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Research Article

Jordan $*$-Derivations on C^*-Algebras and JC^*-Algebras

Jong Su An,[1] Jianlian Cui,[2] and Choonkil Park[3]

[1] *Department of Mathematics Education, Pusan National University, Pusan 609-735, South Korea*
[2] *Department of Mathematical Sciences, Tsinghua University, Beijing 100084, China*
[3] *Department of Mathematics, Hanyang University, Seoul 133-791, South Korea*

Correspondence should be addressed to Choonkil Park, baak@hanyang.ac.kr

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We investigate Jordan $*$-derivations on C^*-algebras and Jordan $*$-derivations on JC^*-algebras associated with the following functional inequality $\|f(x)+f(y)+kf(z)\| \le \|kf((x+y)/k+z)\|$ for some integer k greater than 1. We moreover prove the generalized Hyers-Ulam stability of Jordan $*$-derivations on C^*-algebras and of Jordan $*$-derivations on JC^*-algebras associated with the following functional equation $f((x+y)/k+z) = (f(x)+f(y))/k+f(z)$ for some integer k greater than 1.

1. Introduction and preliminaries

The stability problem of functional equations originated from a question of Ulam [1] concerning the stability of group homomorphisms. Hyers [2] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' theorem was generalized by Aoki [3] for additive mappings and by Th. M. Rassias [4] for linear mappings by considering an unbounded Cauchy difference. In 1982–1994, a generalization of Hyers-Ulam stability result was proved by J. M. Rassias [5–9]. This author assumed that Cauchy-Găvruţa-Rassias inequality

$$\|f(x+y)-f(x)-f(y)\| \le \theta\|x\|^p \cdot \|y\|^q \tag{1.1}$$

is controlled by a product of different powers of norms, where $\theta \ge 0$ and $p,q \in \mathbb{R}$ such that $r = p+q \ne 1$, and retained the condition of continuity of $f(tx)$ in $t \in \mathbb{R}$ for each fixed x. In 1999, Găvruţa [10] studied the singular case $r = p+q = 1$, by constructing a nice counterexample to the above pertinent Ulam stability problem. Also J. M. Rassias [5–9, 11–13] investigated other conditions and still obtained stability results. In all these cases, the approach to the existence question was to prove asymptotic type formulas of the form

$$L(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n} \quad \text{or} \quad L(x) = \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right). \tag{1.2}$$

Theorem 1.1 (see [5–7, 9]). *Let X be a real normed vector space and Y a real complete normed vector space. Assume in addition that $f : X \to Y$ is an approximately additive mapping for which there exist constants $\theta \geq 0$ and $p, q \in \mathbb{R}$ such that $r = p + q \neq 1$ and f satisfies*

$$\|f(x+y) - f(x) - f(y)\| \leq \theta \|x\|^p \cdot \|y\|^q \tag{1.3}$$

for all $x, y \in X$. Then, there exists a unique additive mapping $L : X \to Y$ satisfying

$$\|f(x) - L(x)\| \leq \frac{\theta}{|2 - 2^r|}\|x\|^r \tag{1.4}$$

for all $x \in X$. If, in addition, $f : X \to Y$ is a mapping such that the transformation $t \mapsto f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$, then L is an $\mathbb{R}$-linear mapping.

Theorem 1.2 (see [4]). *Let $f : E \to E'$ be a mapping from a normed vector space E into a Banach space E' subject to the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon\left(\|x\|^p + \|y\|^p\right) \tag{1.5}$$

for all $x, y \in E$, where ε and p are constants with $\varepsilon > 0$ and $p < 1$. Then, the limit

$$L(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n} \tag{1.6}$$

exists for all $x \in E$, and $L : E \to E'$ is the unique additive mapping which satisfies

$$\|f(x) - L(x)\| \leq \frac{2\varepsilon}{2 - 2^p}\|x\|^p \tag{1.7}$$

for all $x \in E$. Also, if for each $x \in E$ the mapping $f(tx)$ is continuous in $t \in \mathbb{R}$, then L is linear.

Th. M. Rassias [14], during the 27th International Symposium on Functional Equations, asked the question whether such a theorem can also be proved for $p \geq 1$. Gajda [15] following the same approach as in Th. M. Rassias [4] gave an affirmative solution to this question for $p > 1$. It was shown by Gajda [15] as well as by Th. M. Rassias and Šemrl [16] that one cannot prove a Th. M. Rassias' type theorem when $p = 1$. The counterexamples of Gajda [15] as well as of Th. M. Rassias and Šemrl [16] have stimulated several mathematicians to invent new definitions of *approximately additive* or *approximately linear* mappings, compare Găvruţa [17] and Jung [18], who among others studied the Hyers-Ulam stability of functional equations. Theorem 1.2 that was introduced for the first time by Th. M. Rassias [4] provided a lot of influence in the development of a generalization of the Hyers-Ulam stability concept. This new concept is known as *generalized Hyers-Ulam stability* of functional equations (cf. the books of Czerwik [19], Hyers et al. [20]).

Găvruţa [17] provided a further generalization of Th. M. Rassias' theorem. Isac and Th. M. Rassias [21] applied the Hyers-Ulam stability theory to prove fixed point theorems and study some new applications in nonlinear analysis. In [22], Hyers et al. studied the asymptoticity aspect of Hyers-Ulam stability of mappings. Beginning around the year 1980, the topic of approximate homomorphisms and their stability theory in the field of functional equations and inequalities was taken up by several mathematicians (see [3–18, 21–51]).

Gilányi [26] showed that if f satisfies the functional inequality

$$\|2f(x) + 2f(y) - f(x-y)\| \leq \|f(x+y)\|, \tag{1.8}$$

then f satisfies the Jordan-von Neumann functional equation

$$2f(x) + 2f(y) = f(x+y) + f(x-y). \tag{1.9}$$

See also [52]. Fechner [53] and Gilányi [27] proved the generalized Hyers-Ulam stability of the functional inequality (1.8). Park et al. [42] introduced and investigated 3-variable Cauchy-Jensen functional inequalities and proved the generalized Hyers-Ulam stability of the 3-variable Cauchy-Jensen functional inequalities.

Definition 1.3. Let A be a C^*-algebra. A $\mathbb{C}$-linear mapping $\delta : A \to A$ is called a Jordan $*$-derivation if

$$\delta(a^2) = a\delta(a) + \delta(a^*)a^* \tag{1.10}$$

for all $a \in A$.

A C^*-algebra A, endowed with the Jordan product $a \circ b := (ab + ba)/2$ on A, is called a JC^*-algebra (see [38, 39]).

Definition 1.4. Let A be a JC^*-algebra. A $\mathbb{C}$-linear mapping $\delta : A \to A$ is called a Jordan $*$-derivation if

$$\delta(a^2) = a \circ \delta(a) + \delta(a^*) \circ a^* \tag{1.11}$$

for all $a \in A$.

This paper is organized as follows. In Section 2, we investigate Jordan $*$-derivations on C^*-algebras associated with the functional inequality

$$\|f(x) + f(y) + kf(z)\| \leq \left\|kf\left(\frac{x+y}{k} + z\right)\right\|, \tag{1.12}$$

and prove the generalized Hyers-Ulam stability of Jordan $*$-derivations on C^*-algebras associated with the functional equation

$$f\left(\frac{x+y}{k} + z\right) = \frac{f(x) + f(y)}{k} + f(z). \tag{1.13}$$

In Section 3, we investigate Jordan $*$-derivations on JC^*-algebras associated with the functional inequality (1.12), and prove the generalized Hyers-Ulam stability of Jordan $*$-derivations on JC^*-algebras associated with the functional equation (1.13).

Throughout this paper, let k be an integer greater than 1.

2. Jordan $*$-derivations on C^*-algebras

Throughout this section, assume that A is a C^*-algebra with norm $\|\cdot\|$.

Lemma 2.1. *Let $f : A \to A$ be a mapping such that*

$$\|f(a)+f(b)+kf(c)\| \le \left\|kf\left(\frac{a+b}{k}+c\right)\right\| \tag{2.1}$$

for all $a,b,c \in A$. Then, f is Cauchy additive, that is, $f(a+b)=f(a)+f(b)$.

Proof. Letting $a=b=c=0$ in (2.1), we get

$$\|(k+2)f(0)\| \le \|kf(0)\|. \tag{2.2}$$

So $f(0)=0$.

Letting $c=0$ and $b=-a$ in (2.1), we get

$$\|f(a)+f(-a)\| \le \|kf(0)\| = 0 \tag{2.3}$$

for all $a \in A$. Hence $f(-a)=-f(a)$ for all $a \in A$.

Letting $c=-(a+b)/k$ in (2.1), we get

$$\left\|f(a)+f(b)-kf\left(\frac{a+b}{k}\right)\right\| = \left\|f(a)+f(b)+kf\left(-\frac{a+b}{k}\right)\right\| \le \|kf(0)\| = 0 \tag{2.4}$$

for all $a,b \in A$. Thus

$$kf\left(\frac{a+b}{k}\right) = f(a)+f(b) \tag{2.5}$$

for all $a,b \in A$. Letting $b=0$ in (2.5), we get $kf(a/k)=f(a)$ for all $a \in A$. So

$$f(a+b) = kf\left(\frac{a+b}{k}\right) = f(a)+f(b) \tag{2.6}$$

for all $a,b \in A$, as desired. □

Theorem 2.2. *Let $r>1$ and θ be a nonnegative real number, and let $f : A \to A$ be a mapping such that*

$$\|\mu f(a)+f(b)+kf(c)\| \le \left\|kf\left(\frac{\mu a+b}{k}+c\right)\right\|, \tag{2.7}$$

$$\|f(a^2)-af(a)-f(a^*)a^*\| \le \theta\|a\|^{2r} \tag{2.8}$$

for all $\mu \in \mathbb{T} := \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$ and all $a,b,c \in A$. Then, the mapping $f : A \to A$ is a Jordan $*$-derivation.

Proof. Let $\mu = 1$ in (2.7). By Lemma 2.1, the mapping $f : A \to A$ is Cauchy additive. So $f(0) = 0$ and $f(x) = \lim_{n\to\infty} 2^n f(a/2^n)$ for all $a \in A$.

Letting $b = -\mu a$ and $c = 0$, we get

$$\|\mu f(a) + f(-\mu a)\| \le \|k f(0)\| = 0 \tag{2.9}$$

for all $a \in A$ and all $\mu \in \mathbb{T}$. So

$$\mu f(a) - f(\mu a) = \mu f(a) + f(-\mu a) = 0 \tag{2.10}$$

for all $a \in A$, and all $\mu \in \mathbb{T}$. Hence, $f(\mu a) = \mu f(a)$ for all $a \in A$ and all $\mu \in \mathbb{T}$. By the same reasoning as in the proof of [39, Theorem 2.1], the mapping $f : A \to A$ is $\mathbb{C}$-linear.

It follows from (2.8) that

$$\begin{aligned} \|f(a^2) - af(a) - f(a^*)a^*\| &= \lim_{n\to\infty} 4^n \left\| f\left(\frac{a^2}{2^n\cdot 2^n}\right) - \frac{a}{2^n} f\left(\frac{a}{2^n}\right) - f\left(\frac{a^*}{2^n}\right)\frac{a^*}{2^n}\right\| \\ &\le \lim_{n\to\infty} \frac{4^n\theta}{4^{nr}} \|a\|^{2r} = 0 \end{aligned} \tag{2.11}$$

for all $a \in A$. Thus

$$f(a^2) = af(a) + f(a^*)a^* \tag{2.12}$$

for all $a \in A$.

Hence the mapping $f : A \to A$ is a Jordan $*$-derivation. □

Theorem 2.3. *Let $r < 1$ and θ be a nonnegative real number, and let $f : A \to A$ be a mapping satisfying* (2.7) *and* (2.8)*. Then, the mapping $f : A \to A$ is a Jordan $*$-derivation.*

Proof. The proof is similar to the proof of Theorem 2.2. □

We prove the generalized Hyers-Ulam stability of Jordan $*$-derivations on C^*-algebras.

Theorem 2.4. *Suppose that $f : A \to A$ is a mapping with $f(0) = 0$ for which there exists a function $\varphi : A \times A \times A \to [0,\infty)$ such that*

$$\lim_{n\to\infty} k^{-n}\varphi(k^n a, k^n b, k^n c) = 0, \tag{2.13}$$

$$\widetilde{\varphi}(a) := \sum_{n=1}^{\infty} k^{-n+1}\varphi(k^n a, 0, 0) < \infty, \tag{2.14}$$

$$\left\| f\left(\frac{\lambda a + \lambda b}{k} + c^2\right) - \frac{\lambda}{k} f(a) - \frac{\lambda}{k} f(b) - cf(c) - f(c^*)c^*\right\| \le \varphi(a,b,c) \tag{2.15}$$

for all $\lambda \in \mathbb{T}$, and all $a, b, c \in A$. Then, there exists a unique Jordan $*$-derivation $\delta : A \to A$ such that

$$\|f(a) - \delta(a)\| \le \widetilde{\varphi}(a) \tag{2.16}$$

for all $a \in A$.

Proof. Putting $b = c = 0$ and $\lambda = 1$, and replacing a by ka in (2.15), we get

$$\left\| f(a) - \frac{f(ka)}{k} \right\| \leq \varphi(ka, 0, 0) \quad (\forall a \in A). \tag{2.17}$$

Using the induction method, we have

$$\left\| \frac{f(k^n a)}{k^n} - \frac{f(k^m a)}{k^m} \right\| \leq \sum_{j=m+1}^{n} k^{-j+1} \varphi(k^j a, 0, 0) \tag{2.18}$$

for all $n > m \geq 0$ and all $a \in A$. It follows that for every $a \in A$, the sequence $\{f(k^n a)/k^n\}$ is Cauchy, and hence it is convergent since A is complete. Set

$$\delta(a) := \lim_{n\to\infty} \frac{f(k^n a)}{k^n} \quad (a \in A). \tag{2.19}$$

Let $c = 0$ and replace a and b by $k^n a$ and $k^n b$, respectively, in (2.15), we get

$$\left\| k^{-n} f\left(k^n \frac{\lambda a + \lambda b}{k}\right) - \frac{\lambda}{k} k^{-n} f(k^n a) - \frac{\lambda}{k} k^{-n} f(k^n b) \right\| \leq k^{-n} \varphi(k^n a, k^n b, 0). \tag{2.20}$$

Taking the limit as $n \to \infty$, we obtain

$$\delta\left(\frac{\lambda a + \lambda b}{k}\right) = \frac{\lambda}{k} \delta(a) + \frac{\lambda}{k} \delta(b) \tag{2.21}$$

for all $a, b \in A$ and all $\lambda \in \mathbb{T}$. Letting $b = 0$ and $\lambda = 1$ in (2.21), we get

$$\delta\left(\frac{a}{k}\right) = \frac{1}{k} \delta(a) \tag{2.22}$$

for all $a \in A$. Hence,

$$\lambda\delta(a) + \lambda\delta(b) = k\delta\left(\frac{\lambda a + \lambda b}{k}\right) = \delta(\lambda a + \lambda b) \tag{2.23}$$

for all $a, b \in A$, in particular, $\delta : A \to A$ is additive. Now similar to the discussion in [54, Theorem 2.1], we show that $\delta : A \to A$ is $\mathbb{C}$-linear. Letting $m = 0$ in (2.18), we get

$$\left\| f(a) - \frac{f(k^n a)}{k^n} \right\| \leq \sum_{j=1}^{n} k^{-j+1} \varphi(k^j a, 0, 0). \tag{2.24}$$

Taking the limit as $n \to \infty$, we have

$$\| f(a) - \delta(a) \| \leq \widetilde{\varphi}(a) \tag{2.25}$$

for all $a \in A$. It follows from [55] that $\delta : A \to A$ is unique.

Letting $\lambda = 1$, $a = b = 0$ and replacing c by $k^n c$ in (2.15), we obtain

$$\| f(k^{2n}(c^2)) - k^n c f(k^n c) - k^n f(k^n c^*) c^* \| \leq \varphi(0, 0, k^n c). \tag{2.26}$$

So

$$\| k^{-2n} f(k^{2n}(c^2)) - k^{-n} c f(k^n c) - k^{-n} f(k^n c^*) c^* \| \leq k^{-2n} \varphi(0, 0, k^n c) \tag{2.27}$$

for all $c \in A$. Letting n tend to infinity, we have

$$\delta(c^2) = c\delta(c) + \delta(c^*) c^* \tag{2.28}$$

for all $c \in A$. Hence $\delta : A \to A$ is a Jordan $*$-derivation. □

Corollary 2.5. *Suppose that $f : A \to A$ is a mapping with $f(0) = 0$ for which there exist constant $\beta \geq 0$ and $p_1, p_2, p_3 \in (-\infty, 1)$ such that*

$$\left\| f\left(\frac{\lambda a + \lambda b}{k} + c^2\right) - \frac{\lambda}{k} f(a) - \frac{\lambda}{k} f(b) - c f(c) - f(c^*) c^* \right\| \leq \beta\left(\|a\|^{p_1} + \|b\|^{p_2} + \|c\|^{p_3}\right) \tag{2.29}$$

for all $\lambda \in \mathbb{T}$ and all $a, b, c \in A$. Then, there exists a unique Jordan $*$-derivation $\delta : A \to A$ such that

$$\|f(a) - \delta(a)\| \leq \frac{k\beta\|a\|^{p_1}}{k^{1-p_1} - 1} \tag{2.30}$$

for all $a \in A$.

Proof. Letting $\varphi(a, b, c) := \beta(\|a\|^{p_1} + \|b\|^{p_2} + \|c\|^{p_3})$ in Theorem 2.4, we obtain the result. □

Theorem 2.6. *Suppose that $f : A \to A$ is a mapping with $f(0) = 0$ for which there exists a function $\varphi : A \times A \times A \to [0, \infty)$ satisfying* (2.15) *such that*

$$\begin{gathered} \lim_{n \to \infty} k^n \varphi(k^{-n} a, k^{-n} b, k^{-n} c) = 0, \\ \widetilde{\varphi}(a) := \sum_{n=1}^{\infty} k^{n+1} \varphi(k^{-n} a, 0, 0) < \infty \end{gathered} \tag{2.31}$$

for all $a, b, c \in A$. Then, there exists a unique Jordan $*$-derivation $\delta : A \to A$ such that

$$\|f(a) - \delta(a)\| \leq \widetilde{\varphi}(a) \tag{2.32}$$

for all $a \in A$.

Proof. Letting $b = c = 0$ and $\lambda = 1$ in (2.15), we get

$$\left\| k f\left(\frac{a}{k}\right) - f(a) \right\| \leq k\varphi(a, 0, 0) \quad (\forall a \in A). \tag{2.33}$$

One can apply the induction method to prove that

$$\|k^n f(k^{-n} a) - k^m f(k^{-m} b)\| \leq \sum_{j=m}^{n-1} k^{j+1} \varphi(k^{-j} a, 0, 0) \tag{2.34}$$

for all $n > m \geq 0$ and $a \in A$. It follows that for every $a \in A$, the sequence $\{k^n f(k^{-n} a)\}$ is Cauchy, and hence it is convergent since A is complete. Set

$$\delta(a) := \lim_{n \to \infty} k^n f(k^{-n} a) \quad (a \in A). \tag{2.35}$$

Letting $c = 0$ and replacing a and b by $k^{-n} a$ and $k^{-n} b$, respectively, in (2.15), we get

$$\left\| k^n f\left(\frac{k^{-n}(a + b)}{k}\right) - \frac{k^n}{k} f(k^{-n} a) - \frac{k^n}{k} f(k^{-n} b) \right\| \leq k^n \varphi(k^{-n} a, k^{-n} b, 0) \tag{2.36}$$

for all $a, b \in A$. Taking the limit as $n \to \infty$, we obtain

$$\delta\left(\frac{a + b}{k}\right) = \frac{\delta(a)}{k} + \frac{\delta(b)}{k} \tag{2.37}$$

for all $a, b \in A$. Hence $\delta : A \to A$ is additive.

The rest of the proof is similar to the proof of Theorem 2.4. □

Corollary 2.7. *Suppose that $f : A \to A$ is a mapping with $f(0) = 0$ for which there exist constant $\beta \geq 0$ and $p_1, p_2, p_3 \in (1, \infty)$ such that*

$$\left\| f\left(\frac{\lambda a + \lambda b}{k} + c^2\right) - \frac{\lambda}{k} f(a) - \frac{\lambda}{k} f(b) - c f(c) - f(c^*) c^* \right\| \leq \beta\left(\|a\|^{p_1} + \|b\|^{p_2} + \|c\|^{p_3}\right) \tag{2.38}$$

for all $\lambda \in \mathbb{T}$ and all $a, b, c \in A$. Then, there exists a unique Jordan $*$-derivation $\delta : A \to A$ such that

$$\|f(a) - \delta(a)\| \leq \frac{k\beta\|a\|^{p_1}}{k^{1-p_1} - 1} \tag{2.39}$$

for all $a \in A$.

Proof. Letting $\varphi(a, b, c) := \beta(\|a\|^{p_1} + \|b\|^{p_2} + \|c\|^{p_3})$ in Theorem 2.6, we obtain the result. □

3. Jordan $*$-derivations on JC^*-algebras

Throughout this section, assume that A is a JC^*-algebra with norm $\|\cdot\|$.

Theorem 3.1. *Let $r > 1$ and θ be a nonnegative real number, and let $f : A \to A$ be a mapping satisfying* (2.7) *such that*

$$\|f(a^2) - a \circ f(a) - f(a^*) \circ a^*\| \leq \theta\|a\|^{2r} \tag{3.1}$$

for all $a \in A$. Then, the mapping $f : A \to A$ is a Jordan $*$-derivation.

Proof. By the same reasoning as in the proof of Theorem 2.2, the mapping $f : A \to A$ is $\mathbb{C}$-linear.

It follows from (3.1) that

$$\begin{aligned} \|f(a^2) - a \circ f(a) - f(a^*) \circ a^*\| &= \lim_{n\to\infty} 4^n \left\| f\left(\frac{a^2}{2^n \cdot 2^n}\right) - \frac{a}{2^n} \circ f\left(\frac{a}{2^n}\right) - f\left(\frac{a^*}{2^n}\right) \circ \frac{a^*}{2^n} \right\| \\ &\leq \lim_{n\to\infty} \frac{4^n\theta}{4^{nr}} \|a\|^{2r} = 0 \end{aligned} \tag{3.2}$$

for all $a \in A$. Thus,

$$f(a^2) = a \circ f(a) + f(a^*) \circ a^* \tag{3.3}$$

for all $a \in A$.

Hence, the mapping $f : A \to A$ is a Jordan $*$-derivation. □

Theorem 3.2. *Let $r < 1$ and θ be a nonnegative real number, and let $f : A \to A$ be a mapping satisfying* (2.7) *and* (3.1)*. Then, the mapping $f : A \to A$ is a Jordan $*$-derivation.*

Proof. The proof is similar to the proofs of Theorems 2.2 and 3.1. □

We prove the generalized Hyers-Ulam stability of Jordan $*$-derivations on JC^*-algebras.

Theorem 3.3. *Suppose that $f : A \to A$ is a mapping with $f(0) = 0$ for which there exists a function $\varphi : A \times A \times A \to [0, \infty)$ satisfying* (2.13) *and* (2.14) *such that*

$$\left\| f\left(\frac{\lambda a + \lambda b}{k} + c^2\right) - \frac{\lambda}{k} f(a) - \frac{\lambda}{k} f(b) - c \circ f(c) - f(c^*) \circ c^* \right\| \leq \varphi(a, b, c) \tag{3.4}$$

for all $\lambda \in \mathbb{T}$ and all $a, b, c \in A$. Then, there exists a unique Jordan $*$-derivation $\delta : A \to A$ such that

$$\| f(a) - \delta(a) \| \leq \tilde{\varphi}(a) \tag{3.5}$$

for all $a \in A$.

Proof. By the same reasoning as in the proof of Theorem 2.4, there exists a unique $\mathbb{C}$-linear mapping $\delta : A \to A$ such that

$$\| f(a) - \delta(a) \| \leq \tilde{\varphi}(a) \tag{3.6}$$

for all $a \in A$. The mapping $\delta : A \to A$ is given by

$$\delta(a) := \lim_{n \to \infty} \frac{f(k^n a)}{k^n} \quad (a \in A). \tag{3.7}$$

Letting $\lambda = 1$, $a = b = 0$ and replacing c by $k^n c$ in (3.4), we obtain

$$\| f(k^{2n} c^2) - k^n c \circ f(k^n c) - k^n f(k^n c^*) \circ c^* \| \leq \varphi(0, 0, k^n c). \tag{3.8}$$

So

$$\| k^{-2n} f(k^{2n} c^2) - k^{-n} c \circ f(k^n c) - k^{-n} f(k^n c^*) \circ c^* \| \leq k^{-2n} \varphi(0, 0, k^n c) \tag{3.9}$$

for all $c \in A$. Letting n tend to infinity, we have

$$\delta(c^2) = c \circ \delta(c) + \delta(c^*) \circ c^* \tag{3.10}$$

for all $c \in A$. Hence, $\delta : A \to A$ is a Jordan $*$-derivation. □

Corollary 3.4. *Suppose that $f : A \to A$ is a mapping with $f(0) = 0$ for which there exist constant $\beta \geq 0$ and $p_1, p_2, p_3 \in (-\infty, 1)$ such that*

$$\left\| f\left(\frac{\lambda a + \lambda b}{k} + c^2\right) - \frac{\lambda}{k} f(a) - \frac{\lambda}{k} f(b) - c \circ f(c) - f(c^*) \circ c^* \right\| \leq \beta(\|a\|^{p_1} + \|b\|^{p_2} + \|c\|^{p_3}) \tag{3.11}$$

for all $\lambda \in \mathbb{T}$ and all $a, b, c \in A$. Then, there exists a unique Jordan $*$-derivation $\delta : A \to A$ such that

$$\| f(a) - \delta(a) \| \leq \frac{k\beta \|a\|^{p_1}}{k^{1-p_1} - 1} \tag{3.12}$$

for all $a \in A$.

Proof. Letting $\varphi(a, b, c) := \beta(\|a\|^{p_1} + \|b\|^{p_2} + \|c\|^{p_3})$ in Theorem 3.3, we obtain the result. □

Theorem 3.5. *Suppose that $f : A \to A$ is a mapping with $f(0) = 0$ for which there exists a function $\varphi : A \times A \times A \to [0, \infty)$ satisfying* (2.31) *and* (3.4)*. Then, there exists a unique Jordan $*$-derivation $\delta : A \to A$ such that*

$$\|f(a) - \delta(a)\| \leq \widetilde{\varphi}(a) \tag{3.13}$$

for all $a \in A$.

Proof. The rest of the proof is similar to the proofs of Theorems 2.4 and 3.3. □

Corollary 3.6. *Suppose that $f : A \to A$ is a mapping with $f(0) = 0$ for which there exist constant $\beta \geq 0$ and $p_1, p_2, p_3 \in (1, \infty)$ such that*

$$\left\| f\left(\frac{\lambda a + \lambda b}{k} + c^2\right) - \frac{\lambda}{k} f(a) - \frac{\lambda}{k} f(b) - c \circ f(c) - f(c^*) \circ c^* \right\| \leq \beta\left(\|a\|^{p_1} + \|b\|^{p_2} + \|c\|^{p_3}\right) \tag{3.14}$$

for all $\lambda \in \mathbb{T}$ and all $a, b, c \in A$. Then, there exists a unique Jordan $*$-derivation $\delta : A \to A$ such that

$$\|f(a) - \delta(a)\| \leq \frac{k\beta\|a\|^{p_1}}{k^{1-p_1} - 1} \tag{3.15}$$

for all $a \in A$.

Proof. Letting $\varphi(a, b, c, d) := \beta(\|a\|^{p_1} + \|b\|^{p_2} + \|c\|^{p_3})$ in Theorem 3.5, we obtain the result. □

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