]_{q^F}} \sum_{a=1}^{F} \chi_n(a)(-1)^a q^a E^*_{n,q^F}\left(\frac{a+pt}{F}\right) \\
&\quad - F^n \frac{[2]_q}{[2]_{q^F}} \sum_{a=1}^{F/p} \chi_n(pa)(-1)^{pa} q^{pa} E^*_{n,q^F}\left(\frac{pa+pt}{F}\right).
\end{aligned}
\tag{3.11}
$$

for $n \in \mathbb{Z}^+$. From Lemma 3.2, we see that

$$
\begin{aligned}
E^*_{n,\chi_n,q}(pt) &= F^n \frac{[2]_q}{[2]_{q^F}} \sum_{a=1}^{F} (-1)^a q^a \chi_n(a) E^*_{n,q^F}\left(\frac{a+pt}{F}\right), \\
E^*_{n,\chi_n,q^p}(t) &= \left(\frac{F}{p}\right)^n \frac{[2]_{q^p}}{[2]_{(q^p)^{F/p}}} \sum_{a=1}^{F/p} (-1)^a (q^p)^a \chi_n(a) E^*_{n,(q^p)^{F/p}}\left(\frac{a+t}{F/p}\right).
\end{aligned}
\tag{3.12}
$$

From (3.3), (3.11), and (3.12), we obtain the following theorem.

Theorem 3.3. *Let F (=odd) be a positive integral multiple of p and d_χ. Then, the two-variable p-adic l_q-function*

$$
l_{p,q}(s,t,\chi) = \frac{[2]_q}{[2]_{q^F}} \sum_{\substack{a=1 \\ (p,a)=1}}^{F} (-1)^a \chi(a) q^a \langle a+pt\rangle^{-s} \sum_{m=0}^{\infty} \binom{-s}{m} \left(\frac{F}{\langle a+pt\rangle}\right)^m E^*_{m,q^F}
\tag{3.13}
$$

admits an analytic function for $t \in \mathbb{C}_p$ with $|t|_p \le 1$ and $s \in D$ and satisfies the relation

$$
l_{p,q}(-n,t,\chi) = E^*_{n,\chi_n,q}(pt) - p^n \chi_n(p) \frac{[2]_q}{[2]_{q^p}} E^*_{n,\chi_n,q^p}(t)
\tag{3.14}
$$

for $n \in \mathbb{Z}^+$ and $t \in \mathbb{C}_p$ with $|t|_p \le 1$.

From (3.5) and Theorem 3.3, it follows that $h_{p,q}(s,t,a \mid F)$ is analytic for $t \in \mathbb{C}_p$ with $|t|_p \leq 1$ and $s \in D$.

Remark 3.4. Let $\langle a+pt\rangle = \omega^{-1}(a)(a+pt)$, and let $t \in \mathbb{C}_p$ with $|t|_p \leq 1$ and $s \in D$. Then the two-variable p-adic l_q-function defined above is redefined by

$$l_{p,q}(s,t,\chi) = \int_{X^*} \chi(a)\langle a+pt\rangle^{-s} d\mu_{-q}(a), \quad \text{compared to [3, 10]}. \tag{3.15}$$

Then, we have

$$\begin{aligned} l_{p,q}(-n,t,\chi) &= \int_X \chi_n(a)(a+pt)^n d\mu_{-q}(a) - \int_X \chi_n(pa)(pa+pt)^n d\mu_{-q}(pa) \\ &\overset{(2.10)}{=} E^*_{n,\chi_n,q}(pt) - p^n \chi_n(p)\frac{[2]_q}{[2]_{q^p}} E^*_{n,\chi_n,q^p}(t), \end{aligned} \tag{3.16}$$

since $X^* = X - pX$ and $[2]_{q^p} d\mu_{-q}(pa) = [2]_q d\mu_{-q^p}(a)$.

If $q \to 1$ in Theorem 3.3 and Remark 3.4, we obtain the following corollary.

Corollary 3.5. *Let F (=odd) be a positive integral multiple of p and d_χ, and let the two-variable p-adic l-function*

$$l_p(s,t,\chi) = \sum_{\substack{a=1 \\ (p,a)=1}}^{F} (-1)^a \chi(a)\langle a+pt\rangle^{-s} \sum_{m=0}^{\infty} \binom{-s}{m} \left(\frac{F}{\langle a+pt\rangle}\right)^m E^*_m. \tag{3.17}$$

Then,

(1) $l_p(s,t,\chi)$ *is analytic for* $t \in \mathbb{C}_p$ *with* $|t|_p \leq 1$ *and* $s \in D$.

(2) $l_p(-n,t,\chi) = E^*_{n,\chi_n}(pt) - p^n \chi_n(p) E^*_{n,\chi_n}(t)$ *for* $n \in \mathbb{Z}^+$.

(3) $l_p(s,t,\chi) = \int_{X^*} \chi(a)\langle a+pt\rangle^{-s} d\mu_{-1}(a)$ *for* $t \in \mathbb{C}_p$ *with* $|t|_p \leq 1$ *and* $s \in D$.

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References

[1] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, vol. 55 of *National Bureau of Standards Applied Mathematics Series*, US Government Printing Office, Washington, DC, USA, 1964.

[2] D. Cvijović and J. Klinowski, "New formulae for the Bernoulli and Euler polynomials at rational arguments," *Proceedings of the American Mathematical Society*, vol. 123, no. 5, pp. 1527–1535, 1995.

[3] T. Kim, M.-S. Kim, L. Jang, and S.-H. Rim, "New q-Euler numbers and polynomials associated with p-adic q-integrals," *Advanced Studies in Contemporary Mathematics*, vol. 15, no. 2, pp. 243–252, 2007.

[4] L. C. Washington, "A note on p-adic L-functions," *Journal of Number Theory*, vol. 8, no. 2, pp. 245–250, 1976.

[5] G. J. Fox, "A method of Washington applied to the derivation of a two-variable p-adic L-function," *Pacific Journal of Mathematics*, vol. 209, no. 1, pp. 31–40, 2003.

[6] T. Kim, "On p-adic interpolating function for q-Euler numbers and its derivatives," *Journal of Mathematical Analysis and Applications*, vol. 339, no. 1, pp. 598–608, 2008.

[7] T. Kim, "Power series and asymptotic series associated with the q-analog of the two-variable p-adic L-function," *Russian Journal of Mathematical Physics*, vol. 12, no. 2, pp. 186–196, 2005.

[8] T. Kim, "On p-adic q-L-functions and sums of powers," *Discrete Mathematics*, vol. 252, no. 1–3, pp. 179–187, 2002.

[9] T. Kim, "On the analogs of Euler numbers and polynomials associated with p-adic q-integral on $\mathbb{Z}_p$ at $q = -1$," *Journal of Mathematical Analysis and Applications*, vol. 331, no. 2, pp. 779–792, 2007.

[10] P. T. Young, "On the behavior of some two-variable p-adic L-functions," *Journal of Number Theory*, vol. 98, no. 1, pp. 67–88, 2003.

[11] L. C. Washington, *Introduction to Cyclotomic Fields*, vol. 83 of *Graduate Texts in Mathematics*, Springer, New York, NY, USA, 2nd edition, 1997.

[12] K. Iwasawa, *Lectures on p-adic L-Functions*, vol. 74 of *Annals of Mathematics Studies*, Princeton University Press, Princeton, NJ, USA, 1972.

[13] Y. Simsek, "On twisted q-Hurwitz zeta function and q-two-variable L-function," *Applied Mathematics and Computation*, vol. 187, no. 1, pp. 466–473, 2007.