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Research Article

On Multiple Twisted p-adic q-Euler ζ-Functions and l-Functions

Min-Soo Kim,[1] Taekyun Kim,[2] and Jin-Woo Son[3]

[1] *National Institute for Mathematical Sciences, Doryong-dong, Yuseong-gu, Daejeon 305-340, South Korea*
[2] *Division of General Education-Mathematics, Kwangwoon University, Seoul 139-701, South Korea*
[3] *Department of Mathematics, Kyungnam University, Masan 631-701, South Korea*

Correspondence should be addressed to Min-Soo Kim, mskim@kyungnam.ac.kr

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We give the existence of multiple twisted p-adic q-Euler ζ-functions and l-functions, which are generalization of the twisted p-adic (h,q)-zeta functions and twisted p-adic (h,q)-Euler l-functions in the work of Ozden and Simsek (2008).

1. Introduction, definitions, and notations

The constants E_n in the Taylor series expansion

$$\frac{2}{e^t+1}=\sum_{n=0}^{\infty}E_n\frac{t^n}{n!} \tag{1.1}$$

are known as the first kind Euler numbers (cf. [1]). From the generating function of the first kind Euler numbers, we note that $E_0=1$ and $E_n=-\sum_{k=0}^{n}\binom{n}{k}E_k$ for $n\in\mathbb{N}$. The first few are $1,-1/2,0,1/4,-1/2,\ldots$ and $E_{2k}=0$ for $k\in\mathbb{N}$. Those numbers play an important role in number theory. For example, the Euler zeta-function essentially equals an Euler numbers at nonpositive integer:

$$\zeta_E(-m)=E_m \quad \text{for } m\geq 0, \tag{1.2}$$

where

$$\zeta_E(s)=\sum_{n=1}^{\infty}\frac{(-1)^n}{n^s},\quad s\in\mathbb{C} \tag{1.3}$$

(see [1–10]).

Throughout this paper $\mathbb{Z}$, $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ will denote the ring of integers, the ring of p-adic rational integers, the field of p-adic rational numbers, and the completion of the algebraic closure of $\mathbb{Q}_p$, respectively. Let v_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = 1/p$. When one talks of q-extension, q is variously considered as an indeterminate, a complex number $q \in \mathbb{C}$, or a p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}_p$, then we normally assume $|1-q|_p < 1$, so that $q^x = \exp(x \log q)$ for $x \in \mathbb{Z}_p$. If $q \in \mathbb{C}$, then we assume that $|q| < 1$. Also we use the following notations:

$$[x]_q = \frac{1-q^x}{1-q}, \qquad [x]_{-q} = \frac{1-(-q)^x}{1+q}, \tag{1.4}$$

cf. [2–4]. For

$$f \in UD(\mathbb{Z}_p, \mathbb{C}_p) = \{ f \mid f : \mathbb{Z}_p \to \mathbb{C}_p \text{ is uniformly differentiable function} \}, \tag{1.5}$$

the p-adic q-integral on $\mathbb{Z}_p$ was defined by Kim (cf. [2–4]) as follows:

$$I_q(f) = \int_{\mathbb{Z}_p} f(a)\, d\mu_q(a) = \lim_{N\to\infty} \frac{1}{[dp^N]_q} \sum_{a=0}^{dp^N-1} f(a) q^a \quad \text{for } |1-q|_p < 1. \tag{1.6}$$

Furthermore, we can consider the fermionic integral in contrast to the conventional bosonic integral. That is, $I_{-1}(f) = \int_{\mathbb{Z}_p} f(a)\, d\mu_{-1}(a)$ (cf. [5]). From this, we derive

$$I_{-1}(f_1) + I_{-1}(f) = 2f(0), \tag{1.7}$$

where $f_1(a) = f(a+1)$. Substitute $f(a) = \xi^a q^{\alpha a} e^{at}$ into (1.7). The twisted (α, q)-extension of Euler numbers is defined by [8]

$$I_{-1}(\xi^a q^{\alpha a} e^{at}) = \frac{2}{\xi q^\alpha e^t + 1} = \sum_{n=0}^{\infty} E_{n,\xi}^{(\alpha)}(q) \frac{t^n}{n!}. \tag{1.8}$$

For $|1-q|_p < 1$, we consider fermionic p-adic q-integral on $\mathbb{Z}_p$ which is the q-extension of $I_{-1}(f)$ as follows:

$$I_{-q}(f) = \int_{\mathbb{Z}_p} f(a)\, d\mu_{-q}(a) = \lim_{N\to\infty} \frac{1}{[dp^N]_{-q}} \sum_{a=0}^{dp^N-1} f(a)(-q)^a \tag{1.9}$$

(cf. [5]). From (1.9), we can derive the following formula [5]:

$$qI_{-q}(f_1) + I_{-q}(f) = [2]_q f(0), \tag{1.10}$$

where $f_1(a)$ is translation with $f_1(a) = f(a+1)$. If we take $f(a) = \xi^a e^{at}$, then we have $f_1(a) = \xi^{a+1} e^{(a+1)t} = \xi^a e^{at} \xi e^t$. From (1.10), we derive $(\xi q e^t + 1) I_{-q}(\xi^a e^{at}) = [2]_q$. Hence, we obtain

$$I_{-q}(\xi^a e^{at}) = \int_{\mathbb{Z}_p} \xi^a e^{at}\, d\mu_{-q}(a) = \frac{[2]_q}{\xi q e^t + 1}. \tag{1.11}$$

By (1.11), we define the twisted q-Euler numbers, $E_{n,q,\xi}$ by means of the following generating function (cf. [5]):

$$\frac{[2]_q}{\xi q e^t + 1} = \sum_{n=0}^{\infty} E_{n,q,\xi} \frac{t^n}{n!}. \tag{1.12}$$

These numbers are interpolated by the twisted Euler q-zeta function which is defined as follows:

$$\zeta_{q,\xi,E}(s) = [2]_q \sum_{n=1}^{\infty} \frac{(-1)^n \xi^n q^n}{n^s}, \quad s \in \mathbb{C}. \tag{1.13}$$

Note that $\zeta_{q,\xi,E}(s)$ is analytic function in the whole complex plane $\mathbb{C}$.

In view of the functional equation for the twisted Euler q-zeta function at nonpositive integers, we have

$$\zeta_{q,\xi,E}(-m) = E_{m,q,\xi} \quad \text{for } m \geq 0 \tag{1.14}$$

(cf. [5]).

Twisted q-Bernoulli and Euler numbers and polynomials are very important not only in practically every field of mathematics, in particular in combinatorial theory, finite difference calculus, numerical analysis, numbers theory, but also probability theory. Recently the q-extensions of those Euler numbers (polynomials) and the multiple of q-extensions of those Euler numbers (polynomials) have been studied by many authors, (cf. [1–15]). In [8], Ozden and Simsek have studied (h,q)-extensions of twisted Euler numbers and polynomials by using p-adic q-integral on the ring of p-adic integers $\mathbb{Z}_p$. From their (h,q)-extensions of twisted Euler numbers and polynomials, they have derived p-adic (h,q)-extensions of Euler zeta function and p-adic (h,q)-extensions of Euler l-functions. They also gave some interesting relations between their (h,q)-Euler numbers and (h,q)-Euler zeta functions, and found the p-adic twisted interpolation function of the generalized twisted (h,q)-Euler numbers. In [11], Jang defined twisted q-Euler numbers and polynomials of higher order, and studied multiple twisted q-Euler zeta functions. He also derived sums of products of q-Euler numbers and polynomials by using fermionic p-adic q-integral. In [7, 9], Ozden et al. defined multivariate Barnes-type Hurwitz q-Euler zeta functions and l-functions. They also gave relation between multivariate Barnes-type Hurwitz q-Euler zeta functions and multivariate q-Euler l-functions. In [16], Kim constructed multiple p-adic L-functions, which interpolate the Bernoulli numbers of higher order. He also derived that the values of the partial derivative of this multiple p-adic L-function at $s = 0$ are given.

In this paper, we consider twisted q-Euler numbers and polynomials of higher order, and study multiple twisted p-adic, q-Euler, ζ-functions, and l-functions, which are generalization of the twisted p-adic (h,q)-zeta functions and twisted p-adic (h,q)-Euler l-functions in [8].

2. Preliminaries

We assume that $q \in \mathbb{C}$ with $|q| < 1$. Let ξ be a primitive r th root of unity.

For an integer h, the twisted q-Euler polynomials of higher order (the index h may be negative), $E_{n,q,\xi}^{(h)}(x)$, are defined by means of the following generating function (cf. [11, 14]):

$$\begin{aligned} F_{q,\xi}^{(h)}(t,x) &= \underbrace{\frac{[2]_q}{1+\xi q e^t}\cdots\frac{[2]_q}{1+\xi q e^t}}_{h\text{-times}} e^{xt} \\ &= [2]_q^h e^{tx}\sum_{l_1=0}^{\infty}(-\xi)^{l_1}q^{l_1}e^{l_1 t}\cdots\sum_{l_h=0}^{\infty}(-\xi)^{l_h}q^{l_h}e^{l_h t} \\ &= [2]_q^h \sum_{l_1,\ldots,l_h=0}^{\infty}(-\xi q)^{l_1+\cdots+l_h}e^{(l_1+\cdots+l_h+x)t} \\ &= \sum_{n=0}^{\infty}E_{n,q,\xi}^{(h)}(x)\frac{t^n}{n!}, \end{aligned} \tag{2.1}$$

where $|t+\log(\xi q)|<\pi$. Note that $[2]_q = 1+q$, so $[2]_q/(1+\xi q e^t) \equiv (1+q)/(1+\xi q) \pmod{t}$. Of course the explicit formulas in (2.1) depend on h which is a positive integer. If $h=1$, $q=q^{\alpha}$ in the above, we obtain the generating function of the twisted (α,q)-extension of Euler polynomials in [8, cf. Section 1, (1.3)]. In fact, if $h>0$ then $-h<0$. Therefore, the generating function $F_{q,\xi}^{(-h)}(t,x)$ is the form

$$F_{q,\xi}^{(-h)}(t,x) = \left(\frac{[2]_q}{1+\xi q e^t}\right)^{-h} e^{tx} = \left(\frac{1+\xi q e^t}{[2]_q}\right)^{h} e^{tx} = \sum_{n=0}^{\infty}E_{n,q,\xi}^{(-h)}(x)\frac{t^n}{n!}. \tag{2.2}$$

The twisted q-Euler numbers of higher order are $E_{n,q,\xi}^{(h)} = E_{n,q,\xi}^{(h)}(0)$. Then, it is immediate that

$$E_{n,q,\xi}^{(h)}(x) = \sum_{k=0}^{n}\binom{n}{k}E_{k,q,\xi}^{(h)}x^{n-k}. \tag{2.3}$$

From now on, we assume $h>0$ and in general whenever h is actually an index then $h>0$. Jang [11] defined the two-variable multiple twisted q-Euler zeta functions as follows.

Definition 2.1. For $s\in\mathbb{C}$ and $x\in\mathbb{R}^+ = \{x\in\mathbb{R} \mid x>0\}$, one defines

$$\zeta_{q,\xi,E}^{(h)}(s,x) = [2]_q^h \sum_{l_1,\ldots,l_h=0}^{\infty}\frac{(-\xi)^{l_1+\cdots+l_h}q^{l_1+\cdots+l_h}}{(l_1+\cdots+l_h+x)^s}. \tag{2.4}$$

$\zeta_{q,\xi,E}^{(h)}(s,x)$ is an analytical function in the whole complex plane.

The value of $\zeta_{q,\xi,E}^{(h)}(s,x)$ at nonpositive integers, $\mathbb{Z}_+ = \mathbb{N}\cup\{0\}$, is given explicitly as follows.

Theorem 2.2 (see [11]). *Let $m\in\mathbb{Z}_+$. Then,* $\zeta_{q,\xi,E}^{(h)}(-m,x) = E_{m,q,\xi}^{(h)}(x)$.

Let χ be a Dirichlet character with odd conductor d. We define a twisted Dirichlet's type q-Euler polynomials of higher order by means of the following generating function (cf. [11, 14]):

$$\begin{aligned}
&F^{(h)}_{q,\xi,\chi}(t,x)\\
&= \frac{1}{[d]^h_{-q}} \sum_{a_1,\dots,a_h=0}^{d-1} \chi(a_1+\cdots+a_h)(-\xi q)^{a_1+\cdots+a_h} e^{(a_1+\cdots+a_h)t} \underbrace{\frac{1+q^d}{1+\xi^d q^d e^{\mathrm{d}t}} \cdots \frac{1+q^d}{1+\xi^d q^d e^{\mathrm{d}t}}}_{h\text{-times}} e^{xt}\\
&= [2]^h_q \sum_{a_1,\dots,a_h=0}^{d-1} \chi(a_1+\cdots+a_h)(-\xi q)^{a_1+\cdots+a_h} e^{(a_1+\cdots+a_h)t} \underbrace{\sum_{x_1=0}^{\infty}\left(-\xi^d q^d e^{\mathrm{d}t}\right)^{x_1} \cdots \sum_{x_h=0}^{\infty}\left(-\xi^d q^d e^{\mathrm{d}t}\right)^{x_h}}_{h\text{-times}} e^{xt}\\
&= [2]^h_q \sum_{x_1,\dots,x_h=0}^{\infty} \sum_{a_1,\dots,a_h=0}^{d-1} \chi(a_1+dx_1+\cdots+a_h+dx_h)(-\xi q)^{a_1+dx_1+\cdots+a_h+dx_h} e^{(a_1+dx_1+\cdots+a_h+dx_h)t} e^{xt}\\
&= [2]^h_q \sum_{l_1,\dots,l_h=0}^{\infty} \chi(l_1+\cdots+l_h)(-\xi q)^{l_1+\cdots+l_h} e^{(x+l_1+\cdots+l_h)t}\\
&= \sum_{n=0}^{\infty} E^{(h)}_{n,q,\xi,\chi}(x)\frac{t^n}{n!}.
\end{aligned} \tag{2.5}$$

We now see that the twisted Dirichlet's type q-Euler polynomials of higher order are easily expressed by the twisted q-Euler polynomials of higher order as follows.

Proposition 2.3. *Let F be an odd multiple of the conductor d. Then,*

$$E^{(h)}_{n,q,\xi,\chi}(x) = F^n \frac{1}{[F]^h_{-q}} \sum_{a_1,\dots,a_h=0}^{F-1} \chi(a_1+\cdots+a_h)(-\xi q)^{a_1+\cdots+a_h} E^{(h)}_{n,q^F,\xi^F}\left(\frac{a_1+\cdots+a_h+x}{F}\right). \tag{2.6}$$

Proof. Let $d(=\text{odd}) \in \mathbb{N}$. By (2.1) and (2.5), we note that

$$F^{(h)}_{q,\xi,\chi}(t,x) = \frac{1}{[d]^h_{-q}} \sum_{a_1,\dots,a_h=0}^{d-1} \chi(a_1+\cdots+a_h)(-\xi q)^{a_1+\cdots+a_h} F^{(h)}_{q^d,\xi^d}\left(\mathrm{d}t, \frac{a_1+\cdots+a_h+x}{d}\right). \tag{2.7}$$

Then, we have

$$E^{(h)}_{n,q,\xi,\chi}(x) = d^n \frac{1}{[d]^h_{-q}} \sum_{a_1,\dots,a_h=0}^{d-1} \chi(a_1+\cdots+a_h)(-\xi q)^{a_1+\cdots+a_h} E^{(h)}_{n,q^d,\xi^d}\left(\frac{a_1+\cdots+a_h+x}{d}\right). \tag{2.8}$$

On the other hand, if $F = dp$, then we get

$$\begin{aligned}
&\frac{1}{[F]_{-q}^{h}}\sum_{a_1,\ldots,a_h=0}^{F-1}\chi(a_1+\cdots+a_h)(-\xi q)^{a_1+\cdots+a_h}F_{q^F,\xi^F}^{(h)}\left(Ft,\frac{a_1+\cdots+a_h+x}{F}\right)\\
&=\sum_{a_1,\ldots,a_h=0}^{F-1}\chi(a_1+\cdots+a_h)(-\xi q)^{a_1+\cdots+a_h}\left(\frac{[2]_q}{\xi^F q^F e^{Ft}+1}\right)^{h}e^{(a_1+\cdots+a_h+x)t}\\
&=\sum_{b_1,\ldots,b_h=0}^{d-1}\sum_{c_1,\ldots,c_h=0}^{p-1}\chi(b_1+c_1d+\cdots+b_h+c_hd)(-\xi q)^{b_1+c_1d+\cdots+b_h+c_hd}\\
&\quad\times\left(\frac{[2]_q}{\xi^F q^F e^{Ft}+1}\right)^{h}e^{(b_1+c_1d+\cdots+b_h+c_hd+x)t}\\
&=\frac{1}{[d]_{-q}^{h}}\sum_{b_1,\ldots,b_h=0}^{d-1}\chi(b_1+\cdots+b_h)(-\xi q)^{b_1+\cdots+b_h}F_{q^d,\xi^d}^{(h)}\left(\mathrm{d}t,\frac{b_1+\cdots+b_h+x}{d}\right).
\end{aligned}\tag{2.9}$$

This completes the proof. □

The two-variable multiple twisted q-Euler l-functions are defined by the following definition.

Definition 2.4 (see [14]). Let χ be a Dirichlet character. For $s \in \mathbb{C}$ and $x \in \mathbb{R}^+$, one has

$$l_{q,\xi,E}^{(h)}(s,x,\chi)=[2]_q^h\sum_{l_1,\ldots,l_h=0}^{\infty}\frac{\chi(l_1+\cdots+l_h)\prod_{i=1}^{h}(-1)^{l_i}\xi^{l_i}q^{l_i}}{(l_1+\cdots+l_h+x)^s}.\tag{2.10}$$

The value of $l_{q,\xi,E}^{(h)}(s,x,\chi)$ at nonpositive integers is given explicitly by the following theorem.

Theorem 2.5 (see [14]). *Let $m \in \mathbb{Z}_+$. Then $l_{q,\xi,E}^{(h)}(-m,x,\chi)=E_{m,q,\xi,\chi}^{(h)}(x)$.*

Proof (cf. [17, 18]). Let χ be a Dirichlet character with odd conductor d and let F be an odd number of multiple d. Set $s \in \mathbb{C}$ and $x \in \mathbb{R}^+$. Beside the multiple twisted q-Euler l-function $l_{q,\xi,E}^{(h)}(s,x,\chi)$, we consider the multiple twisted q-Euler zeta function $\zeta_{q,\xi,E}^{(h)}(s,x)$ in Definition 2.1. Then

$$l_{q,\xi,E}^{(h)}(s,x,\chi)=F^{-s}\frac{1}{[F]_{-q}^{h}}\sum_{a_1,\ldots,a_h=0}^{F-1}\chi(a_1+\cdots+a_h)(-\xi q)^{a_1+\cdots+a_h}\zeta_{q^F,\xi^F,E}^{(h)}\left(s,\frac{a_1+\cdots+a_h+x}{F}\right)\tag{2.11}$$

(cf. [14]).

In the integral for $\Gamma(s)$, we make the change of variable $y=(x+l_1+\cdots+l_h)t$, where $l_1,\ldots,l_h \geq 0$, to obtain

$$\begin{aligned}
\Gamma(s)&=\int_0^{\infty}e^{-y}y^s\frac{\mathrm{d}y}{y}\\
&=(x+l_1+\cdots+l_h)^s\int_0^{\infty}e^{-(x+l_1+\cdots+l_h)t}t^s\frac{\mathrm{d}t}{t},
\end{aligned}\tag{2.12}$$

or

$$(x+l_1+\cdots+l_h)^{-s}\Gamma(s)=\int_0^\infty e^{-(x+l_1+\cdots+l_h)t}t^s\frac{dt}{t}. \tag{2.13}$$

Summing over all $l_1,\ldots,l_h \geq 0$, we find

$$[2]_q^h\sum_{l_1,\ldots,l_h=0}^{\infty}(-\xi q)^{l_1+\cdots+l_h}(x+l_1+\cdots+l_h)^{-s}\Gamma(s)=[2]_q^h\sum_{l_1,\ldots,l_h=0}^{\infty}(-\xi q)^{l_1+\cdots+l_h}\int_0^\infty e^{-(x+l_1+\cdots+l_h)t}t^s\frac{dt}{t}. \tag{2.14}$$

This gives us

$$\begin{aligned}\zeta_{q,\xi,E}^{(h)}(s,x)\Gamma(s)&=[2]_q^h\sum_{l_1,\ldots,l_h=0}^{\infty}(-\xi q)^{l_1+\cdots+l_h}\int_0^\infty e^{-(x+l_1+\cdots+l_h)t}t^s\frac{dt}{t}\\&=\int_0^\infty t^s\underbrace{\frac{[2]_q}{1+\xi qe^{-t}}\cdots\frac{[2]_q}{1+\xi qe^{-t}}}_{h\text{-times}}e^{-xt}\frac{dt}{t}\\&=\int_0^\infty t^sF_{q,\xi}^{(h)}(-t,x)\frac{dt}{t}.\end{aligned} \tag{2.15}$$

If we divide the infinite integral into two parts:

$$\int_0^\infty t^sF_{q,\xi}^{(h)}(-t,x)\frac{dt}{t}=\int_0^1 t^sF_{q,\xi}^{(h)}(-t,x)\frac{dt}{t}+\int_1^\infty t^sF_{q,\xi}^{(h)}(-t,x)\frac{dt}{t}, \tag{2.16}$$

it is easily seen that the second term is an entire function on t.

Consider $\int_0^1 t^sF_{q,\xi}^{(h)}(-t,x)(dt/t)$. By the definition of $E_{n,q,\xi}^{(h)}(x)$, we have

$$F_{q,\xi}^{(h)}(-t,x)=\sum_{n=0}^{\infty}E_{n,q,\xi}^{(h)}(x)(-1)^n\frac{t^n}{n!}=\left(\frac{[2]_q}{1+\xi qe^{-t}}\right)^h e^{-xt}. \tag{2.17}$$

Therefore,

$$\begin{aligned}\int_0^1 t^sF_{q,\xi}^{(h)}(-t,x)\frac{dt}{t}&=\sum_{n=0}^{\infty}\frac{E_{n,q,\xi}^{(h)}(x)}{n!}(-1)^n\int_0^1 t^{s+n-1}dt\\&=\sum_{n=0}^{\infty}\frac{E_{n,q,\xi}^{(h)}(x)}{n!}\frac{(-1)^n}{s+n}.\end{aligned} \tag{2.18}$$

This has an analytic continuation to a meromorphic function s in the entire complex plane. It is holomorphic except at $s=0,-1,-2,\ldots$, where it has a pole of order 1. Note that $\Gamma(s)$ is holomorphic except at $s=0,-1,-2,\ldots$, where it has a pole of order 1. $\Gamma(s)$ does not have a zero. Therefore, $\zeta_{q,\xi,E}^{(h)}(s,x)$ has an analytic continuation to the whole complex plane. For an integer $m\in\mathbb{Z}_+$, we have

$$\lim_{s\to -m}(s+m)\left(\zeta_{q,\xi,E}^{(h)}(s,x)\Gamma(s)\right)=\frac{E_{m,q,\xi}^{(h)}(x)}{m!}(-1)^m. \tag{2.19}$$

If $m \in \mathbb{Z}_+$, we have $\lim_{s \to -m}(s+m)\Gamma(s) = (-1)^m(1/m!)$, and thus we obtain

$$\zeta_{q,\xi,E}^{(h)}(-m,x) = E_{m,q,\xi}^{(h)}(x). \tag{2.20}$$

Consequently, by using Propositions 2.3 and (2.6) and the above equation, we have

$$\begin{aligned} &l_{q,\xi,E}^{(h)}(-m,x,\chi) \\ &\quad = F^m \frac{1}{[F]_{-q}^h} \sum_{a_1,\dots,a_h=0}^{F-1} \chi(a_1+\cdots+a_h)(-\xi q)^{a_1+\cdots+a_h} E_{m,q^F,\xi^F}^{(h)}\left(\frac{a_1+\cdots+a_h+x}{F}\right) = E_{m,q,\xi,\chi}^{(h)}(x). \end{aligned} \tag{2.21}$$

Therefore, we obtain another proof of Theorem 2.5. □

Remark 2.6 (see [11, 14]). We put

$$D = \left(\frac{\mathrm{d}}{\mathrm{d}t}\right). \tag{2.22}$$

Let $m \in \mathbb{Z}_+$, and let $x \in \mathbb{R}^+$. From (2.1) and (2.4), we obtain the following:

$$\begin{aligned} E_{m,q,\xi}^{(h)}(x) &= D^m F_{q,\xi}^{(h)}(t,x)\big|_{t=0} \\ &= [2]_q^h \sum_{l_1,\dots,l_k=0}^{\infty} (-\xi q)^{l_1+\cdots+l_h}(x+l_1+\cdots+l_h)^m \\ &= \zeta_{q,\xi,E}^{(h)}(-m,x). \end{aligned} \tag{2.23}$$

Similarly, by (2.5) and (2.11), we have

$$\begin{aligned} E_{m,q,\xi,\chi}^{(h)}(x) &= D^m F_{q,\xi,\chi}^{(h)}(t,x)\big|_{t=0} \\ &= [2]_q^h \sum_{l_1,\dots,l_h=0}^{\infty} \chi(l_1+\cdots+l_h) \prod_{i=1}^{h} (-1)^{l_i} \xi^{l_i} q^{l_i} (x+l_1+\cdots+l_h)^m \\ &= l_{q,\xi,E}^{(h)}(-m,x,\chi). \end{aligned} \tag{2.24}$$

3. Partial multiple twisted q-Euler ζ-functions

Let $s \in \mathbb{C}$ and $a_i, F \in \mathbb{Z}$ with F as an odd integer and $0 < a_i < F$, where $i = 1,\dots,h$. Then, partial multiple twisted q-Euler ζ-functions are as follows (cf. [14, 16, 18–20]):

$$H_{q,\xi,E}^{(h)}(s,a_1,\dots,a_h,x \mid F) = [2]_q^h \sum_{\substack{l_1,\dots,l_h=0 \\ l_i \equiv a_i \ (\mathrm{mod}\ F), i=1,\dots,h}}^{\infty} \frac{(-\xi q)^{l_1+\cdots+l_h}}{(l_1+\cdots+l_h+x)^s}. \tag{3.1}$$

We give a relationship between $H^{(h)}_{q,\xi,E}(s,a_1,\dots,a_h,x\mid F)$ and $\zeta^{(h)}_{q^F,\xi^F,E}(s,x)$ as follows. For $i=1,\dots,h$, substituting $l_i=a_i+n_iF$ with F as an odd into (3.1), we have

$$
\begin{aligned}
&H^{(h)}_{q,\xi,E}(s,a_1,\dots,a_h,x\mid F)\\
&\quad=[2]_q^h\sum_{n_1,\dots,n_h=0}^{\infty}\frac{(-\xi q)^{a_1+n_1F+\cdots+a_h+n_hF}}{(a_1+n_1F+\cdots+a_h+n_hF+x)^s}\\
&\quad=\frac{1}{[F]_{-q}^h}\frac{(-\xi q)^{a_1+\cdots+a_h}}{F^s}[2]_{q^F}^h\sum_{n_1,\dots,n_h=0}^{\infty}\frac{(-\xi^Fq^F)^{n_1+\cdots+n_h}}{(n_1+\cdots+n_h+(a_1+\cdots+a_h+x)/F)^s}\\
&\quad=\frac{1}{[F]_{-q}^h}\frac{(-\xi q)^{a_1+\cdots+a_h}}{F^s}\zeta^{(h)}_{q^F,\xi^F,E}\left(s,\frac{a_1+\cdots+a_h+x}{F}\right).
\end{aligned}
\tag{3.2}
$$

By using (2.3) and Theorem 2.2 and substituting $s=-m$, $m\in\mathbb{Z}_+$ in the above, we arrive at the following theorem.

Theorem 3.1. *Let F be an odd integer, $s\in\mathbb{C}$ and let $x\in\mathbb{R}^+$. Then*

$$
H^{(h)}_{q,\xi,E}(s,a_1,\dots,a_h,x\mid F)=\frac{1}{[F]_{-q}^h}\frac{(-\xi q)^{a_1+\cdots+a_h}}{F^s}\zeta^{(h)}_{q^F,\xi^F,E}\left(s,\frac{a_1+\cdots+a_h+x}{F}\right).
\tag{3.3}
$$

In particular, if $m\in\mathbb{Z}_+$, then

$$
\begin{aligned}
&H^{(h)}_{q,\xi,E}(-m,a_1,\dots,a_h,x\mid F)\\
&\quad=\frac{1}{[F]_{-q}^h}(-\xi q)^{a_1+\cdots+a_h}(a_1+\cdots+a_h+x)^m\sum_{k=0}^{m}\binom{m}{k}\left(\frac{F}{a_1+\cdots+a_h+x}\right)^kE^{(h)}_{k,q^F,\xi^F}.
\end{aligned}
\tag{3.4}
$$

By using Theorem 3.1 and (2.11), we arrive at the following theorem.

Theorem 3.2. *Let χ be a Dirichlet character with conductor d and F as an odd multiple of d. Then,*

$$
l^{(h)}_{q,\xi,E}(s,x,\chi)=\sum_{a_1,\dots,a_h=0}^{F-1}\chi(a_1+\cdots+a_h)H^{(h)}_{q,\xi,E}(s,x,a_1,\dots,a_h,x\mid F),
\tag{3.5}
$$

where $s\in\mathbb{C}$ and $x\in\mathbb{R}^+$.

4. Multiple twisted p-adic q-Euler l-functions

Let p be an odd prime. $\mathbb{Z}_p,\mathbb{Q}_p$ and $\mathbb{C}_p$ will always denote, respectively, the ring of p-adic integers, the field of p-adic numbers, and the completion of the algebraic closure of $\mathbb{Q}_p$. Let $v_p:\mathbb{C}_p\to\mathbb{Q}\cup\{\infty\}$ ($\mathbb{Q}$ the field of rational numbers) denote the p-adic valuation of $\mathbb{C}_p$ normalized so that $v_p(p)=1$. The absolute value on $\mathbb{C}_p$ will be denoted as $|\cdot|_p$, and $|x|_p=p^{-v_p(x)}$ for $x\in\mathbb{C}_p$. We let $\mathbb{Z}_p^\times=\{x\in\mathbb{Z}_p\mid 1/x\in\mathbb{Z}_p\}$. A p-adic integer in $\mathbb{Z}_p^\times$ is sometimes called a p-adic unit. For each integer $N\geq 0$, C_{p^N} will denote the multiplicative group of the primitive p^N-th roots of unity in $\mathbb{C}_p^\times=\mathbb{C}_p\setminus\{0\}$. Set

$$
\mathbf{T}_p=\{\xi\in\mathbb{C}_p\mid \xi^{p^N}=1,\ \text{for some } N\geq 0\}=\bigcup_{N\geq 0}C_{p^N}.
\tag{4.1}
$$

The dual of $\mathbb{Z}_p$, in the sense of p-adic Pontrjagin duality, is $\mathbf{T}_p = C_{p^\infty}$, the direct limit (under inclusion) of cyclic groups C_{p^N} of order p^N $(N \geq 0)$, with the discrete topology.

When one talks of q-extension, q is variously considered as an indeterminate, a complex number $q \in \mathbb{C}$, or a p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}_p$, then we normally assume $|1-q|_p < 1$.

We will consider the p-adic analogue of the $l^{(h)}_{q,\xi,E}$-functions which are introduced in the previous section. In order to consider p-adic and complex $l^{(h)}_{q,\xi,E}$-functions simultaneously, we will use an isomorphism, σ, between the algebraic closure of the rational numbers in $\mathbb{C}_p$ and the algebraic closure of the rational numbers within the complex numbers $\mathbb{C}$. Our purpose is to discuss the values of $l^{(h)}_{q,\xi,E}$-functions, so we will consider σ as fixed throughout this section and use σ to identify p-adic algebraic numbers with complex algebraic numbers. We will write $x = y$, when $x \in \mathbb{C}_p$, $y \in \mathbb{C}$ and $y = \sigma(x)$.

Let ω be denoted as the Teichmüller character having conductor p. For an arbitrary character χ, let $\chi_n = \chi\omega^{-n}$, where $n \in \mathbb{Z}$, in sense of the product of characters. We put

$$\langle a \rangle = \omega^{-1}(a)a = \frac{a}{\omega(a)}, \tag{4.2}$$

whenever $(a,p) = 1$. We then have $\langle a \rangle \equiv 1 \pmod{p\mathbb{Z}_p}$ for these values of a. Note that we extend this notation by defining

$$\langle a + pt \rangle = \omega^{-1}(a)(a + pt) \tag{4.3}$$

for all $a \in \mathbb{Z}$ with $(a,p) = 1$, and $t \in \mathbb{C}_p$ such that $|t|_p \leq 1$. Thus, $\langle a + pt \rangle = \langle a \rangle + p\omega^{-1}(a)t$, so that $\langle a + pt \rangle \equiv 1 \pmod{p\mathbb{Z}_p[t]}$ (cf. [21, 22]).

The significance of Theorem 3.1 lies in the fact that the right-hand side is essentially a liner combination of terms of the form

$$\sum_{k=0}^{m} \binom{m}{k} \left(\frac{F}{a_1 + \cdots + a_h + pt} \right)^k E^{(h)}_{k,q^F,\xi^F}, \tag{4.4}$$

which makes sense when m is replaced by a p-adic variable and $p|F$. Set

$$D = \left\{ s \in \mathbb{C}_p \mid |s|_p \leq p^{(p-2)/(p-1)} \right\}, \tag{4.5}$$

(cf. [8, 16, 18–22]). Let F be an odd integer, and let $(a_1 + \cdots + a_h, p) = 1$, $a_i \in \mathbb{Z}$ with $0 < a_i < F$ for $i = 1, \ldots, h$. Suppose that $s \in D$ and $t \in \mathbb{C}_p$ with $|t|_p \leq 1$. We apply [18, Proposition 5.8, page 53] to the series

$$\sum_{k=0}^{\infty} \binom{s}{k} \left(\frac{F}{a_1 + \cdots + a_h + pt} \right)^k E^{(h)}_{k,q^F,\xi^F}. \tag{4.6}$$

Let $q \in 1 + \mathbf{M}_p$ where $\mathbf{M}_p = \{ z \in \mathbb{C}_p \mid |z| < 1 \}$. In [13], we see that

$$\left| E^{(h)}_{k,q^F,\xi^F} \right|_p \leq 1, \tag{4.7}$$

since $\xi \in \mathbf{T}_p$. Observe that we have for odd $p|F$,

$$\left|\left(\frac{F}{a_1+\cdots+a_h+pt}\right)^k E^{(h)}_{k,q^F,\xi^F}\right|_p \le p^{-k} = \left(\frac{1}{p}\right)^k, \tag{4.8}$$

so that we can take $r = 1/p$ and $M = 1$ in [18, Proposition 5.8]. This prove that (4.6) is analytic in D. Note that $\langle a_1+\cdots+a_h+pt\rangle^{-s}$ is analytic in D for $(a_1+\cdots+a_h, p) = 1$ and $t \in \mathbb{C}_p$ such that $|t|_p \le 1$.

Definition 4.1. Let F be an odd integer, and let $(a_1+\cdots+a_h, p) = 1$, $a_i \in \mathbb{Z}$ with $0 < a_i < F$ for $i = 1,\ldots,h$. Suppose that $s \in D$ and $t \in \mathbb{C}_p$ with $|t|_p \le 1$. One defines the partial multiple twisted p-adic q-Euler ζ-functions for $p \mid F$:

$$\begin{aligned} &H^{(h)}_{p,q,\xi,E}(s,a_1,\ldots,a_h,pt \mid F) \\ &\quad = \frac{(-\xi q)^{a_1+\cdots+a_h}}{[F]^h_{-q}}\langle a_1+\cdots+a_h+pt\rangle^{-s}\sum_{k=0}^{\infty}\binom{-s}{k}\left(\frac{F}{a_1+\cdots+a_h+pt}\right)^k E^{(h)}_{k,q^F,\xi^F}. \end{aligned} \tag{4.9}$$

Theorem 4.2. *Let F be an odd integer with $p \mid F$, and let $(a_1+\cdots+a_h, p) = 1$, $a_i \in \mathbb{Z}$ with $0 < a_i < F$ for $i = 1,\ldots,h$. Suppose that $s \in D$ and $t \in \mathbb{C}_p$ with $|t|_p \le 1$. Then $H^{(h)}_{p,q,\xi,E}(s,a_1,\ldots,a_h,pt \mid F)$ is a p-adic analytic function on D such that*

$$H^{(h)}_{p,q,\xi,E}(-m,a_1,\ldots,a_h,pt \mid F) = \omega^{-m}(a_1+\cdots+a_h)H^{(h)}_{q,\xi,E}(-m,a_1,\ldots,a_h,pt \mid F) \tag{4.10}$$

for $m \in \mathbb{Z}_+$. In particular, if $m \equiv 0 \pmod{p-1}$, then

$$H^{(h)}_{p,q,\xi,E}(-m,a_1,\ldots,a_h,pt \mid F) = H^{(h)}_{q,\xi,E}(-m,a_1,\ldots,a_h,pt \mid F). \tag{4.11}$$

Proof. We have already remarked that $\langle a_1+\cdots+a_h+pt\rangle^{-s}$ is analytic in D for $(a_1+\cdots+a_h, p) = 1$ and $t \in \mathbb{C}_p$ such that $|t|_p \le 1$. Also we see that (4.6) is analytic in D. It is clear that $H^{(h)}_{p,q,\xi,E}$ is a p-adic analytic function on D. For $s = -m$, $m \in \mathbb{Z}_+$ one has

$$\begin{aligned} &H^{(h)}_{p,q,\xi,E}(-m,a_1,\ldots,a_h,pt \mid F) \\ &\quad = \frac{(-\xi q)^{a_1+\cdots+a_h}}{[F]^h_{-q}}\left(\frac{a_1+\cdots+a_h+pt}{\omega(a_1+\cdots+a_h)}\right)^m\sum_{k=0}^{m}\binom{m}{k}\left(\frac{F}{a_1+\cdots+a_h+pt}\right)^k E^{(h)}_{k,q^F,\xi^F} \\ &\quad \text{(where we use Theorem 3.1)} \\ &\quad = \omega^{-m}(a_1+\cdots+a_h)H^{(h)}_{q,\xi,E}(-m,a_1,\ldots,a_h,pt \mid F). \end{aligned} \tag{4.12}$$

This completes the proof. □

Definition 4.3. Let χ be a Dirichlet character with odd conductor d, and let F be a positive multiple of p and d. We can define the multiple twisted p-adic q-Euler l-function:

$$l^{(h)}_{p,q,\xi,E}(s,t,\chi)=\sum_{\substack{a_1,\dots,a_h=0\\(a_1+\cdots+a_h,p)=1}}^{F-1}\chi(a_1+\cdots+a_h)H^{(h)}_{p,q,\xi,E}(s,a_1,\dots,a_h,pt\mid F). \tag{4.13}$$

Theorem 4.4. *Let χ be a Dirichlet's character with an odd conductor d, and let F be a positive multiple of p and d. Then $l^{(h)}_{q,\xi,E}(s,t,\chi)$ is a p-adic analytic function on D with*

$$l^{(h)}_{p,q,\xi,E}(-m,t,\chi)=E^{(h)}_{m,q,\xi,\chi_m}(pt)-\frac{F^m}{[F]^h_{-q}}\chi_m(p)\sum_{\beta\in I_0}\chi_m(\beta)\left(-\xi^p q^p\right)^{\beta}E^{(h)}_{m,q^F,\xi^F}\left(\frac{\beta+t}{F/p}\right), \tag{4.14}$$

where $m\in\mathbb{Z}_+$ and

$$I_0=\left\{\frac{1}{p}x\;\middle|\;\begin{array}{c}x=a_1+\cdots+a_h\equiv 0\ (\mathrm{mod}\ p)\\ \text{for some } a_1,\dots,a_h \text{ with } 0\le a_1,\dots,a_h\le F-1\end{array}\right\}, \tag{4.15}$$

and in $\sum_{\beta\in I_0}$ one sums over $\beta=(1/p)x$ as many times as x is expressed in the form $x=a_1+\cdots+a_h$ by various a_j's, and $\chi_m=\chi\omega^{-m}$ with ω the Teichmüller in the sense of the product of characters.

Remark 4.5. Theorem 4.4 can be extended to obtain similar results for the multiple p-adic L-function in [16]. In the case $h=1$, we note that

$$\sum_{\beta\in I_0}=\sum_{a=0}^{F/p-1}. \tag{4.16}$$

Observe that if $h=1$, then

$$\begin{aligned}
&\lim_{h\to 1} l^{(h)}_{p,q,\xi,E}(-m,t,\chi)\\
&=E_{m,q,\xi,\chi_m}(pt)-\frac{F^m}{[F]_{-q}}\chi_m(p)\sum_{a=0}^{F/p-1}\chi_m(a)\left(-\xi^p q^p\right)^{a}E_{m,q^F,\xi^F}\left(\frac{a+t}{F/p}\right)\\
&=E_{m,q,\xi,\chi_m}(pt)-p^m\chi_m(p)\frac{[F/p]_{-q^p}}{[F]_{-q}}\left(\frac{F}{p}\right)^m\frac{1}{[F/p]_{-q^p}}\sum_{a=0}^{F/p-1}\chi_m(a)\left(-\xi^p q^p\right)^{a}E_{m,(q^p)^{F/p},(\xi^p)^{F/p}}\left(\frac{a+t}{F/p}\right)\\
&\text{(where we use Proposition 2.3)}\\
&=E_{m,q,\xi,\chi_m}(pt)-p^m\frac{1}{[p]_{-q}}\chi_m(p)E_{m,q^p,\xi^p,\chi_m}(t).
\end{aligned} \tag{4.17}$$

This function interpolates the twisted generalized q-Euler polynomials at negative integers. For $l^{(1)}_{p,q,\xi,E}(-m,t,\chi)$, the twisted p-adic q-Euler l-functions similar results were obtained (cf. see for detail [8, Theorem 9]). If $q\to 1$ in the above, then

$$\lim_{q\to 1} l^{(1)}_{p,q,\xi,E}(s,t,\chi)=l_{p,\xi,E}(s,t,\chi)=2\sum_{\substack{l=0\\(l,p)=1}}^{\infty}\frac{\chi(l)(-1)^l\xi^l}{(t+l)^s},\quad |t|_p\le 1, \tag{4.18}$$

which is called the twisted p-adic l_E-function of two variables.

Proof. The formula for $l^{(h)}_{q,\xi,E}(s,x,\chi)$ is p-adic analytic function in D by the Theorem 4.2. On the other hand, by substituting $s=-m, m\in\mathbb{Z}_+$, into Definition 4.3, we have

$$\begin{aligned} l^{(h)}_{p,q,\xi,E}(-m,t,\chi) &= \sum_{\substack{a_1,\ldots,a_h=0\\(a_1+\cdots+a_h,p)=1}}^{F-1} \chi(a_1+\cdots+a_h)H^{(h)}_{p,q,\xi,E}(-m,a_1,\ldots,a_h,pt\mid F)\\ &= \sum_{a_1,\ldots,a_h=0}^{F-1}\chi(a_1+\cdots+a_h)H^{(h)}_{p,q,\xi,E}(-m,a_1,\ldots,a_h,pt\mid F)\\ &\quad-\sum_{\substack{a_1,\ldots,a_h=0\\(a_1+\cdots+a_h,p)\neq 1}}^{F-1}\chi(a_1+\cdots+a_h)H^{(h)}_{p,q,\xi,E}(-m,a_1,\ldots,a_h,pt\mid F). \end{aligned} \tag{4.19}$$

From Theorems 2.2, 3.1, and 4.2, we obtain

$$\begin{aligned} &\sum_{\substack{a_1,\ldots,a_h=0\\(a_1+\cdots+a_h,p)\neq 1}}^{F-1}\chi(a_1+\cdots+a_h)H^{(h)}_{p,q,\xi,E}(-m,a_1,\ldots,a_h,pt\mid F)\\ &\quad=\sum_{\substack{a_1,\ldots,a_h=0\\(a_1+\cdots+a_h,p)\neq 1}}^{F-1}\chi_m(a_1+\cdots+a_h)(-\xi q)^{a_1+\cdots+a_h}\frac{F^m}{[F]^h_{-q}}E^{(h)}_{m,q^F,\xi^F}\left(\frac{a_1+\cdots+a_h+pt}{F}\right)\\ &\quad=\frac{F^m}{[F]^h_{-q}}\chi_m(p)\sum_{\beta\in I_0}\chi_m(\beta)\left(-\xi^p q^p\right)^{\beta}E^{(h)}_{m,q^F,\xi^F}\left(\frac{\beta+t}{F/p}\right). \end{aligned} \tag{4.20}$$

Therefore,

$$l^{(h)}_{p,q,\xi,E}(-m,t,\chi)=E^{(h)}_{m,q,\xi,\chi_m}(pt)-\frac{F^m}{[F]^h_{-q}}\chi_m(p)\sum_{\beta\in I_0}\chi_m(\beta)\left(-\xi^p q^p\right)^{\beta}E^{(h)}_{m,q^F,\xi^F}\left(\frac{\beta+t}{F/p}\right). \tag{4.21}$$

This completes the proof. □

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References

[1] T. Kim, "Euler numbers and polynomials associated with zeta functions," *Abstract and Applied Analysis*, vol. 2008, Article ID 581582, 11 pages, 2008.

[2] T. Kim, "On the q-extension of Euler and Genocchi numbers," *Journal of Mathematical Analysis and Applications*, vol. 326, no. 2, pp. 1458–1465, 2007.

[3] T. Kim, "On p-adic interpolating function for q-Euler numbers and its derivatives," *Journal of Mathematical Analysis and Applications*, vol. 339, no. 1, pp. 598–608, 2008.

[4] T. Kim, "On the multiple q-Genocchi and Euler numbers," to appear in *Russian Journal of Mathematical Physics*.

[5] T. Kim, M.-S. Kim, L. Jang, and S.-H. Rim, "New q-Euler numbers and polynomials associated with p-adic q-integrals," *Advanced Studies in Contemporary Mathematics*, vol. 15, no. 2, pp. 243–252, 2007.

[6] T. Kim, L.-C. Jang, and C.-S. Ryoo, "Note on q-extensions of Euler numbers and polynomials of higher order," *Journal of Inequalities and Applications*, vol. 2008, Article ID 371295, 9 pages, 2008.

[7] H. Ozden, I. N. Cangul, and Y. Simsek, "Multivariate interpolation functions of higher-order q-Euler numbers and their applications," *Abstract and Applied Analysis*, vol. 2008, Article ID 390857, 16 pages, 2008.

[8] H. Ozden and Y. Simsek, "Interpolation function of the (h, q)-extension of twisted Euler numbers," *Computers & Mathematics with Applications*, vol. 56, no. 4, pp. 898–908, 2008.

[9] H. Ozden, I. N. Cangul, and Y. Simsek, "Remarks on sum of products of (h, q)-twisted Euler polynomials and numbers," *Journal of Inequalities and Applications*, vol. 2008, Article ID 816129, 8 pages, 2008.

[10] Y. Simsek, "q-analogue of twisted l-series and q-twisted Euler numbers," *Journal of Number Theory*, vol. 110, no. 2, pp. 267–278, 2005.

[11] L.-C. Jang, "Multiple twisted q-Euler numbers and polynomials associated with p-adic q-integrals," *Advances in Difference Equations*, vol. 2008, Article ID 738603, 11 pages, 2008.

[12] L.-C. Jang and C.-S. Ryoo, "A note on the multiple twisted Carlitz's type q-Bernoulli polynomials," *Abstract and Applied Analysis*, vol. 2008, Article ID 498173, 7 pages, 2008.

[13] T. Kim, "A note on q-Euler numbers and polynomials," preprint 2006, http://arxiv.org/abs/math.NT/0608649.

[14] M.-S. Kim, T. Kim, and J.-W. Son, "Multivariate p-adic fermionic q-integral on $\mathbb{Z}_p$ and related multiple zeta-type functions," *Abstract and Applied Analysis*, vol. 2008, Article ID 304539, 13 pages, 2008.

[15] T. Kim, "q-Bernoulli numbers and polynomials associated with Gaussian binomial coefficients," *Russian Journal of Mathematical Physics*, vol. 15, no. 1, pp. 51–57, 2008.

[16] T. Kim, "Multiple p-adic L-function," *Russian Journal of Mathematical Physics*, vol. 13, no. 2, pp. 151–157, 2006.

[17] K. Kato, N. Kurokawa, and T. Saito, *Number Theory 1: Fermat's Dream*, vol. 186 of *Translations of Mathematical Monographs*, American Mathematical Society, Providence, RI, USA, 2000.

[18] L. C. Washington, *Introduction to Cyclotomic Fields*, vol. 83 of *Graduate Texts in Mathematics*, Springer, New York, NY, USA, 2nd edition, 1997.

[19] L. C. Washington, "A note on p-adic L-functions," *Journal of Number Theory*, vol. 8, no. 2, pp. 245–250, 1976.

[20] L. C. Washington, "p-adic L-functions and sums of powers," *Journal of Number Theory*, vol. 69, no. 1, pp. 50–61, 1998.

[21] G. J. Fox, "A method of Washington applied to the derivation of a two-variable p-adic L-function," *Pacific Journal of Mathematics*, vol. 209, no. 1, pp. 31–40, 2003.

[22] P. T. Young, "On the behavior of some two-variable p-adic L-functions," *Journal of Number Theory*, vol. 98, no. 1, pp. 67–88, 2003.