

Research Article

Extension of The Best Approximation Operator in Orlicz Spaces

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Let $(\Omega, \mathcal{A}, \mu)$ be a probability space and $\mathcal{L} \subset \mathcal{A}$ a sub- σ -lattice of the σ -algebra \mathcal{A} . We study an extension of the best ϕ -approximation operator from an Orlicz space L^ϕ to the space $L^{\phi'}$, where ϕ' denotes the derivative of the convex, but not necessarily a strictly convex function ϕ . We obtain convergence results when a sequence of σ -algebras \mathcal{B}_n converges to \mathcal{B}_∞ in a suitable way.

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1. Introduction and notations

Let $(\Omega, \mathcal{A}, \mu)$ be a probability space and let $\mathcal{M} = \mathcal{M}(\Omega, \mathcal{A}, \mu)$ be the set of all \mathcal{A} -measurable real-valued functions defined on Ω . Given a C^1 convex function $\phi : [0, \infty) \rightarrow [0, \infty)$ such that $\phi(0) = 0$, $\phi(t) > 0$, when $t > 0$, let $L^\phi = L^\phi(\Omega, \mathcal{A}, \mu)$ be the space of all functions $f \in \mathcal{M}$ such that

$$\int_{\Omega} \phi(\lambda|f|)d\mu < \infty, \quad (1.1)$$

for some $\lambda > 0$. Since we only deal with a Δ_2 function ϕ , that is, there exists a constant $K > 0$ such that $\phi(2x) \leq K\phi(x)$ for all $x \geq 0$, the space L^ϕ can be defined as the space of all function $f \in \mathcal{M}$, where (1.1) holds for every positive number λ . The space $L^{\phi'}$ is analogously defined, where ϕ' is the derivative of the function ϕ . Besides, observe that for a Δ_2 function ϕ it holds the next inequality

$$\phi(x) \leq x\phi'(x) \leq \phi(2x) \leq K\phi(x) \quad (1.2)$$

for all $x \geq 0$, and therefore $L^\phi \subset L^{\phi'}$. Moreover φ is a Δ_2 function if and only if φ' is a Δ_2 function.

We say, according to [1], that a collection \mathcal{L} of sets in \mathcal{A} is a σ -lattice if it is closed under countable unions and intersections and contains \emptyset and Ω . Given a σ -lattice \mathcal{L} , we denote by $\overline{\mathcal{L}}$ the σ -lattice of all the complementary sets of \mathcal{L} , that is, $\overline{\mathcal{L}} = \{A^c : A \in \mathcal{L}\}$. Denote by $L^\phi(\mathcal{L})$ all \mathcal{L} -measurable functions in L^ϕ .

A set $C \subset L^\phi$ is called ϕ -closed if and only if $f_n \in C$ and $f_n \nearrow f \in L^\phi$ or $(f_n \searrow f \in L^\phi)$ then $f \in C$. Then $L^\phi(\mathcal{L})$ is a ϕ -closed convex set and a lattice, that is closed for the maximum and minimum of functions.

We will use the notation $f \wedge g = \min(f, g)$ and $f \vee g = \max(f, g)$. A set $C \subset L^\phi$ is called a σ -complete lattice if and only if $\bigwedge_{n \in \mathbb{N}} f_n = \inf_{n \in \mathbb{N}} f_n \in C$ and $\bigvee_{n \in \mathbb{N}} f_n = \sup_{n \in \mathbb{N}} f_n \in C$ for all sequence $\{f_n\}_{n \in \mathbb{N}} \subset C$.

It is well known, see [1], that for every $f \in L^\phi$ there exists an element $g \in L^\phi(\mathcal{L})$ such that

$$\int_{\Omega} \phi(|f - g|) d\mu = \inf_{h \in L^\phi(\mathcal{L})} \int_{\Omega} \phi(|f - h|) d\mu. \quad (1.3)$$

Denote by $\mu_{\phi}^{\mathcal{L}}(f)$ the set of all $g \in L^\phi(\mathcal{L})$ satisfying (1.3). Each element of $\mu_{\phi}^{\mathcal{L}}(f)$ will be called a best ϕ -approximation of f given $L^\phi(\mathcal{L})$, and we will refer to the mapping $f \rightarrow \mu_{\phi}^{\mathcal{L}}(f)$ defined on L^ϕ as the best approximation operator.

It is showed in [1] that for $f \in L^\phi$ the set $\mu_{\phi}^{\mathcal{L}}(f)$ is a nonvoid σ -complete lattice. Also it was proved that if $f \leq g$, both in L^ϕ , $f_1 \in \mu_{\phi}^{\mathcal{L}}(f)$ and $g_1 \in \mu_{\phi}^{\mathcal{L}}(g)$ then we have $f_1 \wedge g_1 \in \mu_{\phi}^{\mathcal{L}}(f)$ and $f_1 \vee g_1 \in \mu_{\phi}^{\mathcal{L}}(g)$. In this case, we say that the multivalued operator $\mu_{\phi}^{\mathcal{L}}(f)$ is a monotone operator.

The main purpose of this paper is to extend the best approximation operator to the set $L^{\phi'}$. The case $\varphi(t) = t^p$, $p > 1$, was extensively treated in [2] and the best L^1 approximation operator is extended to all measurable functions in [3]. The extension from L^ϕ to $L^{\phi'}$ is considered in [4] for a C^1 function φ which is strictly convex and $\varphi'(0) = 0$.

Now in this paper we consider a C^1 convex function ϕ , $\phi'(0) = 0$, but not necessarily a strictly convex function. Extension of best approximation operator when the approximation classes are the constants is treated in [5–7].

The extension of the best approximation operator is $\mu_{\phi}^{\mathcal{L}}(f)$; for $f \in L^{\phi'}$, we will be denoted by $\tilde{\mu}_{\phi}^{\mathcal{L}}(f)$. In Theorem 2.12, we prove that $\tilde{\mu}_{\phi}^{\mathcal{L}}(f) \neq \emptyset$ for every $f \in L^{\phi'}$, while in Theorem 2.16 it is proved that it is indeed an extension, that is, $\tilde{\mu}_{\phi}^{\mathcal{L}}(f) = \mu_{\phi}^{\mathcal{L}}(f)$ for $f \in L^\phi$. Additional properties are obtained for the set $\tilde{\mu}_{\phi}^{\mathcal{L}}(f)$ when the σ -lattice \mathcal{L} is a σ -algebra (see Theorem 2.17) and similar results hold when \mathcal{L} is the class of monotone functions in $L^{\phi'}$ (see Theorem 3.2). A martingale-type result is given in Theorem 4.1 which generalizes [8, Theorem 2.8] for the particular family of measures considered in this paper.

2. Extension of the best ϕ -approximation operator

We begin with some definitions and auxiliary results. The proof of the next two lemmas can be found in [4].

Lemma 2.1. *A necessary and sufficient condition for a continuously differentiable and convex function ϕ to satisfy the Δ_2 condition is that there exists a constant $\alpha > 1$ such that*

$$u\phi'(u) \leq \alpha \phi(u) \quad \forall u \geq 0. \quad (2.1)$$

Lemma 2.2. *Let f, g be in L^ϕ . Then $\phi'(|f|)g$ is an integrable function.*

According to Brunk and Johansen [8], we set the following definitions.

Definition 2.3. Let ν be a signed measure on \mathcal{A} and let \mathcal{L} be a σ -lattice contained in \mathcal{A} . Say that $P \in \mathcal{L}$ is a ν -positive set, if for all $D \in \overline{\mathcal{L}}$, then $\nu(P \cap D) \geq 0$. A set $N \in \overline{\mathcal{L}}$ is called ν -negative if for all $C \in \mathcal{L}$ one has $\nu(N \cap C) \leq 0$.

Definition 2.4. Let $\{\nu_a\}_{a \in \mathbb{R}}$ be a family of measures on \mathcal{A} , and let \mathcal{L} be a σ -lattice contained in \mathcal{A} . An \mathcal{L} -measurable function g is called a Lebesgue-Radon-Nikodym function (LRN function) for $\{\nu_a\}$ given \mathcal{L} if and only if the set $\{g > a\}$ is ν_a -positive for all $a \in \mathbb{R}$ and the set $\{g < a\}$ is ν_a -negative for all $a \in \mathbb{R}$.

Remark 2.5. We note that in Definition 2.4 it is sufficient to impose the conditions for all a in a dense set in \mathbb{R} , see [8, page 588].

For $f \in L^\phi$, $g \in L^\phi(\mathcal{L})$, and $a \in \mathbb{R}$, we define the following measures on \mathcal{A} :

$$\mu_g(A) = \int_A \underline{\phi}'(f - g) d\mu, \quad \mu_a(A) = \int_A \underline{\phi}'(f - a) d\mu, \quad (2.2)$$

where $\underline{\phi}'(x) = \phi'(|x|)\text{sign}(x)$. Note that when $f \in L^\phi$ and $g \in L^\phi(\mathcal{L})$, the measure μ_g and μ_a are well defined.

The next theorem is a characterization of $\mu_{\underline{\phi}}^{\mathcal{L}}(f)$, see [9, Theorem 3.2].

Theorem 2.6. *Let $f \in L^\phi$, $\mathcal{L} \subset \mathcal{A}$ be a σ -lattice and $g \in L^\phi(\mathcal{L})$. Then the following statement are equivalent.*

- (1) $g \in \mu_{\underline{\phi}}^{\mathcal{L}}(f)$.
- (2) (a) a set $\{g > a\}$ is μ_g -positive for all $a \in \mathbb{R}$; and
(b) a set $\{g < a\}$ is μ_g -negative for all $a \in \mathbb{R}$.
- (3) g is an LRN function for the family $\{\mu_a\}_{a \in \mathbb{R}}$ given \mathcal{L} .

Now we extend the operator $\mu_{\underline{\phi}}^{\mathcal{L}}(\cdot)$ to the space L^ϕ .

Definition 2.7. Let \mathcal{L} be a σ -lattice and let $f \in L^\phi$. Then g is an extended best ϕ -approximation if and only if $g \in L^\phi(\mathcal{L})$ and

- (i) the set $\{g > a\}$ is μ_g -positive for all $a \in \mathbb{R}$;
- (ii) the set $\{g < a\}$ is μ_g -negative for all $a \in \mathbb{R}$.

For $f \in L^\phi$ we denote by $\tilde{\mu}_{\underline{\phi}}^{\mathcal{L}}(f)$ the set of all extended best ϕ -approximation functions.

Remark 2.8. Let $f \in L^\phi$ and let g be a function in $L^\phi(\mathcal{L})$ such that the set $\{g > a\}$ is μ_g -positive and the set $\{g < a\}$ is μ_g -negative for all $a \in \mathbb{R}$. Then we have the following.

- (i) For all $h \in L^\infty(\overline{\mathcal{L}})$ and $h \geq 0$,

$$\int_{\{g > a\}} \underline{\phi}'(f - g) h d\mu \geq 0. \quad (2.3)$$

For all $h \in L^\infty(\mathcal{L})$ and $h \geq 0$,

$$\int_{\{g < a\}} \underline{\phi}'(f - g)h \, d\mu \leq 0. \quad (2.4)$$

(ii)

$$\int \underline{\phi}'(f - g) \, d\mu = 0. \quad (2.5)$$

Proof. We prove inequality (2.3). Since the set $\{g > a\}$ is μ_g -positive, that is, for each $D \in \overline{\mathcal{L}}$ and $a \in \mathbb{R}$, we have

$$\int_{\{g > a\} \cap D} \underline{\phi}'(f - g) \, d\mu \geq 0. \quad (2.6)$$

For $h = \sum_{k=1}^N c_k \mathcal{X}_{D_k}$, where $D_k \in \overline{\mathcal{L}}$ and $c_k \geq 0$, $k = 1, \dots, N$, then by (2.6), we have

$$\int_{\{g > a\}} \underline{\phi}'(f - g)h \, d\mu = \sum_{k=1}^N c_k \int_{\{g > a\} \cap D_k} \underline{\phi}'(f - g) \, d\mu \geq 0. \quad (2.7)$$

All nonnegative $h \in L^\infty(\overline{\mathcal{L}})$ can be obtained as a limit of functions of the above type. The proof of inequality (2.4) is similar.

The equality (2.5) is obtained using Lebesgue's theorem when $a \rightarrow \infty$ with $h = 1$ in (2.4) and if $a \rightarrow -\infty$ consider in (2.3) also $h = 1$. \square

As a reference, we note that (2) is equivalent to (3) in Theorem 2.6 for $f \in L^{\phi'}$, $g \in L^{\phi'}$. We have the next remark.

Remark 2.9. For $f \in L^{\phi'}$ and $g \in L^{\phi'}(\mathcal{L})$, the following statements are equivalent:

- (1) $g \in \tilde{\mu}_\phi^{\mathcal{L}}(f)$;
- (2) g is an LRN function for the family $\{\mu_a\}_{a \in \mathbb{R}}$ given \mathcal{L} .

The next lemma is a particular case of [8, Theorem 1.8].

Lemma 2.10. *Let $f \in L^{\phi'}$ and $g \in L^{\phi'}(\mathcal{L})$. Then the following statements are equivalent.*

- (1) g is an LRN function for the family $\{\mu_a\}_{a \in \mathbb{R}}$ given \mathcal{L} .
- (2) There exists a countable set D such that $\{g \leq a\}$ is μ_a -negative for all $a \in D^c$ and the set $\{g \geq a\}$ is μ_a -positive for all $a \in D^c$.

We need the following auxiliary result.

Lemma 2.11. *Let $f_n \in L^{\phi'}$ be a sequence of functions such that $f_n \nearrow f$ ($f_n \searrow f$), where $f \in L^{\phi'}$. Let $g_n \in \tilde{\mu}_\phi^{\mathcal{L}}(f_n)$ be such that $g_n \nearrow g$ ($g_n \searrow g$). Then $g \in \tilde{\mu}_\phi^{\mathcal{L}}(f)$.*

Proof. We will prove the result just for the increasing case, the proof for the decreasing case follows the same pattern. The function $g = \lim_{n \rightarrow \infty} g_n$ is obviously \mathcal{L} -measurable function. Now, we prove that $g \in L^{\phi'}$, and it satisfies (i) and (ii) of Definition 2.7. We have that

$$f_n - g_n \leq f - g_n \leq f - g_1. \quad (2.8)$$

Using (2.5),

$$\int_{\Omega} \underline{\phi}'(f_n - g_n) d\mu = 0 \quad \text{for each } n \in \mathbb{N}. \quad (2.9)$$

According to (2.8) and (2.9), we have, since $\underline{\phi}'(x)$ is an increasing function,

$$\int_{\Omega} \underline{\phi}'(f - g_n) d\mu \geq 0. \quad (2.10)$$

Since ϕ' is a continuous function and $\phi'(0) = 0$, we have by (2.8) and Lebesgue's theorem

$$\int_{\Omega} (\underline{\phi}'(f - g) \vee 0) d\mu \leq \int_{\Omega} (\underline{\phi}'(f - g_1) \vee 0) d\mu. \quad (2.11)$$

Now by (2.10) it holds

$$\int_{\Omega} (-\underline{\phi}'(f - g_n) \vee 0) d\mu \leq \int_{\Omega} (\underline{\phi}'(f - g_n) \vee 0) d\mu. \quad (2.12)$$

Using Fatou in (2.12), we have

$$\int_{\Omega} (-\underline{\phi}'(f - g) \vee 0) d\mu \leq \int_{\Omega} (\underline{\phi}'(f - g) \vee 0) d\mu. \quad (2.13)$$

Therefore, using (2.11) and (2.12), we get $g \in L^{\phi'}(\Omega)$ and $\int_{\Omega} \underline{\phi}'(f - g) d\mu \geq 0$.

Let $a \in \mathbb{R}$, $D \in \overline{\mathcal{L}}$, and $C \in \mathcal{L}$ we know for each n

$$\int_{\{g_n > a\} \cap D} \underline{\phi}'(f_n - g_n) d\mu \geq 0, \quad \int_{\{g_n < a\} \cap C} \underline{\phi}'(f_n - g_n) d\mu \leq 0. \quad (2.14)$$

Since $\{g_n\}_{n \in \mathbb{N}}$ is an increasing sequence, we get $\bigcup_{n \in \mathbb{N}} \{g_n > a\} = \{g > a\}$, and by (2.14), we have

$$\int_{\{g > a\} \cap D} \underline{\phi}'(f - g) d\mu \geq 0. \quad (2.15)$$

Hence, the set $\{g > a\}$ is μ_g -positive for all $a \in \mathbb{R}$.

Now, for $a \in \mathbb{R}$ and $k, n \in \mathbb{N}$, we define $B_n = \{g_n < a\}$ and $B_{n,k} = \{g_n < a - 1/k\}$. We have that, for $n \rightarrow \infty$, $B_{n,k} \searrow A_k$ for some \mathcal{A} -measurable set such that

$$\left\{ g < a - \frac{1}{k} \right\} \subset A_k \subset \left\{ g \leq a - \frac{1}{k} \right\}. \quad (2.16)$$

We observe that $\mathcal{X}_{A_k} \rightarrow \mathcal{X}_{\{g < a\}}$, a.e. Then taking limit as $k \rightarrow \infty$, $n \rightarrow \infty$ and using Lebesgue's theorem, we obtain

$$\int_{\{g < a\} \cap C} \underline{\phi}'(f - g) d\mu = \lim_{k \rightarrow \infty} \int \underline{\phi}'(f - g) \mathcal{X}_{A_k \cap C} d\mu = \lim_{k \rightarrow \infty} \lim_{n \rightarrow \infty} \int_{B_{n,k} \cap C} \underline{\phi}'(f - g) d\mu \leq 0. \quad (2.17)$$

□

Theorem 2.12. *Let \mathcal{L} be a σ -lattice and $f \in L^\phi$, then $\tilde{\mu}_\phi^\ell(f) \neq \emptyset$.*

Proof. For $f \in L^\phi$ we can define the following sequences. For each $m \in \mathbb{N}$, let $f_m = f \vee (-m)$ and when $m \rightarrow \infty$ we have $f_m \searrow f$. Set $f_m^n = (f \vee (-m)) \wedge n$ for all n, m in \mathbb{N} , then we have $f_{mn} \nearrow f_m$, when $n \rightarrow \infty$. Since for each $m, n \in \mathbb{N}$ we have $f_m^n \in L^\phi$, there exist $g_m^n \in \mu_\phi^\ell(f_m^n)$. As $\mu_\phi^\ell(\cdot)$ is a mono-tone operator over L^ϕ we can take a new sequence that we call again $g_m^n \in \mu_\phi^\ell(f_m^n)$, such that $g_m^n \leq g_m^{n+1}$ for all $n \in \mathbb{N}$.

Since $f_m^n \geq f_{m+1}^n$ and using again that $\mu_\phi^\ell(\cdot)$ is a monotone operator, we have $\tilde{g}_{m+1}^n \leq \tilde{g}_m^n$, where $\tilde{g}_m^n \in \mu_\phi^\ell(f_m^n)$ is the sequence defined by $\tilde{g}_1^n = g_1^n$ and $\tilde{g}_{m+1}^n = \tilde{g}_m^n \wedge g_{m+1}^n$. Furthermore, it is easy to check that $\tilde{g}_m^n \leq \tilde{g}_m^{n+1}$.

Then, we have that for each $m \in \mathbb{N}$ that $f_m^n \nearrow f_m$, when $n \rightarrow \infty$, and since $\mu_\phi^\ell(f_m^n) \subset \tilde{\mu}_\phi^\ell(f_m^n)$ we have $\tilde{g}_m^n \in \tilde{\mu}_\phi^\ell(f_m^n)$ and if we define $g_m = \lim_{n \rightarrow \infty} \tilde{g}_m^n$ by Lemma 2.11 we obtain $g_m \in \tilde{\mu}_\phi^\ell(f_m)$ and $g_m \geq g_{m+1}$ for all $m \in \mathbb{N}$. If we take $m \rightarrow \infty$, we have $f_m \searrow f$ and by Lemma 2.11 we get $g \in \tilde{\mu}_\phi^\ell(f)$, where $g = \lim_{m \rightarrow \infty} g_m$. □

To see that the extended best ϕ -approximation is an extension of the best ϕ -approximation operator, we must prove $\tilde{\mu}_\phi^\ell(f) = \mu_\phi^\ell(f)$ for every $f \in L^\phi$. First, we need to prove the following lemmas.

Lemma 2.13. *Let ϕ be a C^1 convex function and assume that it satisfies the Δ_2 -condition. Then*

$$\phi(a) + \frac{K}{2} a \phi'(x - a) \leq \frac{K+2}{2} \phi(x), \quad (2.18)$$

for $a, x \geq 0$, where K is the constant for the Δ_2 condition.

Proof. We consider two cases. First, we assume $0 < x \leq a$. Since ϕ is Δ_2 -convex function, we have that $\phi(a) = \phi(a - x + x) \leq (K/2)(\phi(a - x) + \phi(x))$. Using $x\phi'(x) \geq \phi(x)$ for all x , we get

$$\begin{aligned} \phi(x) + a\phi'(a - x) &= \phi(x) + (a - x + x)\phi'(a - x) \\ &\geq \phi(x) + \phi(a - x) + x\phi'(a - x) \\ &\geq \frac{2}{K}\phi(a) + x\phi'(a - x) \geq \frac{2}{K}\phi(a). \end{aligned} \quad (2.19)$$

Then we obtain

$$\phi(a) - \frac{K}{2} a\phi'(a - x) \leq \frac{K}{2} \phi(x). \quad (2.20)$$

For $0 \leq a < x$, we have

$$\phi(a) + a\frac{K}{2}\phi'(x - a) \leq \phi(a) + \frac{K}{2} \int_{x-a}^x \phi'(t) dt \leq \phi(x) + \frac{K}{2} \phi(x) = \frac{K+2}{2} \phi(x). \quad (2.21)$$

□

Lemma 2.14. *Let $f \in L^\phi$ and $g \in \tilde{\mu}_\phi^{\underline{L}}(f)$, then*

$$\int_{\{g>0\}} \underline{\phi}'(f-g)g \, d\mu \geq 0. \quad (2.22)$$

Proof. Since $\{g > a\}$ is μ_g -positive for all $a \in \mathbb{R}$, then for all $D \in \overline{\mathcal{L}}$, we have that

$$\int_{\{g>a\} \cap D} \underline{\phi}'(f-g) \, d\mu \geq 0. \quad (2.23)$$

In particular, it holds that for all $a \in \mathbb{R}$,

$$\int_{\{g>a\}} \underline{\phi}'(f-g) \, d\mu \geq 0, \quad (2.24)$$

that is,

$$\int_{\{g>a\} \cap \{f>g\}} \phi'(|f-g|) \, d\mu \geq \int_{\{g>a\} \cap \{f \leq g\}} \phi'(|f-g|) \, d\mu. \quad (2.25)$$

Now, we have

$$\int_0^\infty \int_{\{g>a\} \cap \{f>g\}} \phi'(|f-g|) \, d\mu \, da \geq \int_0^\infty \int_{\{g>a\} \cap \{f \leq g\}} \phi'(|f-g|) \, d\mu \, da. \quad (2.26)$$

By the Fubini's theorem, we get

$$\int_{\{g>0\}} \int_0^g \phi'(|f-g|) \mathcal{X}_{\{f>g\}} \, da \, d\mu \geq \int_{\{g>0\}} \int_0^g \phi'(|f-g|) \mathcal{X}_{\{f \leq g\}} \, da \, d\mu. \quad (2.27)$$

Thus

$$\int_{\{g>0\}} \phi'(|f-g|) \mathcal{X}_{\{f>g\}} g \, d\mu \geq \int_{\{g>0\}} \phi'(|f-g|) \mathcal{X}_{\{f \leq g\}} g \, d\mu. \quad (2.28)$$

To see that inequality (2.22) is equivalent to (2.28) we will prove that $\phi'(f-g) \mathcal{X}_{\{f>g\}} g \in L^1(\{g > 0\})$. In fact

$$\int_{\{g>0\}} \phi'(f-g) \mathcal{X}_{\{f>g\}} g \, d\mu \leq \int_{\{g>0\}} \phi'(f-g) \mathcal{X}_{\{f>g\}} f \, d\mu \leq \int_{\{g>0\}} \phi'(f) f. \quad (2.29)$$

Since $f \in L^\phi$ by Lemma 2.2, the last integral is finite. \square

The following properties of the set $\tilde{\mu}_\phi^{\underline{L}}(f)$ can be easily proved.

Proposition 2.15. *Let $f \in L^\phi$, then*

- (1) $-\tilde{\mu}_\phi^{\underline{L}}(-f) = \tilde{\mu}_\phi^{\underline{L}}(f)$,
- (2) $\tilde{\mu}_\phi^{\underline{L}}(f+t) = \tilde{\mu}_\phi^{\underline{L}}(f) + t$ for all $h \in \mathbb{R}$.

Now we prove that the operator $\tilde{\mu}_\phi^{\underline{L}}(f)$ is in fact an extension of the operator $\mu_\phi^{\underline{L}}(f)$.

Theorem 2.16. *Let $f \in L^\phi$, then $\tilde{\mu}_\phi^\ell(f) = \mu_\phi^\ell(f)$.*

Proof. For $f \in L^\phi$, we will prove only that $\tilde{\mu}_\phi^\ell(f) \subset \mu_\phi^\ell(f)$. The other inclusion follows from Theorem 2.6. Let $g \in \tilde{\mu}_\phi^\ell(f)$ and again using Theorem 2.6 it remains to prove that $g \in L^\phi$. Recall that $\phi(0) = 0$, then

$$\int_{\Omega} \phi(|g|) d\mu = \int_{\{g>0\}} \phi(g) d\mu + \int_{\{g<0\}} \phi(-g) d\mu. \quad (2.30)$$

By Lemma 2.13 we obtain the following inequality:

$$\int_{\{g>0\}} \phi(g) d\mu + \frac{K}{2} \int_{\{g>0\}} \underline{\phi}'(|f| - g) g d\mu \leq \frac{K+2}{2} \int_{\{g>0\}} \phi(|f|) d\mu. \quad (2.31)$$

Applying Lemma 2.14 and taking into account that $\underline{\phi}'(x)$ is an increasing function, we get

$$0 \leq \int_{\{g>0\}} \underline{\phi}'(f - g) g d\mu \leq \int_{\{g>0\}} \underline{\phi}'(|f| - g) g d\mu. \quad (2.32)$$

Thus using (2.32) in (2.31), we have

$$\int_{\{g>0\}} \phi(g) d\mu \leq \frac{K+2}{2} \int_{\Omega} \phi(|f|) d\mu. \quad (2.33)$$

For the set $\{g < 0\}$, again by Lemma 2.13, we obtain

$$\int_{\{g<0\}} \phi(-g) d\mu + \frac{K}{2} \int_{\{g<0\}} \underline{\phi}'(|f| + g) (-g) d\mu \leq \frac{K+2}{2} \int_{\{g<0\}} \phi(|f|) d\mu. \quad (2.34)$$

Since $-f + g \leq |f| + g$, we have $\underline{\phi}'(-f + g) \leq \underline{\phi}'(|f| + g)$. Thus

$$\int_{\{g<0\}} \underline{\phi}'(-f + g) (-g) d\mu \leq \int_{\{g<0\}} \underline{\phi}'(|f| + g) (-g) d\mu. \quad (2.35)$$

By (1) in Proposition 2.15, $-g \in \tilde{\mu}_\phi^\ell(-f)$, and by Lemma 2.14, we have that

$$\int_{\{g<0\}} \underline{\phi}'(-f + g) (-g) d\mu \geq 0. \quad (2.36)$$

Therefore, $\int_{\{g<0\}} \phi(-g) d\mu \leq ((K+2)/2) \int \phi(|f|) d\mu$. By (2.33), we have

$$\int_{\Omega} \phi(|g|) d\mu \leq \frac{K+2}{2} \int_{\Omega} \phi(|f|) d\mu, \quad (2.37)$$

and therefore $g \in L^\phi$. □

Now, if consider a σ -subalgebra \mathcal{B} instead of a σ -lattice \mathcal{L} , the extended best ϕ -approximation operator has the following properties.

Theorem 2.17. *Let f , f_1 , and f_2 be in L^ϕ , if \mathcal{B} is a sub- σ -algebra of the σ -algebra \mathcal{A} , then the following hold.*

- (1) *The set-valued function $\tilde{\mu}_\phi^\mathcal{B}(f)$ is a monotone operator.*
- (2) *The set $\tilde{\mu}_\phi^\mathcal{B}(f)$ is a σ -complete lattice, and there exist $U_\mathcal{B}, V_\mathcal{B} \in \tilde{\mu}_\phi^\mathcal{B}(f)$ such that $U_\mathcal{B} \leq g \leq V_\mathcal{B}$ a.e. for every $g \in \tilde{\mu}_\phi^\mathcal{B}(f)$.*

Proof. To prove (1), recall that this set-valued operator is monotone if $f_1 \leq f_2$; then if $g_1 \in \tilde{\mu}_\phi^\mathcal{B}(f_1)$ and $g_2 \in \tilde{\mu}_\phi^\mathcal{B}(f_2)$, we have that $g_1 \wedge g_2 \in \tilde{\mu}_\phi^\mathcal{B}(f_1)$ and $g_1 \vee g_2 \in \tilde{\mu}_\phi^\mathcal{B}(f_2)$. Since $L^\phi(\mathcal{B})$ is a lattice, we know $g_1 \wedge g_2 \in L^\phi(\mathcal{B})$ and $g_1 \vee g_2 \in L^\phi(\mathcal{B})$. We will prove first that $g_1 \wedge g_2 \in \tilde{\mu}_\phi^\mathcal{B}(f_1)$. Set $\mu_a^{f_i}(A) = \int_A \phi'(f_i - a) d\mu$, where $a \in \mathbb{R}$ and $i = 1, 2$. We will see that $g_1 \wedge g_2$ is an LRN function for the family of measures $\{\mu_a^{f_1}\}_{a \in \mathbb{R}}$ given \mathcal{B} . First, we will see that for each $a \in \mathbb{R}$ and for all $B \in \mathcal{B}$, we have

$$\mu_a^{f_1}(\{g_1 \wedge g_2 > a\} \cap B) \geq 0. \quad (2.38)$$

Since $\{g_1 \wedge g_2 > a\} \cap B = \{g_1 > a\} \cap \{g_2 > a\} \cap B$ and $\{g_2 > a\} \cap B \in \mathcal{B}$ and using that g_1 is an LRN function of the family $\{\mu_a^{f_1}\}_{a \in \mathbb{R}}$, we obtain that for all $B \in \mathcal{B}$

$$\int_{\{g_1 \wedge g_2 > a\} \cap B} \phi'(f_1 - a) d\mu = \int_{\{g_1 > a\} \cap \{g_2 > a\} \cap B} \phi'(f_1 - a) d\mu \geq 0. \quad (2.39)$$

Now, we see that $\{g_1 \wedge g_2 < a\}$ is $\mu_a^{f_1}$ -negative for all $a \in \mathbb{R}$. For $B \in \mathcal{B}$, we have

$$\{g_1 \wedge g_2 < a\} \cap B = (\{g_1 < a\} \cap B) \cup (\{g_1 \geq a\} \cap \{g_2 < a\} \cap B). \quad (2.40)$$

Using $f_1 \leq f_2$ and that $\phi'(\cdot)$ is a nondecreasing function, we obtain

$$\begin{aligned} \int_{\{g_1 \wedge g_2 < a\} \cap B} \phi'(f_1 - a) d\mu &= \int_{\{g_1 < a\} \cap B} \phi'(f_1 - a) d\mu + \int_{\{g_1 \geq a\} \cap \{g_2 < a\} \cap B} \phi'(f_1 - a) d\mu \\ &\leq \int_{\{g_1 < a\} \cap B} \phi'(f_1 - a) d\mu + \int_{\{g_1 \geq a\} \cap \{g_2 < a\} \cap B} \phi'(f_2 - a) d\mu \leq 0. \end{aligned} \quad (2.41)$$

Thus

$$\int_{\{g_1 \wedge g_2 < a\} \cap B} \phi'(f_1 - a) d\mu \leq 0. \quad (2.42)$$

By (2.39) and (2.42), we have $g_1 \wedge g_2 \in \tilde{\mu}_\phi^\mathcal{B}(f_1)$.

Now we show $g_1 \vee g_2 \in \tilde{\mu}_\phi^\mathcal{B}(f_2)$. Since $f_1 \leq f_2$ and $\{g_1 > a\}$ is a $\mu_a^{f_1}$ -positive for all $a \in \mathbb{R}$ and for all $B \in \mathcal{B}$, we have

$$\begin{aligned} \int_{\{g_1 \vee g_2 > a\} \cap B} \phi'(f_2 - a) d\mu &= \int_{\{g_2 > a\} \cap B} \phi'(f_2 - a) d\mu + \int_{\{g_1 > a\} \cap \{g_2 \leq a\} \cap B} \phi'(f_2 - a) d\mu \\ &\geq \int_{\{g_2 > a\} \cap B} \phi'(f_2 - a) d\mu + \int_{\{g_1 > a\} \cap \{g_2 \leq a\} \cap B} \phi'(f_1 - a) d\mu \geq 0. \end{aligned} \quad (2.43)$$

Since

$$\int_{\{g_1 \vee g_2 < a\} \cap B} \underline{\phi}'(f_2 - a) d\mu = \int_{\{g_1 < a\} \cap \{g_2 < a\} \cap B} \underline{\phi}'(f_2 - a) d\mu \leq 0, \tag{2.44}$$

the inequalities (2.43) and (2.44) prove that $g_1 \vee g_2 \in \tilde{\mu}_\phi^B(f_2)$.

As the statement (1) proves in particular that $\tilde{\mu}_\phi^B(f)$ is a lattice, we will see that the set is a σ -complete lattice. Given a sequence $\{g_n\}_{n \in \mathbb{N}}$ in $\tilde{\mu}_\phi^B(f)$, we have that $\bigvee_1^n g_k \in \tilde{\mu}_\phi^B(f)$; then, from Lemma 2.11 we obtain that $\bigvee_{n \in \mathbb{N}} g_n = \lim_{n \rightarrow \infty} \bigvee_1^n g_n \in \tilde{\mu}_\phi^B(f)$. The proof $\bigwedge_{n \in \mathbb{N}} g_n \in \tilde{\mu}_\phi^B(f)$ is similar.

By [10, Proposition II.4.1] there exists a sequence $g_n \in \tilde{\mu}_\phi^B(f)$ such that $\inf g_n \leq g \leq \sup g_n$, for every $g \in \tilde{\mu}_\phi^B(f)$. Set $U_B = \inf g_n$ and $V_B = \sup g_n$, then U_B and V_B are in $\tilde{\mu}_\phi^B(f)$ since this set is a σ -complete lattice. □

3. Extended best ϕ -approximation with nondecreasing functions

When the approximation class is the monotone functions defined on $[0, 1]$ we can obtain similar results as those of Theorem 2.17. Now $\Omega = [0, 1]$, μ is Lebesgue measure on the measurable sets, and $\mathcal{L} = \{(a, 1), [a, 1), \emptyset, \mathbb{R}\}_{a \in \mathbb{R}}$. Therefore, $L^\phi(\mathcal{L})$ is the set of nondecreasing functions in $L^\phi[0, 1]$.

Remark 3.1. Let g be a nondecreasing function on $[0, 1]$. Given $a \in \mathbb{R}$, the set $\{g < a\}$ is one of the intervals $[0, \alpha_a)$ or $[0, \alpha_a]$ and similarly the set $\{g > a\}$ is $(\beta_a, 1]$ or $[\beta_a, 1]$ with $0 \leq \alpha_a \leq \beta_a \leq 1$. Then $H_g = \{a \in \mathbb{R} : \alpha_a = \beta_a\}$ is a dense set in \mathbb{R} . In fact, the complement set of H_g is a countable set.

Note that each $C \in \mathcal{L}$ is of the form $(c, 1]$ or $[c, 1]$ and $D \in \overline{\mathcal{L}}$ is $D = [0, d)$ or $[0, d]$. Thus

$$\int_{\{g > a\} \cap D} \underline{\phi}'(f - a) d\mu = \int_{\beta_a}^d \underline{\phi}'(f - a) d\mu, \quad \int_{\{g < a\} \cap C} \underline{\phi}'(f - a) d\mu = \int_c^{\alpha_a} \underline{\phi}'(f - a) d\mu. \tag{★}$$

Theorem 3.2. *Let $L^\phi(\mathcal{L})$ be the class of the ϕ' -integrable nondecreasing functions in $[0, 1]$. Then the following hold.*

- (1) *The set mapping $\tilde{\mu}_\phi^{\mathcal{L}}(f)$ is a monotone operator.*
- (2) *For every $f \in L^\phi$, the set $\tilde{\mu}_\phi^{\mathcal{L}}(f)$ is a σ -complete lattice.*

Proof. First we prove (1), that is given f_1, f_2 in L^ϕ with $f_1 \leq f_2$, for each $g_i \in \tilde{\mu}_\phi^{\mathcal{L}}(f_i)$, $i = 1, 2$ we will see that $g_1 \wedge g_2 \in \tilde{\mu}_\phi^{\mathcal{L}}(f_1)$ and $g_1 \vee g_2 \in \tilde{\mu}_\phi^{\mathcal{L}}(f_2)$. Let H be $H_{g_1} \cap H_{g_2}$, where H_{g_i} is the set given in Remark 3.1. Recall that $g_1 \wedge g_2 \in \tilde{\mu}_\phi^{\mathcal{L}}(f_1)$ if and only if for each $a \in H$ and $c, d \in \mathbb{R}$

$$\int_{\{g_1 \wedge g_2 > a\} \cap (0, d)} \underline{\phi}'(f_1 - a) dx \geq 0, \quad \int_{\{g_1 \wedge g_2 < a\} \cap (c, 1)} \underline{\phi}'(f_1 - a) dx \leq 0. \tag{3.1}$$

Also $g_1 \vee g_2 \in \tilde{\mu}_\phi^{\mathcal{L}}(f_2)$ if and only if for each $a \in H$, and $c, d \in \mathbb{R}$,

$$\int_{\{g_1 \vee g_2 > a\} \cap (0, d)} \underline{\phi}'(f_2 - a) dx \geq 0, \quad \int_{\{g_1 \vee g_2 < a\} \cap (c, 1)} \underline{\phi}'(f_2 - a) dx \leq 0. \tag{3.2}$$

First we prove (3.1). Now we see that

$$\int_{\{g_1 \wedge g_2 > a\} \cap (0, d)} \underline{\phi}'(f_1 - a) dx \geq 0, \quad (3.3)$$

with $\{g_1 \wedge g_2 > a\} = (\beta_1^a, 1] \cap (\beta_2^a, 1]$. Since $\int_{(\beta_1^a, d)} \underline{\phi}'(f_1 - a) dx \geq 0$ to prove (3.3), we have to see that

$$\int_{(\beta_2^a, d)} \underline{\phi}'(f_1 - a) dx \geq 0, \quad (3.4)$$

where $\beta_1^a \leq \beta_2^a$. Indeed by (\star) , we get

$$0 \leq \int_{\beta_1^a}^{\beta_2^a} a^d \underline{\phi}'(f_1 - a) dx = \int_{\beta_1^a}^{\beta_2^a} \underline{\phi}'(f_1 - a) dx + \int_{\beta_2^a}^d \underline{\phi}'(f_1 - a) dx. \quad (3.5)$$

Since $\underline{\phi}'(\cdot)$ is a nondecreasing function, we have

$$0 \leq \int_{\beta_1^a}^{\beta_2^a} \underline{\phi}'(f_2 - a) dx + \int_{\beta_2^a}^d \underline{\phi}'(f_1 - a) dx. \quad (3.6)$$

As $\int_{\beta_1^a}^{\beta_2^a} \underline{\phi}'(f_2 - a) dx \leq 0$ ($\beta_2^a = \alpha_2^a$), we have $\int_{\beta_2^a}^d \underline{\phi}'(f_1 - a) dx \geq 0$, that is (3.4).

Now we will prove that

$$\int_{\{g_1 \wedge g_2 < a\} \cap (c, 1)} \underline{\phi}'(f_1 - a) dx \leq 0. \quad (3.7)$$

In fact

$$\begin{aligned} \int_{\{g_1 \wedge g_2 < a\} \cap (c, 1)} \underline{\phi}'(f_1 - a) dx &= \int_{\{g_1 < a\} \cap (c, 1)} \underline{\phi}'(f_1 - a) dx + \int_{\{g_1 \geq a\} \cap \{g_2 < a\} \cap (c, 1)} \underline{\phi}'(f_1 - a) dx \\ &\leq \int_{\{g_1 < a\} \cap (c, 1)} \underline{\phi}'(f_1 - a) dx + \int_{\{g_2 < a\} \cap [\{g_1 \geq a\} \cap (c, 1)]} \underline{\phi}'(f_2 - a) dx. \end{aligned} \quad (3.8)$$

The last two integrals in (3.8) are less or equal than zero, so (3.7) holds. A similar argument shows that $g_1 \vee g_2 \in \tilde{\mu}_\phi^d(f_2)$. Therefore, the extended best ϕ -approximation operator is a monotone operator. By (1), we have that $\tilde{\mu}_\phi^d(f)$ is a lattice, just setting $f = f_1 = f_2$. Now by Lemma 2.11 we obtain that $\tilde{\mu}_\phi^d(f)$ is a σ -complete lattice. \square

4. A limit theorem for extended best ϕ -approximations

Given a sequence $\{\mathcal{B}_n\}_{n \in \mathbb{N}}$ of σ -algebras contained in the σ -algebra \mathcal{A} , we consider two cases, $\mathcal{B}_n \subset \mathcal{B}_{n+1}$ for all $n \in \mathbb{N}$ and we set \mathcal{B}_∞ for the σ -algebra generated by $\bigcup_n \mathcal{B}_n$, and if $\mathcal{B}_n \supseteq \mathcal{B}_{n+1}$ for all $n \in \mathbb{N}$, we set $\mathcal{B}_\infty = \bigcap_{n \in \mathbb{N}} \mathcal{B}_n$.

The next result is a particular case of [8, Theorem 2.8] when ϕ is a strictly convex function. This assumption on the function ϕ assures that the family of measures $\{\mu_a\}_{a \in \mathbb{R}}$ decreases at zero as required by Brunk and Johansen in that theorem.

Theorem 4.1. Let $\{\mathcal{B}_n\}_{n \in \mathbb{N}}$ be an increasing or decreasing sequence of σ -algebras in \mathcal{A} , and let \mathcal{B}_∞ be the limit of the sequence. If $f \in L^{\phi'}$, then we have for all $g_n \in \tilde{\mu}_\phi^{\mathcal{B}_n}(f)$, $n \in \mathbb{N}$, that $\liminf_{n \rightarrow \infty} g_n$, and $\limsup_{n \rightarrow \infty} g_n$ are in $\tilde{\mu}_\phi^{\mathcal{B}_\infty}(f)$.

Proof. Define $\bar{g} = \limsup_{n \rightarrow \infty} g_n$ and $\underline{g} = \liminf_{n \rightarrow \infty} g_n$, then we only prove that $\underline{g}, \bar{g} \in \tilde{\mu}_\phi^{\mathcal{B}_\infty}(f)$, when $\{\mathcal{B}_n\}_{n \in \mathbb{N}}$ is an increasing sequence of σ -algebras, the proof for the decreasing case is similar.

First, we prove for each $f \in L^{\phi'}$ that the set $\{\bar{g} \geq a\}$ is μ_a -positive for all $a \in \mathbb{R}$. Let $B \in \mathcal{B}_m$, and for $H_n = \bigcup_{k \geq n} \{g_k > a - \epsilon_n\}$, where ϵ_n decreases to zero, we have that $\{\bar{g} \geq a\} = \bigcap_{n \geq 1} H_n$, and for all $n \in \mathbb{N}$, $H_{n+1} \subset H_n$. Now for each $n \in \mathbb{N}$, we define the following disjoint sets. For $p \geq n$, set $H_{n,n} = \{g_n > a - \epsilon_n\}, \dots, H_{n,p} = \{g_p > a - \epsilon_n\} \cap \{g_{p-1} \leq a - \epsilon_n\} \cap \dots \cap \{g_n \leq a - \epsilon_n\}$. Thus, $H_n = \bigcup_{p \geq n} H_{n,p}$, and then

$$\int_{H_n \cap B} \underline{\phi}'(f - (a - \epsilon_n)) d\mu = \sum_{p=n}^{\infty} \int_{H_{n,p} \cap B} \underline{\phi}'(f - (a - \epsilon_n)) d\mu. \quad (4.1)$$

As $\mathcal{B}_n \subset \mathcal{B}_{n+1}$, and for $p \geq n$, we have that $\{g_{p-1} \leq a - \epsilon_n\} \cap \dots \cap \{g_n \leq a - \epsilon_n\} \in \mathcal{B}_p$. As $B \in \mathcal{B}_m$ then $B \in \mathcal{B}_p$ and $\int_{H_{n,p} \cap B} \underline{\phi}'(f - (a - \epsilon_n)) d\mu \geq 0$ for $m \leq n \leq p$. Thus $\int_{H_n \cap B} \underline{\phi}'(f - (a - \epsilon_n)) d\mu \geq 0$, and by Lebesgue's theorem, we get

$$\int_{\{\bar{g} \geq a\} \cap B} \underline{\phi}'(f - a) d\mu = \lim_{n \rightarrow \infty} \int_{H_n \cap B} \underline{\phi}'(f - (a - \epsilon_n)) d\mu \geq 0 \quad (4.2)$$

for all $B \in \bigcup_{n \in \mathbb{N}} \mathcal{B}_n$. Now we have (4.2) for all $B \in \mathcal{B}_\infty$. In fact, the set $D = \{B \in \mathcal{A} : \int_{\{\bar{g} \geq a\} \cap B} \underline{\phi}'(f - a) d\mu \geq 0\}$ is a monotone class, that is, the set D is closed for increasing and decreasing sequences of sets. As $\bigcup_{n \in \mathbb{N}} \mathcal{B}_n \subset D$ and this union is an algebra of sets, the monotone class generated by it is \mathcal{B}_∞ , that is, $\mathcal{B}_\infty \subset D$.

Now let us prove that the set $\{\bar{g} \leq a\}$ is μ_a -negative for all $a \in \mathbb{R}$. As $\{\bar{g} \leq a\} = \bigcap_{n \in \mathbb{N}} \{\bar{g} < a + 1/n\}$ and

$$\int_{\{\bar{g} \leq a\} \cap B} \underline{\phi}'(f - a) d\mu = \lim_{n \rightarrow \infty} \int_{\{\bar{g} < a + 1/n\} \cap B} \underline{\phi}'(f - a) d\mu, \quad (4.3)$$

then we have to prove that for all $a \in \mathbb{R}$, the set $\{\bar{g} < a\}$ is μ_a -negative. Since $\{\bar{g} < a\} = \bigcup_{n \geq 1} \bigcap_{k \geq n} \{g_k < a - \epsilon_n\} = \bigcup_{n \geq 1} H_n$, where $H_n = \bigcap_{k \geq n} \{g_k < a - \epsilon_n\}$ and $\epsilon_n \searrow 0$, we have for all $n \in \mathbb{N}$ that $H_n \subset H_{n+1}$. Then

$$\int_{\{\bar{g} < a\} \cap B} \underline{\phi}'(f - a) d\mu = \lim_{n \rightarrow \infty} \int_{H_n \cap B} \underline{\phi}'(f - (a - \epsilon_n)) d\mu, \quad (4.4)$$

for a fixed set $B \in \mathcal{B}_m$.

Set $G_n = \bigcap_{k \geq n+1} \{g_k < a - \epsilon_n\}$ and note that $H_n = G_n \cap \{g_n < a - \epsilon_n\}$. Then for $m \leq n$, we have

$$\begin{aligned} & \int_{\{g_n < a - \epsilon_n\} \cap B \cap G_n} \underline{\phi}'(f - (a - \epsilon_n)) d\mu + \int_{\{g_n < a - \epsilon_n\} \cap B \cap G_n^c} \underline{\phi}'(f - (a - \epsilon_n)) d\mu \\ &= \int_{\{g_n < a - \epsilon_n\} \cap B} \underline{\phi}'(f - (a - \epsilon_n)) d\mu \leq 0. \end{aligned} \quad (4.5)$$

Now, we prove the following inequality:

$$\int_{\{g_n < a - \epsilon_n\} \cap B \cap G_n^c} \underline{\phi}'(f - (a - \epsilon_n)) d\mu \geq 0. \quad (4.6)$$

We can see that

$$\{g_n < a - \epsilon_n\} \cap B \cap G_n^c = \bigcup_{k \geq n+1} A_k, \quad (4.7)$$

where A_k are the following disjoint sets

$$\begin{aligned} A_{n+1} &= \{g_{n+1} \geq a - \epsilon_n\} \cap B \cap \{g_n < a - \epsilon_n\}, \\ &\vdots \\ A_k &= \{g_k \geq a - \epsilon_n\} \cap \bigcap_{i=n+1}^{k-1} \{g_i < a - \epsilon_n\} \cap \{g_n < a - \epsilon_n\} \cap B. \end{aligned} \quad (4.8)$$

Then

$$\int_{\{g_n < a - \epsilon_n\} \cap B \cap G_n^c} \underline{\phi}'(f - (a - \epsilon_n)) d\mu = \sum_{k \geq n+1} \int_{A_k} \underline{\phi}'(f - (a - \epsilon_n)) d\mu. \quad (4.9)$$

Since $A_k = \{g_k \geq a - \epsilon_n\} \cap B_k$, where $B_k \in \mathcal{B}_k$, we have (4.6). Therefore by (4.5), we have

$$\int_{H_n \cap B} \underline{\phi}'(f - (a + \epsilon_n)) d\mu \leq 0, \quad (4.10)$$

for all $B \in \bigcup_{n \in \mathbb{N}} \mathcal{B}_n$. Thus by (4.10), (4.4), and (4.3), we get

$$\int_{\{\bar{g} \leq a\} \cap B} \underline{\phi}'(f - a) d\mu \leq 0 \quad (4.11)$$

for all $B \in \bigcup_{n \in \mathbb{N}} \mathcal{B}_n$. Therefore, the result is satisfied for all $B \in \mathcal{B}_\infty$. Thus $\bar{g} \in \tilde{\mu}_\phi^{\mathcal{B}_\infty}(f)$.

We have $\{\underline{g} \leq a\} = \bigcap_{n \geq 1} H_n$, where $H_n = \bigcup_{k \geq n} \{g_k < a + \epsilon_n\}$ and $\epsilon_n \searrow 0$, then $H_{n+1} \subset H_n$ for all $n \in \mathbb{N}$. Since $f \in L^{\phi'}$ we have for all $B \in \mathcal{B}_\infty$ that

$$\int_{\{\underline{g} \leq a\} \cap B} \underline{\phi}'(f - a) d\mu = \lim_{n \rightarrow \infty} \int_{H_n \cap B} \underline{\phi}'(f - (a + \epsilon_n)) d\mu. \quad (4.12)$$

For $p > n$ define the following disjoint sets $H_{n,n} = \{g_n < a + \epsilon_n\}$ and $H_{n,p} = \{g_p < a + \epsilon_n\} \cap \{g_{p-1} \geq a + \epsilon_n\} \cap \dots \cap \{g_n \geq a + \epsilon_n\}$. Then for $B \in \mathcal{B}_m$ we have

$$\int_{H_n \cap B} \underline{\phi}'(f - (a + \epsilon_n)) d\mu = \sum_{p \geq n} \int_{H_{n,p} \cap B} \underline{\phi}'(f - (a + \epsilon_n)) d\mu. \quad (4.13)$$

Now if $m \leq n \leq p$, $H_{n,p} \cap B = \{g_k < a + \epsilon_n\} \cap B^*$, where $B^* \in \mathcal{B}_p$. Thus

$$\int_{H_{n,p} \cap B} \underline{\phi}'(f - (a + \epsilon_n)) d\mu \leq 0. \quad (4.14)$$

Then by (4.12) and (4.14), we have for all $B \in \bigcup_{n \in \mathbb{N}} \mathcal{B}_n$ that

$$\int_{\{\underline{g} \leq a\} \cap B} \underline{\phi}'(f - a) d\mu \leq 0. \quad (4.15)$$

Therefore, we have (4.15) for all $B \in \mathcal{B}_\infty$.

Let us see now that

$$\int_{\{\underline{g} \geq a\} \cap B} \underline{\phi}'(f - a) d\mu \geq 0 \quad (4.16)$$

for all $B \in \bigcup_{n \in \mathbb{N}} \mathcal{B}_n$.

As $\{\underline{g} \geq a\} = \bigcap_{n \in \mathbb{N}} \{\underline{g} > a - 1/n\}$, we have to prove that for all $a \in \mathbb{R}$, the set $\{\underline{g} > a\}$ is μ_a -positive. We have that $\{\underline{g} > a\} = \bigcup_{n \geq 1} \bigcap_{k \geq n} \{g_k \geq a + \epsilon_n\} = \bigcup_{n \geq 1} H_n$, where H_n is the increasing sequence $H_n = \bigcap_{k \geq n} \{g_k \geq a + \epsilon_n\}$ and $\epsilon_n \searrow 0$. Then we have

$$\int_{\{\underline{g} > a\} \cap B} \underline{\phi}'(f - a) d\mu = \lim_{n \rightarrow \infty} \int_{H_n \cap B} \underline{\phi}'(f - (a + \epsilon_n)) d\mu. \quad (4.17)$$

Set $G_n = \bigcap_{k \geq n+1} \{g_k \geq a + \epsilon_n\}$ and note that $H_n = G_n \cap \{g_n \geq a + \epsilon_n\}$. Then for $B \in \mathcal{B}_m$, $m \leq n$, we have

$$\begin{aligned} 0 &\leq \int_{\{g_n \geq a + \epsilon_n\} \cap B} \underline{\phi}'(f - (a + \epsilon_n)) d\mu \\ &= \int_{\{g_n \geq a + \epsilon_n\} \cap B \cap G_n} \underline{\phi}'(f - (a + \epsilon_n)) d\mu + \int_{\{g_n \geq a + \epsilon_n\} \cap B \cap G_n^c} \underline{\phi}'(f - (a + \epsilon_n)) d\mu. \end{aligned} \quad (4.18)$$

Now, we prove

$$\int_{\{g_n \geq a + \epsilon_n\} \cap B \cap G_n^c} \underline{\phi}'(f - (a + \epsilon_n)) d\mu \leq 0. \quad (4.19)$$

We can see that

$$\{g_n \geq a + \epsilon_n\} \cap B \cap G_n^c = \bigcup_{k \geq n+1} A_k, \quad (4.20)$$

where A_k are the following disjoint sets:

$$\begin{aligned} A_{n+1} &= \{g_{n+1} < a + \epsilon_n\} \cap B \cap \{g_n \geq a + \epsilon_n\}, \\ &\vdots \\ A_k &= \{g_k < a + \epsilon_n\} \cap \bigcap_{i=n+1}^{k-1} \{g_i \geq a + \epsilon_n\} \cap \{g_n \geq a + \epsilon_n\} \cap B. \end{aligned} \quad (4.21)$$

Then

$$\int_{\{g_n \geq a + \epsilon_n\} \cap B \cap G_n^c} \underline{\phi}'(f - (a + \epsilon_n)) d\mu = \sum_{k \geq n+1} \int_{A_k} \underline{\phi}'(f - (a + \epsilon_n)) d\mu. \quad (4.22)$$

Since $A_k = \{g_k < a + \epsilon_n\} \cap B_k$, where $B_k \in \mathcal{B}_k$, we have (4.19). Therefore by (4.18), we have

$$\int_{H_n \cap B} \underline{\phi}'(f - (a + \epsilon_n)) d\mu \geq 0 \quad (4.23)$$

for all $B \in \bigcup_{n \in \mathbb{N}} \mathcal{B}_n$. Thus by (4.23) and (4.17), we get

$$\int_{\{g \geq a\} \cap B} \underline{\phi}'(f - a) d\mu \geq 0 \quad (4.24)$$

for all $B \in \bigcup_{n \in \mathbb{N}} \mathcal{B}_n$. Therefore, the result is satisfied for all $B \in \mathcal{B}_\infty$. Thus $\underline{g} \in \tilde{\mu}_{\phi}^{\mathcal{B}_\infty}(f)$. \square

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References

- [1] D. Landers and L. Rogge, "Best approximants in L_ϕ -spaces," *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete*, vol. 51, no. 2, pp. 215–237, 1980.
- [2] D. Landers and L. Rogge, "Isotonic approximation in L_s ," *Journal of Approximation Theory*, vol. 31, no. 3, pp. 199–223, 1981.
- [3] F. Mazzone and H. Cuenya, "Isotonic approximations in L_1 ," *Journal of Approximation Theory*, vol. 117, no. 2, pp. 279–300, 2002.
- [4] S. Favier and F. Zó, "Extension of the best approximation operator in Orlicz spaces and weak-type inequalities," *Abstract and Applied Analysis*, vol. 6, no. 2, pp. 101–114, 2001.
- [5] S. Favier and F. Zó, "A Lebesgue type differentiation theorem for best approximations by constants in Orlicz spaces," *Real Analysis Exchange*, vol. 30, no. 1, pp. 29–42, 2005.
- [6] S. Favier and F. Zó, "Sharp conditions for maximal inequalities of the best approximation operator," preprint.
- [7] F. Mazzone and H. Cuenya, "Maximal inequalities and Lebesgue's differentiation theorem for best approximant by constant over balls," *Journal of Approximation Theory*, vol. 110, no. 2, pp. 171–179, 2001.
- [8] H. D. Brunk and S. Johansen, "A generalized Radon-Nikodym derivative," *Pacific Journal of Mathematics*, vol. 34, pp. 585–617, 1970.
- [9] F. Mazzone and H. Cuenya, "A characterization of best ϕ -approximants with applications to multidimensional isotonic approximation," *Constructive Approximation*, vol. 21, no. 2, pp. 207–223, 2005.
- [10] J. Neveu, *Mathematical Foundations of the Calculus of Probability*, Holden-Day, San Francisco, Calif, USA, 1965.