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Research Article

Various Half-Eigenvalues of Scalar p-Laplacian with Indefinite Integrable Weights

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Consider the half-eigenvalue problem $(\phi_p(x'))' + \lambda a(t)\phi_p(x_+) - \lambda b(t)\phi_p(x_-) = 0$ a.e. $t \in [0,1]$, where $1 < p < \infty$, $\phi_p(x) = |x|^{p-2}x$, $x_\pm(\cdot) = \max\{\pm x(\cdot), 0\}$ for $x \in \mathcal{C}^0 := C([0,1],\mathbb{R})$, and $a(t)$ and $b(t)$ are indefinite integrable weights in the Lebesgue space $\mathcal{L}^\gamma := L^\gamma([0,1],\mathbb{R}), 1 \le \gamma \le \infty$. We characterize the spectra structure under periodic, antiperiodic, Dirichlet, and Neumann boundary conditions, respectively. Furthermore, all these half-eigenvalues are continuous in $(a,b) \in (\mathcal{L}^\gamma, w_\gamma)^2$, where w_γ denotes the weak topology in $\mathcal{L}^\gamma$ space. The Dirichlet and the Neumann half-eigenvalues are continuously Fréchet differentiable in $(a,b) \in (\mathcal{L}^\gamma, \|\cdot\|_\gamma)^2$, where $\|\cdot\|_\gamma$ is the L^γ norm of $\mathcal{L}^\gamma$.

1. Introduction

Fučik spectrum and half-eigenvalues are useful for solving problems with "jumping nonlinearities." Compared with Fučik spectrum, half-eigenvalues have not been paid much research. However, as half-eigenvalues are concerned with only one parameter, we think the theory on half-eigenvalues will lead to better knowledge for Fučik spectrum and jumping nonlinearities.

Given an exponent $p \in (1,\infty)$, let $\phi_p(x) := |x|^{p-2}x$ for $x \in \mathbb{R}$ which is used to define the scalar p-Laplacian. Denote $x_\pm = \max\{\pm x, 0\}$ for $x \in \mathbb{R}$ and $x_\pm(\cdot) = \max\{\pm x(\cdot), 0\}$ for any $x \in \mathcal{C}^0 := C([0,1],\mathbb{R})$. For a pair of indefinite (sign-changing) integrable weights (a,b) with $a, b \in \mathcal{L}^\gamma := L^\gamma([0,1],\mathbb{R})$, $1 \le \gamma \le \infty$, we are concerned with the half-eigenvalue problem

$$\left(\phi_p(x')\right)' + \lambda a(t)\phi_p(x_+) - \lambda b(t)\phi_p(x_-) = 0, \quad \text{a.e. } t \in [0,1], \tag{1.1}$$

associated with the periodic boundary condition

$$x(0) - x(1) = x'(0) - x'(1) = 0, \tag{1.2}$$

the antiperiodic boundary condition

$$x(0) + x(1) = x'(0) + x'(1) = 0, \tag{1.3}$$

the Dirichlet boundary condition

$$x(0) = x(1) = 0, \tag{1.4}$$

or the Neumann boundary condition

$$x'(0) = x'(1) = 0, \tag{1.5}$$

respectively. Values of λ for which (1.1) has nontrivial solutions x satisfying the boundary condition (1.2), (1.3), (1.4), or (1.5) will be called periodic, antiperiodic, Dirichlet, or Neumann half-eigenvalues, respectively, while the corresponding solutions x will be called half-eigenfunctions. Subscripts P, A, D, and N will be used to indicate periodic, antiperiodic, Dirichlet and Neumann boundary conditions, respectively. The set of the half-eigenvalues under each of the preceding boundary conditions will be denoted by Σ_P, Σ_A, Σ_D, and Σ_N, respectively. Notice that these spectra are dependent on weights (a, b).

This article studies the half-eigenvalue problem (1.1) from two aspects. One is the structure of Σ_P, Σ_A, Σ_D, and Σ_N, and the other is the dependence of all these spectra on weights.

Generally speaking, for the p-Laplacian eigenvalue problem, if separated boundary conditions (such as Dirichlet and Neumann) are imposed and if the weight is definite, the spectrum structure is similar to that of the classical linear case. However, compared with separated boundary conditions, definite weights, and $\mathcal{L}^\infty$ weights (or potentials), more difficulty will always be brought on by nonseparated boundary conditions, indefinite weights, and $\mathcal{L}^\gamma$ $(1 \le \gamma < \infty)$ weights (or potentials), respectively. It has been proved very recently in [1] that with an indefinite weight $a \in \mathcal{L}^\gamma$, $1 \le \gamma \le \infty$, the spectra structures of

$$(\phi_p(x'))' + \lambda a(t)\phi_p(x) = 0 \quad \text{a.e. } t \in [0,1] \tag{1.6}$$

under both boundary conditions (1.4) and (1.5) are the same as when $p = 2$. Special case of (1.6) with indefinite weight $a \in \mathcal{L}^\infty$ is also studied by Anane et al. [2], Cuesta [3], and Eberhard and Elbert [4]. Various spectra (including half-eigenvalues) of p-Laplacian with potentials are studied by Binding and Rynne in [5]. It is shown that compared with the linear case $p = 2$, some new phenomena occur in the periodic problems when $p \ne 2$. The existence of nonvariational periodic and antiperiodic eigenvalues of p-Laplacian for some potentials in [6] can partially explain these new phenomena.

We will use the shooting method to obtain spectra structures. To this end, (1.1) is studied in p-polar coordinates (2.5), and the associated argument function plays a crucial role. Careful analysis on the argument function leads to partial characterization of Σ_P and Σ_A and complete characterization of Σ_D and Σ_N, see Theorems 3.9, 4.2, and 5.2. Some quasimonotonicity of the argument function is proved by using the Fréchet derivatives and the boundary conditions, see Lemmas 3.3, 4.1, and 5.1. The uniformly asymptotical result is obtained from the w^* compactness in the Banach-Alaoglu theorem as done in [1], see

Lemma 2.3. Besides the properties of the argument functions, the Hamiltonian structure of the problem is essential for us to obtain variational periodic and antiperiodic half-eigenvalues, see Lemma 3.2. After introducing the rotation number function, we can easily obtain the ordering of these variational periodic half-eigenvalues, see (3.38).

For regular self-adjoint linear Sturm-Liouville problems, the continuous dependence of eigenvalues on weights or potentials in the usual L^γ topology is well understood, and so is the Fréchet differentiable dependence. Many of these results are summarized in [7]. However, since the space of potentials (or weights) is infinite-dimensional, such a continuity result cannot answer many basic questions. For example, if potentials or weights are confined to a bounded set or a noncompact set, are the eigenvalues finite? To answer such kind of questions, a stronger continuity result is obtained in [8] for Sturm-Liouville operators and Hill's operators. That is, the eigenvalues are continuous in potentials in weak topology w_γ. Based on such a stronger continuity and the differentiability, variational method and singular integrals are applied in [9] to obtain the extremal value of smallest eigenvalues of Hill's operators with potentials confined to L^1 balls. The continuity result in weak topology are generalized to scalar p-Laplacian for eigenvalues on potentials (see [10]), for (separated) eigenvalues on indefinite weights (see [1]), and for half-eigenvalues on potentials (see [11]). Some elementary applications are also presented in [1, 10].

In this paper, we will prove that the variational periodic or antiperiodic half-eigenvalues $\underline{\lambda}_m^L$ and $\overline{\lambda}_m^R$ (defined by (3.22) and (3.25), resp.), and all the half-eigenvalues in Σ_D and Σ_N, are continuous in weights $(a,b) \in (\mathcal{L}^\gamma, w_\gamma)^2$. See Theorems 3.12, 4.3, and 5.3. Moreover, the Dirichlet and the Neumann half-eigenvalues are continuously Fréchet differentiable in weights $(a,b) \in (\mathcal{L}^\gamma, \|\cdot\|_\gamma)^2$, see Theorems 4.4 and 5.3. Due to the so-called parametric resonance [12] or the so-called coexistence of periodic and antiperiodic eigenvalues [13], periodic and antiperiodic half-eigenvalues are, in general, not differentiable in weights (a,b).

If λ is a half-eigenvalue of (1.1) corresponding to weights (a,b) and satisfying the boundary condition (1.2), (1.3), (1.4), or (1.5), then $-\lambda$ is also a half-eigenvalue of (1.1) corresponding to weights $(-a,-b)$ and satisfying the same boundary condition. So we need only consider nonnegative half-eigenvalues of (1.1). Some preliminary results are given in Section 2. However, Sections 3, 4, and 5 are devoted to $\Sigma_{A/P}$, Σ_D, and Σ_N, respectively.

2. Preliminary Results

Given $p \in (1,\infty)$, denote by $(C_p(\theta), S_p(\theta))$ the unique solution of the initial value problem

$$\frac{dx}{d\theta} = -\phi_{p^*}(y), \qquad \frac{dy}{d\theta} = \phi_p(x), \qquad (x(0), y(0)) = (1,0). \tag{2.1}$$

The functions $C_p(\theta)$ and $S_p(\theta)$ are the so-called p-cosine and p-sine, because they possess some properties similar to those of cosine and sine functions, such as

(i) both $C_p(\theta)$ and $S_p(\theta)$ are $2\pi_p$-periodic, where

$$\pi_p = \frac{2\pi(p-1)^{1/p}}{p\sin(\pi/p)}; \tag{2.2}$$

(ii) $C_p(\theta) = 0$ if and only if $\theta = \pi_p/2 + m\pi_p$, $m \in \mathbb{Z}$, and $S_p(\theta) = 0$ if and only if $\theta = m\pi_p$, $m \in \mathbb{Z}$;

(iii) one has

$$|C_p(\theta)|^p + (p-1)|S_p(\theta)|^{p^*} \equiv 1. \tag{2.3}$$

Given $a, b \in \mathcal{L}^\gamma$, $\gamma \in [1, \infty]$, let

$$y = -\phi_p(x'). \tag{2.4}$$

In the p-polar coordinates

$$x = r^{2/p} C_p(\theta), \qquad y = r^{2/p^*} S_p(\theta), \tag{2.5}$$

the scalar equation

$$(\phi_p(x'))' + a(t)\phi_p(x_+) - b(t)\phi_p(x_-) = 0, \quad \text{a.e. } t \in [0,1], \tag{2.6}$$

is transformed into the following equations for r and θ

$$\begin{aligned} \theta' &= A(t, \theta; a, b) \\ &:= \begin{cases} a(t)|C_p(\theta)|^p + (p-1)|S_p(\theta)|^{p^*}, & \text{if } C_p(\theta) \geq 0, \\ b(t)|C_p(\theta)|^p + (p-1)|S_p(\theta)|^{p^*}, & \text{if } C_p(\theta) < 0, \end{cases} \end{aligned} \tag{2.7}$$

$$\begin{aligned} (\log r)' &= G(t, \theta; a, b) \\ &:= \begin{cases} \dfrac{p}{2}(a(t) - 1)\phi_p(C_p(\theta))\phi_{p^*}(S_p(\theta)), & \text{if } C_p(\theta) \geq 0, \\ \dfrac{p}{2}(b(t) - 1)\phi_p(C_p(\theta))\phi_{p^*}(S_p(\theta)), & \text{if } C_p(\theta) < 0. \end{cases} \end{aligned} \tag{2.8}$$

For any $\vartheta_0 \in \mathbb{R}$, denote by $(\theta(t; \vartheta_0, a, b), r(t; \vartheta_0, a, b))$, $t \in [0,1]$, the unique solution of (2.7) + (2.8) satisfying $\theta(0; \vartheta_0, a, b) = \vartheta_0$ and $r(0; \vartheta_0, a, b) = 1$. Let

$$\begin{aligned} \Theta(\vartheta_0, a, b) &:= \theta(1; \vartheta_0, a, b), \\ R(\vartheta_0, a, b) &:= r(1; \vartheta_0, a, b). \end{aligned} \tag{2.9}$$

As $A(t, \theta; a, b)$ is independent of r and is $2\pi_p$-periodic in θ, we have

$$\theta(t; \vartheta_0 + 2m\pi_p, a, b) \equiv \theta(t; \vartheta_0, a, b) + 2m\pi_p \tag{2.10}$$

for all $t \in [0,1]$, $\vartheta_0 \in \mathbb{R}$, and $m \in \mathbb{Z}$.

An important property of the argument solution θ is the quasimonotonicity as in the following lemma.

Lemma 2.1 (see [14]). *Let $\theta(t) = \theta(t; \vartheta_0, a, b)$ be a solution of* (2.7). *Then*

$$\theta(t) \geq -\frac{\pi_p}{2} + m\pi_p \quad \text{at } t \in [0,1) \Longrightarrow \theta(s) > -\frac{\pi_p}{2} + m\pi_p \ \forall s \in (t,1]. \tag{2.11}$$

Denote by $\mathcal{C}^0 := C([0,1],\mathbb{R})$ the space of continuous functions from $[0,1]$ to $\mathbb{R}$. Some dependence results of solutions r and θ on (a,b) are collected in the following theorem.

Theorem 2.2 (see [11]). (i) *The functional*

$$\mathbb{R} \times \left(\mathcal{L}^\gamma, w_\gamma\right)^2 \longrightarrow \mathbb{R}, \qquad (\vartheta, a, b) \longmapsto \Theta(\vartheta, a, b) \tag{2.12}$$

is continuous. Here w_γ denotes the weak topology in $\mathcal{L}^\gamma$.

(ii) *The functional*

$$\mathbb{R} \times \left(\mathcal{L}^\gamma, \|\cdot\|_\gamma\right)^2 \longrightarrow \mathbb{R}, \qquad (\vartheta, a, b) \longmapsto \Theta(\vartheta, a, b) \tag{2.13}$$

is continuously differentiable. The derivatives of $\Theta(\vartheta, a, b)$ at ϑ, at $a \in \mathcal{L}^\gamma$, and at $b \in \mathcal{L}^\gamma$ (in the Fréchet sense), denoted, respectively, by $\partial_\vartheta\Theta$, $\partial_a\Theta$, and $\partial_b\Theta$, are

$$\partial_\vartheta\Theta(\vartheta, a, b) = \frac{1}{R^2(\vartheta, a, b)}, \tag{2.14}$$

$$\partial_a\Theta(\vartheta, a, b) = X_+^p \in \mathcal{C}^0 \subset \left(\mathcal{L}^\gamma, \|\cdot\|_\gamma\right)^*, \tag{2.15}$$

$$\partial_b\Theta(\vartheta, a, b) = X_-^p \in \mathcal{C}^0 \subset \left(\mathcal{L}^\gamma, \|\cdot\|_\gamma\right)^*, \tag{2.16}$$

where

$$X = X(t) = X(t; \vartheta, a, b) := \frac{\{r(t; \vartheta, a, b)\}^{2/p} C_p(\theta(t; \vartheta, a, b))}{\{r(1; \vartheta, a, b)\}^{2/p}} \tag{2.17}$$

is a solution of (2.6).

To characterize the spectrum structure of (1.1) via shooting method, we need careful analysis on the associated argument function $\Theta(\vartheta, \lambda a, \lambda b)$. Given $a_1, a_2, b_1, b_2 \in \mathcal{L}^\gamma$, write $a_1 \succ b_1$ if $a_1 \geq b_1$ and $a_1(t) > b_1(t)$ holds for t in a subset of $[0,1]$ of positive measure. Write $(a_1, b_1) \geq (a_2, b_2)$ if $a_1 \geq a_2$ and $b_1 \geq b_2$. Write $(a_1, b_1) \succ (a_2, b_2)$ if $(a_1, b_1) \geq (a_2, b_2)$ and both $a_1(t) > a_2(t)$ and $b_1(t) > b_2(t)$ hold for t in a *common* subset of $[0,1]$ of positive measure. Denote

$$\mathcal{W}_+^\gamma := \{(a,b) \mid a, b \in \mathcal{L}^\gamma, (a_+, b_+) \succ (0,0)\}. \tag{2.18}$$

The following asymptotical property of $\Theta(\vartheta, \lambda a, \lambda b)$ in $\lambda \in \mathbb{R}$ plays a crucial role in the characterization of half-eigenvalues.

Lemma 2.3. *Given* $(a,b) \in \mathcal{W}_+^{\gamma}$, $1 \le \gamma \le \infty$, *one has*

$$\lim_{\lambda\to\infty} (\Theta(\vartheta, \lambda a, \lambda b) - \vartheta) = \infty \tag{2.19}$$

uniformly in $\vartheta \in \mathbb{R}$.

Proof. The proof is a slight extension of that in [1, Lemma 3.4]. We write it out in detail for the convenience of the readers.

Besides (2.4), we introduce another transformation

$$x_\lambda = x, \qquad y_\lambda = -\lambda^{-1/p^*}\phi_p(x') = \lambda^{-1/p^*} y \tag{2.20}$$

when $\lambda > 0$. However, (1.1) becomes

$$x_\lambda = -\lambda^{1/p}\phi_{p^*}(y_\lambda), \qquad y'_\lambda = \lambda^{1/p}\{a(t)\phi_p((x_\lambda)_+) + b(t)\phi_p((x_\lambda)_-)\}. \tag{2.21}$$

In the p-polar coordinates of the (x_λ, y_λ)-plane

$$x_\lambda := r_\lambda^{2/p} C_p(\theta_\lambda), \qquad y_\lambda := r_\lambda^{2/p^*} S_p(\theta_\lambda), \tag{2.22}$$

the equation for the new argument θ_λ is

$$\theta'_\lambda = \lambda^{1/p}\left\{1 + a(t)(C_p(\theta_\lambda))_+^p + b(t)(C_p(\theta_\lambda))_-^p - \left|C_p(\theta_\lambda)\right|^p\right\}. \tag{2.23}$$

Denote by $\theta_\lambda(t;\vartheta,a,b)$ the solution of (2.23) satisfying the initial condition $\theta_\lambda(0) = \vartheta$. By (2.20), the argument functions of (1.1), $\theta_\lambda(t) = \theta_\lambda(t;\vartheta,a,b)$ and $\theta(t) = \theta(t;\vartheta,\lambda a,\lambda b)$ are related by an orientation-preserving homeomorphism $\mathscr{H} : \mathbb{R} \to \mathbb{R}$, which fixes the points $\{m\pi_p, m\pi_p + \pi_p/2 : m \in \mathbb{Z}\}$, in the following relation

$$\theta(t;\mathscr{H}(\vartheta),a,b) = \mathscr{H}(\theta_\lambda(t;\vartheta,a,b)). \tag{2.24}$$

If lemma is not true, then there exist $\{\lambda_n\}_{n\in\mathbb{N}}$ and $\{\vartheta_n\}_{n\in\mathbb{N}}$ such that $\lambda_n \uparrow \infty$ and $\{\Theta(\vartheta_n,\lambda_n a,\lambda_n b) - \vartheta_n\}_{n\in\mathbb{N}}$ is bounded from above. Combining (2.11), there exists some $m_0 \in \mathbb{N}$ such that

$$-\pi_p < \Theta(\vartheta_n,\lambda_n a,\lambda_n b) - \vartheta_n < m_0\pi_p, \quad \forall n \in \mathbb{N}. \tag{2.25}$$

By (2.11) again, we have

$$-\pi_p < \theta(t;\vartheta_n,\lambda_n a,\lambda_n b) - \vartheta_n < (m_0+1)\pi_p, \quad \forall t \in [0,1],\ \forall n \in \mathbb{N}. \tag{2.26}$$

By (2.10), we may assume

$$\vartheta_n \in [0, 2\pi_p], \quad \forall n \in \mathbb{N}. \tag{2.27}$$

Hence,

$$-\pi_p < \theta(t;\vartheta_n,\lambda_n a,\lambda_n b) < (m_0+3)\pi_p, \quad \forall t \in [0,1],\ \forall n \in \mathbb{N}. \tag{2.28}$$

Denote $\theta_n(t) := \theta_{\lambda_n}(t;\vartheta_n,a,b)$ for simplicity. By the conjugacy (2.24) and the estimate (2.28), we have

$$-\pi_p < \theta_n(t) < (m_0+3)\pi_p, \quad \forall t \in [0,1],\ \forall n \in \mathbb{N}. \tag{2.29}$$

Denote $I_0 := \{f \in \mathcal{L}^\infty : 0 \le f \le 1\}$ and define

$$\begin{aligned} f_n(t) &:= \left(C_p(\theta_n(t))\right)_+^p \in I_0, \\ g_n(t) &:= \left(C_p(\theta_n(t))\right)_-^p \in I_0, \\ h_n(t) &:= \left|C_p(\theta_n(t))\right|^p \in I_0, \end{aligned} \tag{2.30}$$

for any $n \in \mathbb{N}$. By the Banach-Alaoglu theorem [15, pages 229-230], the unit ball $B_1 := \{f \in \mathcal{L}^\infty : \|f\|_\infty \le 1\}$ is sequentially compact in $(\mathcal{L}^\infty, w_\infty)$ by considering $\mathcal{L}^\infty$ as the dual space of the (separable) Banach space $(\mathcal{L}^1, \|\cdot\|_1)$. The order interval I_0 is a closed subset of $(\mathcal{L}^\infty, w_\infty)$. Hence, $I_0(\subset B_1)$ is also sequentially compact in $(\mathcal{L}^\infty, w_\infty)$. Consequently, passing to a subsequence, we may assume

$$f_n \longrightarrow f_0 \in I_0, \quad g_n \longrightarrow g_0 \in I_0, \quad h_n \longrightarrow h_0 \in I_0, \quad \text{in } (\mathcal{L}^\infty, w_\infty). \tag{2.31}$$

From (2.23), $\theta_n(t)$ satisfies

$$\theta_n' = \lambda_n^{1/p}\left\{1 + a(t)f_n(t) + b(t)g_n(t) - h_n(t)\right\} \quad \text{a.e. } t \in [0,1]. \tag{2.32}$$

For any $t \in [0,1]$, we have

$$\begin{aligned} \frac{\theta_n(t)-\vartheta_n}{\lambda_n^{1/p}} &= \int_0^t \left\{1 + a(s)f_n(s) + b(s)g_n(s) - h_n(s)\right\} ds \\ &= t + \int_0^1 a(s)\chi_{[0,t]}(s)\cdot f_n(s)ds + \int_0^1 b(s)\chi_{[0,t]}(s)\cdot g_n(s)ds - \int_0^1 \chi_{[0,t]}(s)\cdot h_n(s)ds. \end{aligned} \tag{2.33}$$

Let $n \to \infty$, by (2.27) and (2.29), the left-hand side tends to 0. Since

$$a(\cdot)\chi_{[0,t]}(\cdot), \quad b(\cdot)\chi_{[0,t]}(\cdot), \quad \chi_{[0,t]}(\cdot) \in \mathcal{L}^1, \tag{2.34}$$

we can use (2.31) to deduce from the above equality

$$\begin{aligned} 0 &= t + \int_0^1 a(s)\chi_{[0,t]}(s)\cdot f_0(s)ds + \int_0^1 b(s)\chi_{[0,t]}(s)\cdot g_0(s)ds - \int_0^1 \chi_{[0,t]}(s)\cdot h_0(s)ds \\ &= \int_0^t \{1 + a(s)f_0(s) + b(s)g_0(s) - h_0(s)\}ds \end{aligned} \tag{2.35}$$

for all $t \in [0,1]$. Thus

$$1 + a(t)f_0(t) + b(t)g_0(t) - h_0(t) = 0, \quad \text{a.e. } t \in [0,1]. \tag{2.36}$$

As $h_n = f_n + g_n$ and the weak* limits f_0, g_0, h_0 satisfy $0 \le f_0(t), g_0(t), h_0(t) \le 1$ a.e. $t \in [0,1]$, this equality implies that

$$h_0(t) = f_0(t) + g_0(t) \neq 0, \quad \text{a.e. } t \in [0,1]. \tag{2.37}$$

Therefore,

$$a(t)f_0(t) + b(t)g_0(t) = h_0(t) - 1 \le 0, \quad \text{a.e. } t \in [0,1], \tag{2.38}$$

which contradicts the assumption $(a,b) \in \mathcal{W}_+^{\gamma}$. □

Remark 2.4. Let $a \succ 0$ and $b(t) = 0$, a.e. $t \in [0,1]$ in (1.1). Then the equation for the corresponding argument function is

$$\theta' = \lambda a(t)\left(C_p(\theta)\right)_+^p + (p-1)\left|S_p(\theta)\right|^{p^*} \tag{2.39}$$

(cf. (2.7)). The solution $\theta(t)$ takes a constant length of time, say $T_0 > 0$, to start from $\pi_p/2 + 2m\pi_p$ and to reach at $3\pi_p/2+2m\pi_p$ for any $m \in \mathbb{Z}$. Consequently, $\Theta(-\pi_p/2, \lambda a, \lambda\cdot 0)$ is bounded for $\lambda \in [0,\infty)$ and (2.19) does not hold in this case.

Notice that in the potential case [11], the associated argument function $\Theta(\vartheta, \lambda + a, \lambda + b)$ is strictly increasing in λ. However, as we are considering sign-changing weights (a,b), the monotonicity of $\Theta(\vartheta, \lambda a, \lambda b)$ in λ does not hold any more. We will develop some quasi-monotonicity of $\Theta(\vartheta, \lambda a, \lambda b)$ in λ by employing the Fréchet derivatives of Θ and the boundary conditions, see Lemmas 3.3, 4.1, and 5.1.

3. Periodic and Antiperiodic Spectrum

3.1. Structure of Periodic and Antiperiodic Half-Eigenvalues

Given $a, b \in \mathcal{L}^\gamma$, $1 \le \gamma \le \infty$, by (2.9), (2.10) and Theorem 2.2(i), one sees that

$$\underline{\Theta}(a,b) := \max_{\vartheta_0 \in [0,2\pi_p]} (\Theta(\vartheta_0, a, b) - \vartheta_0) = \max_{\vartheta_0 \in \mathbb{R}} (\Theta(\vartheta_0, a, b) - \vartheta_0), \tag{3.1}$$

$$\overline{\Theta}(a,b) := \min_{\vartheta_0 \in [0,2\pi_p]} (\Theta(\vartheta_0, a, b) - \vartheta_0) = \min_{\vartheta_0 \in \mathbb{R}} (\Theta(\vartheta_0, a, b) - \vartheta_0) \tag{3.2}$$

are well-defined. Moreover, we have the following lemma.

Lemma 3.1. *Given $\gamma \in [1, \infty]$, the functionals*

$$\begin{aligned} (\mathcal{L}^\gamma, w_\gamma)^2 &\longrightarrow \mathbb{R}, \qquad (a,b) \longmapsto \underline{\Theta}(a,b), \\ (\mathcal{L}^\gamma, w_\gamma)^2 &\longrightarrow \mathbb{R}, \qquad (a,b) \longmapsto \overline{\Theta}(a,b) \end{aligned} \tag{3.3}$$

are continuous.

Given $(a,b) \in \mathcal{W}_+^\gamma$, $1 \le \gamma \le \infty$, we write

$$\underline{\Theta}(\lambda) := \underline{\Theta}(\lambda a, \lambda b), \qquad \overline{\Theta}(\lambda) := \overline{\Theta}(\lambda a, \lambda b) \tag{3.4}$$

for simplicity if there is no ambiguity. By Lemma 3.1, $\underline{\Theta}(\lambda)$ and $\overline{\Theta}(\lambda)$ are continuous in $\lambda \in \mathbb{R}$. By Lemma 2.3, one has

$$\lim_{\lambda \to \infty} \underline{\Theta}(\lambda) = \lim_{\lambda \to \infty} \overline{\Theta}(\lambda) = \infty. \tag{3.5}$$

Moreover, by setting $a = b = 0$ in (2.7), we know that

$$\underline{\Theta}(0) \in (0, \pi_p), \qquad \overline{\Theta}(0) = 0. \tag{3.6}$$

Now we are considering the following two sequences of equations for λ

$$\underline{\Theta}(\lambda) = m\pi_p, \quad m \in \mathbb{N}, \tag{3.7}$$

$$\overline{\Theta}(\lambda) = m\pi_p, \quad m \in \mathbb{Z}^+ := \mathbb{N} \cup \{0\}. \tag{3.8}$$

Lemma 3.2. *Given $(a,b) \in \mathcal{W}_+^\gamma$, $1 \le \gamma \le \infty$.*

(i) *For any m specified,* (3.7) *and* (3.8) *always have solutions in $\mathbb{R}^+ := \{\lambda \in \mathbb{R} \mid \lambda \ge 0\}$.*

(ii) *All solutions of* (3.7) *and* (3.8) *are periodic half-eigenvalues or antiperiodic half-eigenvalues of* (1.1) *if m is even or odd, respectively, while the corresponding half-eigenfunctions have precisely m zeroes in the interval* $[0, 1)$.

Proof. (i) The solvability of (3.7) and (3.8) follows immediately from (3.5), (3.6) and the continuity of $\underline{\Theta}(\lambda)$ and $\overline{\Theta}(\lambda)$ in $\lambda \in \mathbb{R}^+$.

(ii) Suppose $\lambda \in \mathbb{R}^+$ satisfies (3.7) or (3.8). Then there exists $\widehat{\vartheta}_0 \in [0, 2\pi_p]$ such that

$$\Theta\left(\widehat{\vartheta}_0, \lambda a, \lambda b\right) - \widehat{\vartheta}_0 = m\pi_p, \tag{3.9}$$

$$\frac{d}{d\vartheta_0}\Theta(\vartheta_0, \lambda a, \lambda b)\Big|_{\vartheta_0 = \widehat{\vartheta}_0} = 1. \tag{3.10}$$

By (2.14), the latter equation is

$$R\left(\widehat{\vartheta}_0, \lambda a, \lambda b\right) = 1. \tag{3.11}$$

Geometrically, (3.9) and (3.11) imply that in p-polar coordinates (2.5), the solution of (1.1) starting at the point $(r, \theta) = (1, \widehat{\vartheta}_0)$ arrives at $(r, \theta) = (1, \widehat{\vartheta}_0 + m\pi_p)$ after one period. Thus λ is a periodic or an antiperiodic half-eigenvalue of (1.1) if m is even or odd, respectively.

Denote by x the half-eigenfunction corresponding to λ. Then $x(t) = 0$ if and only if

$$\theta(t) := \theta\left(t; \widehat{\vartheta}_0, \lambda a, \lambda b\right) = -\frac{\pi_p}{2} + k\pi_p, \quad k \in \mathbb{Z}. \tag{3.12}$$

Notice that $\theta(0) = \widehat{\vartheta}_0$, $\theta(1) = \widehat{\vartheta}_0 + m\pi_p$ (see (3.9)), and $\theta(t)$ is quasimonotone in t (see (2.11)). Consequently, if $\widehat{\vartheta}_0 = -\pi_p/2 + l_0\pi_p$ for some $l_0 \in \mathbb{Z}$, then (3.12) holds only for $k = l_0, l_0+1, l_0+2, \ldots, l_0+m$ at $(0 =)\ t_0 < t_1 < t_2 < \cdots < t_m\ (= 1)$. If $-\pi_p/2 + l_0\pi_p < \widehat{\vartheta}_0 < -\pi_p/2 + (l_0+1)\pi_p$, then (3.12) holds only for $k = l_0+1, l_0+2, \ldots, l_0+m$ at $(0 <)\ t_1 < t_2 < \cdots < t_m\ (< 1)$. In both cases x has precisely m zeroes in $[0, 1)$. □

The following lemma will be used for further study on solutions of (3.7) and (3.8).

Lemma 3.3. *Given $(a, b) \in \mathcal{W}_+^\gamma$, $1 \le \gamma \le \infty$.*

(i) *If $\underline{\Theta}(\mu) = m\pi_p$ for some $\mu > 0$ and $m \in \mathbb{N}$, then there exists $\delta > 0$ such that*

$$\underline{\Theta}(\lambda) > m\pi_p, \quad \forall \lambda \in (\mu, \mu + \delta). \tag{3.13}$$

(ii) *If $\overline{\Theta}(\mu) = m\pi_p$ for some $\mu > 0$ and $m \in \mathbb{Z}^+$, then there exists $\delta \in (0, \mu)$ such that*

$$\overline{\Theta}(\lambda) < m\pi_p, \quad \forall \lambda \in (\mu - \delta, \mu). \tag{3.14}$$

Proof. We only prove (i) and the proof of (ii) is analogous.

Assume that $\underline{\Theta}(\mu) = m\pi_p$. By Lemma 3.2, μ is a periodic or an antiperiodic half-eigenvalue of (1.1). Then there exists some $\vartheta_0 \in [0, 2\pi_p]$ such that

$$\Theta(\vartheta_0, \mu a, \mu b) - \vartheta_0 = m\pi_p, \tag{3.15}$$

and the corresponding half-eigenfunction $X(t) = X(t; \vartheta_0, \mu a, \mu b)$ satisfies

$$\left(\phi_p(X')\right)' + \mu a(t)\phi_p(X_+) - \mu b(t)\phi_p(X_-) = 0, \quad \text{a.e. } t \in [0,1], \tag{3.16}$$

and the boundary condition (1.2) or (1.3). Applying (2.15) and (2.16), we have

$$\frac{d}{d\lambda}\Theta(\vartheta_0, \lambda a, \lambda b)\bigg|_{\lambda=\mu} = \int_0^1 \left(aX_+^p + bX_-^p\right) dt, \tag{3.17}$$

Multiplying (3.16) by X and integrating over $[0,1]$, one has

$$\begin{aligned}\int_0^1 \left(aX_+^p + bX_-^p\right) dt &= \frac{1}{\mu}\left(\int_0^1 |X'|^p dt - X(t)\phi_p(X'(t))\Big|_{t=0}^1\right) \\ &= \frac{1}{\mu}\int_0^1 |X'|^p dt \\ &> 0,\end{aligned} \tag{3.18}$$

and, therefore,

$$\frac{d}{d\lambda}\Theta(\vartheta_0, \lambda a, \lambda b)\bigg|_{\lambda=\mu} > 0. \tag{3.19}$$

Then there exists $\delta > 0$ such that

$$\Theta(\vartheta_0, \lambda a, \lambda b) - \vartheta_0 > m\pi_p \quad \forall \lambda \in (\mu, \mu + \delta). \tag{3.20}$$

Consequently, we have

$$\underline{\Theta}(\lambda) \geq \Theta(\vartheta_0, \lambda a, \lambda b) - \vartheta_0 > m\pi_p, \quad \forall \lambda \in (\mu, \mu + \delta). \tag{3.21}$$

□

Definition 3.4. Given $(a,b) \in \mathcal{W}_+^\gamma$, $1 \leq \gamma \leq \infty$, define

$$\underline{\lambda}_m^L := \min\{\lambda > 0 \mid \underline{\Theta}(\lambda) = m\pi_p\}, \quad m \in \mathbb{N}, \tag{3.22}$$

$$\underline{\lambda}_m^R := \max\{\lambda > 0 \mid \underline{\Theta}(\lambda) = m\pi_p\}, \quad m \in \mathbb{N}, \tag{3.23}$$

$$\overline{\lambda}_m^L := \min\left\{\lambda \geq 0 \mid \overline{\Theta}(\lambda) = m\pi_p\right\}, \quad m \in \mathbb{Z}^+, \tag{3.24}$$

$$\overline{\lambda}_m^R := \max\left\{\lambda \geq 0 \mid \overline{\Theta}(\lambda) = m\pi_p\right\}, \quad m \in \mathbb{Z}^+. \tag{3.25}$$

These values $\underline{\lambda}_m^{L/R}$ and $\overline{\lambda}_m^{L/R}$ are well defined. By Lemma 3.2, they are periodic or antiperiodic half-eigenvalues. These are what we are interested in. By Lemma 3.3, we have the following results.

Corollary 3.5. *Given* $(a,b) \in \mathcal{W}_+^\gamma$, $1 \le \gamma \le \infty$, *one has*

$$\underline{\Theta}(\lambda)\begin{cases} < m\pi_p, & \text{if } 0 \le \lambda < \underline{\lambda}_m^L \\ \in [m\pi_p, m\pi_p + \pi_p), & \text{if } \underline{\lambda}_m^L \le \lambda \le \underline{\lambda}_m^R \\ > m\pi_p, & \text{if } \lambda > \underline{\lambda}_m^R \end{cases} \quad \forall m \in \mathbb{N},$$
$$\overline{\Theta}(\lambda)\begin{cases} < m\pi_p, & \text{if } 0 \le \lambda < \overline{\lambda}_m^L \\ \in (m\pi_p - \pi_p, m\pi_p], & \text{if } \overline{\lambda}_m^L \le \lambda \le \overline{\lambda}_m^R \\ > m\pi_p, & \text{if } \lambda > \overline{\lambda}_m^R. \end{cases} \quad \forall m \in \mathbb{Z}^+. \tag{3.26}$$

Combining (3.5), (3.6), and (3.26), one has the following ordering for these half-eigenvalues:

$$(0 <)\, \underline{\lambda}_1^L \le \underline{\lambda}_1^R < \underline{\lambda}_2^L \le \underline{\lambda}_2^R < \cdots < \underline{\lambda}_m^L \le \underline{\lambda}_m^R < \cdots (\longrightarrow \infty),$$
$$(0 =)\, \overline{\lambda}_0^L \le \overline{\lambda}_0^R < \overline{\lambda}_1^L \le \overline{\lambda}_1^R < \cdots < \overline{\lambda}_m^L \le \overline{\lambda}_m^R < \cdots (\longrightarrow \infty). \tag{3.27}$$

Moreover, by the definition of $\underline{\lambda}_m^{L/R}$ and $\overline{\lambda}_m^{L/R}$, one has

$$\underline{\lambda}_m^L \le \overline{\lambda}_m^R, \quad \underline{\lambda}_m^R, \overline{\lambda}_m^L \in \left[\underline{\lambda}_m^L, \overline{\lambda}_m^R\right], \quad \forall m \in \mathbb{N}. \tag{3.28}$$

Notice that the ordering between $\underline{\lambda}_m^R$ and $\overline{\lambda}_m^L$ is not determinate.

Till now, we still do not know the ordering between $\overline{\lambda}_m^R$ and $\underline{\lambda}_n^L$ with $n > m$. This will be partially settled by associating (1.1) with the rotation number function defined by

$$\rho(\lambda) = \rho(\lambda, a, b) := \lim_{n \to \infty} \frac{\Theta^n(\vartheta_0, \lambda a, \lambda b) - \vartheta_0}{2n\pi_p}, \tag{3.29}$$

where Θ^n is the nth iteration of Θ, namely,

$$\Theta^n(\vartheta_0, a, b) = \Theta\Big(\Theta^{n-1}(\vartheta_0, a, b), a, b\Big). \tag{3.30}$$

It can be proved that the rotation number function $\rho(\lambda)$ in (3.29) is well defined and is independent of ϑ_0 (cf. [16, Theorem 2.1]).

Lemma 3.6. *Given* $(a,b) \in \mathcal{W}_+^{\gamma}$, $1 \leq \gamma \leq \infty$, *one has*

(i) $\rho(\lambda) \geq 0$ *and* $\rho(0) = 0$;

(ii) $\lim_{\lambda\to\infty}\rho(\lambda) = \infty$;

(iii) $\lambda \in [0, \underline{\lambda}_m^L)$, $m \in \mathbb{N} \Rightarrow \rho(\lambda) < m/2$;

(iv) $\lambda \in (\overline{\lambda}_m^R, \infty)$, $m \in \mathbb{Z}^+$, $\Rightarrow \rho(\lambda) > m/2$;

(v) $\lambda \in [\underline{\lambda}_{2m}^L, \overline{\lambda}_{2m}^R]$, $m \in \mathbb{N} \Rightarrow \rho(\lambda) = m$;

(vi) $\lambda = \overline{\lambda}_0^R \Rightarrow \rho(\lambda) = 0$.

Proof. (i) By (2.11), one has $\Theta^n(0,\lambda a,\lambda b) > -\pi_p/2$ for any $n \in \mathbb{N}$. Thus $\rho(\lambda) \geq 0$ since $\rho(\lambda)$ defined by (3.29) is independent of ϑ_0. If $\lambda = 0$, the equation for the argument function associated with (1.1) becomes $\theta' = (p-1)|S_p(\theta)|^{p^*}$, which has equilibria $\theta = k\pi_p$, $k \in \mathbb{Z}$. Thus $\Theta^n(-\pi_p/2, 0\cdot a, 0\cdot b) \in (-\pi_p/2, 0)$ for all $n \in \mathbb{N}$ and consequently $\rho(0) = 0$.

(ii) Given $M > 0$ arbitrarily large, it follows from (3.5) that there exists $\Lambda > 0$ such that

$$\Theta(\vartheta_0,\lambda a,\lambda b) - \vartheta_0 > 2M\pi_p, \quad \forall\lambda > \Lambda,\ \forall\vartheta_0 \in \mathbb{R}. \tag{3.31}$$

Thus

$$\begin{aligned}\Theta^n(\vartheta_0,\lambda a,\lambda b) - \vartheta_0 &= \left[\Theta\left(\Theta^{n-1}(\vartheta_0,\lambda a,\lambda b),\lambda a,\lambda b\right) - \Theta\left(\Theta^{n-2}(\vartheta_0,\lambda a,\lambda b),\lambda a,\lambda b\right)\right]\\ &\quad + \left[\Theta\left(\Theta^{n-2}(\vartheta_0,\lambda a,\lambda b),\lambda a,\lambda b\right) - \Theta\left(\Theta^{n-3}(\vartheta_0,\lambda a,\lambda b),\lambda a,\lambda b\right)\right]\\ &\quad + \cdots + [\Theta(\vartheta_0,\lambda a,\lambda b) - \vartheta_0]\\ &\geq 2nM\pi_p\end{aligned} \tag{3.32}$$

for any $\lambda > \Lambda$ and any $\vartheta_0 \in \mathbb{R}$. Consequently, $\rho(\lambda) \geq M$ for any $\lambda > \Lambda$.

(iii) Given $\lambda \in [0, \underline{\lambda}_m^L)$, $m \in \mathbb{N}$, it follows from (3.26) that there exists $\delta = \delta(\lambda) > 0$ such that

$$\underline{\Theta}(\lambda) < (m-\delta)\pi_p. \tag{3.33}$$

By the definition of $\underline{\Theta}(\lambda) = \underline{\Theta}(\lambda a,\lambda b)$ (see (3.1)), this implies that

$$\Theta(\vartheta_0,\lambda a,\lambda b) - \vartheta_0 < (m-\delta)\pi_p, \quad \forall\vartheta_0 \in \mathbb{R}. \tag{3.34}$$

Now similar arguments as in (ii) show that $\rho(\lambda) \leq (m-\delta)/2 < m/2$.

(iv) The proof is analogous to that of (iii).

(v) Given $\lambda \in [\underline{\lambda}^L_{2m}, \overline{\lambda}^R_{2m}]$, $m \in \mathbb{N}$, it follows from (3.26) that

$$\overline{\Theta}(\lambda) \le 2m\pi_p \le \underline{\Theta}(\lambda). \tag{3.35}$$

Then there exists ϑ_0 such that $\Theta(\vartheta_0, \lambda a, \lambda b) - \vartheta_0 = 2m\pi_p$, and therefore $\rho(\lambda) = m$.

(vi) By (3.25), if $\lambda = \overline{\lambda}^R_0$, then there exists ϑ_0 such that $\Theta(\vartheta_0, \lambda a, \lambda b) - \vartheta_0 = 0$. Hence $\rho(\lambda) = 0$. □

Corollary 3.7. *Given* $(a,b) \in \mathcal{W}^\gamma_+$, $1 \le \gamma \le \infty$, *one has*

$$\underline{\lambda}^L_{2m} = \min\{\lambda > 0 \mid \rho(\lambda) = m\}, \quad \forall m \in \mathbb{N}, \tag{3.36}$$

$$\overline{\lambda}^R_{2m} = \max\{\lambda \ge 0 \mid \rho(\lambda) = m\}, \quad \forall m \in \mathbb{Z}^+, \tag{3.37}$$

and the ordering

$$(0 \le)\, \overline{\lambda}^R_0 < \underline{\lambda}^L_2 \le \overline{\lambda}^R_2 < \cdots < \underline{\lambda}^L_{2m} \le \overline{\lambda}^R_{2m} < \underline{\lambda}^L_{2m+2} \le \overline{\lambda}^R_{2m+2} \cdots (\longrightarrow \infty). \tag{3.38}$$

Proof. The characterization (3.36) follows from Lemma 3.6(iii) and (v), while (3.37) follows from Lemma 3.6(iv), (v), and (vi).

To prove (3.38), by (3.27) and (3.28), we need only prove $\overline{\lambda}^R_{2m} < \underline{\lambda}^L_{2m+2}$ for any $m \in \mathbb{N}$. Assume on the contrary that $\overline{\lambda}^R_{2m} \ge \underline{\lambda}^L_{2m+2}$ for some m. Then

$$\underline{\lambda}^L_{2m} < \underline{\lambda}^L_{2m+2} \le \overline{\lambda}^R_{2m}. \tag{3.39}$$

Now Lemma 3.6(v) shows that $m = \rho(\underline{\lambda}^L_{2m+2}) = m + 1$, a contradiction. □

Corollary 3.8. *Given* $(a,b) \in \mathcal{W}^\gamma_+$, $1 \le \gamma \le \infty$, *one has*

$$\lambda \in \left(\overline{\lambda}^R_{2m}, \underline{\lambda}^L_{2m+2}\right), \quad m \in \mathbb{N} \Longrightarrow \rho(\lambda) \in (m, m+1). \tag{3.40}$$

In summary, we can now partially characterize the set of periodic half-eigenvalues Σ_P and the set of antiperiodic half-eigenvalues Σ_A of (1.1). Denote $\Sigma^+_P := \Sigma_P \cap \mathbb{R}^+$ and $\Sigma^+_A := \Sigma_A \cap \mathbb{R}^+$. For any $m \in \mathbb{Z}^+$, denote by $\Sigma^+_m = \Sigma^+_m(a,b)$ the set of nonnegative periodic or antiperiodic half-eigenvalues for which the corresponding half-eigenfunctions have precisely m zeroes in the interval $[0,1)$. Then

$$\Sigma^+_P = \bigcup_{m=0}^{\infty} \Sigma^+_{2m}, \qquad \Sigma^+_A = \bigcup_{m=0}^{\infty} \Sigma^+_{2m+1}. \tag{3.41}$$

Theorem 3.9. *Given* $(a,b) \in \mathcal{W}_+^{\gamma}$, $1 \le \gamma \le \infty$.

(i) *For periodic half-eigenvalues, one has*

$$\begin{gathered} \left\{0, \overline{\lambda}_0^R\right\} \subset \Sigma_0^+ \subset \left[0, \overline{\lambda}_0^R\right], \\ \left\{\underline{\lambda}_{2m}^L, \underline{\lambda}_{2m}^R, \overline{\lambda}_{2m}^L, \overline{\lambda}_{2m}^R\right\} \subset \Sigma_{2m}^+ \subset \left[\underline{\lambda}_{2m}^L, \overline{\lambda}_{2m}^R\right], \quad \forall m \in \mathbb{N}. \end{gathered} \tag{3.42}$$

Moreover, $\underline{\lambda}_{2m}^{L/R}$ *and* $\overline{\lambda}_{2m}^{L/R}$ *satisfy* (3.27), (3.28), *and* (3.38).

(ii) *For antiperiodic half-eigenvalues, one has*

$$\left\{\underline{\lambda}_{2m+1}^L, \underline{\lambda}_{2m+1}^R, \overline{\lambda}_{2m+1}^L, \overline{\lambda}_{2m+1}^R\right\} \subset \Sigma_{2m+1}^+, \quad \forall m \in \mathbb{Z}^+. \tag{3.43}$$

Moreover, $\underline{\lambda}_{2m+1}^{L/R}$ *and* $\overline{\lambda}_{2m+1}^{L/R}$ *satisfy* (3.27) *and* (3.28).

Proof. By Lemma 3.2 and the definitions of $\underline{\lambda}_m^{L/R}$ and $\overline{\lambda}_m^{L/R}$ (see (3.22)–(3.25)), we need only prove

$$\Sigma_0^+ \subset \left[0, \overline{\lambda}_0^R\right], \qquad \Sigma_{2m}^+ \subset \left[\underline{\lambda}_{2m}^L, \overline{\lambda}_{2m}^R\right], \quad \forall m \in \mathbb{N}. \tag{3.44}$$

If $\lambda \in \Sigma_{2m}^+$, $m \in \mathbb{Z}^+$, then λ is a periodic half-eigenvalue of (1.1) and the corresponding half-eigenfunction has precisely $2m$ zeroes $[0,1)$. Consequently, there exists $\vartheta_0 \in \mathbb{R}$ such that

$$\theta(\vartheta_0, \lambda a, \lambda b) = \vartheta_0 + 2m\pi_p. \tag{3.45}$$

Hence $\rho(\lambda) = m$. Combining (3.36), and the fact $\rho(0) = 0$ we obtain (3.44). □

The following theorem gives the necessary and sufficient condition for $\overline{\lambda}_0^R > 0$.

Theorem 3.10. *Given* $(a,b) \in \mathcal{W}_+^{\gamma}$, $1 \le \gamma \le \infty$, *then*

$$\overline{\lambda}_0^R > 0 \Longleftrightarrow \int_0^1 a(t)dt < 0 \quad \text{or} \quad \int_0^1 b(t)dt < 0. \tag{3.46}$$

Proof. Assume that $\overline{\lambda}_0^R > 0$. By Lemma 3.2, the half-eigenfunction $X = X(t)$ corresponding to $\overline{\lambda}_0^R$ is nowhere vanishing and satisfies the periodic boundary condition (1.2). If $X(t) > 0$ for any $t \in [0,1]$, then

$$(\phi_p(X'))' + \overline{\lambda}_0^R a(t)\phi_p(X) = 0, \quad X \not\equiv \text{const}. \tag{3.47}$$

Therefore,

$$\begin{aligned}\int_0^1 a(t)dt &= -\frac{1}{\overline{\lambda}_0^R}\int_0^1 \frac{(\phi_p(X'))'}{\phi_p(X)}dt \\ &= -\frac{1}{\overline{\lambda}_0^R}\int_0^1 \phi_p(X')d\left(\frac{1}{\phi_p(X)}\right) + \frac{\phi_p(X'(t))}{\overline{\lambda}_0^R\phi_p(X(t))}\Bigg|_{t=0}^1 \\ &= -\frac{(p-1)}{\overline{\lambda}_0^R}\int_0^1 \frac{|X'|^p}{X^p}dt \\ &< 0.\end{aligned} \tag{3.48}$$

Similarly, if $X(t) < 0$ for any $t \in [0,1]$, then $\int_0^1 b(t)dt < 0$.

On the other hand, assume that $\int_0^1 a(t)dt < 0$. Since $x(t) \equiv 1$ is the periodic half-eigenfunction corresponding to $\lambda = 0$, by (2.4), we have $y(t) \equiv 0$. Hence in the p-polar coordinates (2.5) $\vartheta_0 = 0$ and the argument solution $\theta(t;\vartheta_0, 0\cdot a, 0\cdot b) \equiv 0$. Then it follows from (2.15) and (2.16) that

$$\frac{d}{d\lambda}\Theta(0,\lambda a,\lambda b)\bigg|_{\lambda=0} = \int_0^1 a(t)dt < 0. \tag{3.49}$$

Thus there exists $\delta > 0$ such that

$$\Theta(0,\lambda a,\lambda b) < 0, \quad \forall \lambda \in (0,\delta). \tag{3.50}$$

By the definition of $\overline{\Theta}(\lambda) = \overline{\Theta}(\lambda a, \lambda b)$ (see (3.2)), we get

$$\overline{\Theta}(\lambda) < 0, \quad \forall \lambda \in (0,\delta). \tag{3.51}$$

By (3.5), (3.6) and the continuity of $\overline{\Theta}(\lambda)$ in $\lambda \in \mathbb{R}^+$ (see Lemma 3.1), one has $\overline{\lambda}_0^R > 0$. Similarly one can obtain $\overline{\lambda}_0^R > 0$ if $\int_0^1 b(t)dt < 0$. □

Notice that for the antiperiodic half-eigenvalues, it does not hold

$$\Sigma_{2m+1} \subset \left[\underline{\lambda}_{2m+1}^L, \overline{\lambda}_{2m+1}^R\right], \quad \forall m \in \mathbb{Z}^+, \tag{3.52}$$

for general $a, b \in \mathcal{L}^\gamma$. This is because $A(t,\theta;a,b)$ in (2.7) is $2\pi_p$-periodic in θ but not π_p-periodic in θ. Hence (2.10) holds but one can not obtain

$$\theta(t;\vartheta_0 + m\pi_p, a, b) \equiv \theta(t;\vartheta_0, a, b) + m\pi_p, \quad \forall t \in [0,1],\ \vartheta_0 \in \mathbb{R},\ m \in \mathbb{Z}. \tag{3.53}$$

However, if $a = b$, then $A(t,\theta;a,a)$ is π_p-periodic in θ and

$$\theta(t;\vartheta_0 + m\pi_p, a, a) \equiv \theta(t;\vartheta_0, a, a) + m\pi_p, \quad \forall t \in [0,1],\ \vartheta_0 \in \mathbb{R},\ m \in \mathbb{Z}. \tag{3.54}$$

The following theorem characterizes periodic and antiperiodic eigenvalues of (1.6). Now $\Sigma_m^+ = \Sigma_m^+(a,a)$, $m \in \mathbb{Z}^+$, is the set of all those nonnegative periodic or antiperiodic eigenvalues of (1.6) for which the corresponding eigenfunctions have precisely m zeroes in $[0,1)$.

Theorem 3.11. *Given $\gamma \in [1,\infty]$ and $a = b \in \mathcal{L}^\gamma$ with $a_+ \succ 0$.*

(i) *All solutions of* (3.7) *and* (3.8) *are periodic or antiperiodic eigenvalues of* (1.6) *if m is even or odd, respectively.*

(ii) *One has*

$$\begin{gathered}\left\{0, \overline{\lambda}_0^R\right\} \subset \Sigma_0^+ \subset \left[0, \overline{\lambda}_0^R\right], \\ \left\{\underline{\lambda}_m^L, \underline{\lambda}_m^R, \overline{\lambda}_m^L, \overline{\lambda}_m^R\right\} \subset \Sigma_m^+ \subset \left[\underline{\lambda}_m^L, \overline{\lambda}_m^R\right], \quad \forall m \in \mathbb{N}.\end{gathered} \tag{3.55}$$

(iii) *One has the ordering*

$$(0 \le)\, \overline{\lambda}_0^R < \underline{\lambda}_1^L \le \overline{\lambda}_1^R < \underline{\lambda}_2^L \le \overline{\lambda}_2^R < \cdots < \underline{\lambda}_m^L \le \overline{\lambda}_m^R < \cdots (\longrightarrow \infty). \tag{3.56}$$

(iv) *Finally* $\overline{\lambda}_0^R > 0 \Leftrightarrow \int_0^1 a(t)dt < 0$.

Proof. By Theorems 3.9 and 3.10, we need only prove (3.52) and

$$\overline{\lambda}_{2m}^R < \underline{\lambda}_{2m+1}^L, \quad \overline{\lambda}_{2m+1}^R < \underline{\lambda}_{2m+2}^L, \quad \forall m \in \mathbb{Z}^+. \tag{3.57}$$

If $\lambda \in \Sigma_m^+$, $m \in \mathbb{N}$, then there exists $\vartheta_0 \in \mathbb{R}$ such that $\Theta(\vartheta_0, \lambda a, \lambda a) - \vartheta_0 = m\pi_p$. By (3.54) and the definition of $\rho(\cdot)$ in (3.29), one has

$$\lambda \in \Sigma_m^+ \Longrightarrow \rho(\lambda) = \frac{m}{2}, \tag{3.58}$$

in particular

$$\rho\left(\underline{\lambda}_{2m+1}^L\right) = \rho\left(\overline{\lambda}_{2m+1}^R\right) = \frac{2m+1}{2}, \quad \forall m \in \mathbb{Z}^+. \tag{3.59}$$

Then it follows from Lemma 3.6(iii) and (iv) that

$$\underline{\lambda}^L_{2m+1} = \min\left\{\lambda \geq 0 \mid \rho(\lambda) = \frac{m+1}{2}\right\}, \quad m \in \mathbb{Z}^+,$$
$$\overline{\lambda}^R_{2m+1} = \max\left\{\lambda \geq 0 \mid \rho(\lambda) = \frac{m+1}{2}\right\}, \quad m \in \mathbb{Z}^+. \tag{3.60}$$

Hence (3.52) follows immediately from (3.58) and (3.60).

Suppose $\overline{\lambda}^R_{2m} \geq \underline{\lambda}^L_{2m+1}$ for some $m \in \mathbb{Z}^+$. By Lemma 3.6(iii) and (v) we obtain $\rho(\underline{\lambda}^L_{2m+1}) \leq m$, which contradicts (3.59). Thus $\overline{\lambda}^R_{2m} > \underline{\lambda}^L_{2m+1}$ for any $m \in \mathbb{Z}^+$. Similarly one can prove $\overline{\lambda}^R_{2m+1} < \underline{\lambda}^L_{2m+2}$, $\forall m \in \mathbb{Z}^+$, proving the desired results in (3.57). □

3.2. Continuity of Periodic and Antiperiodic Half-Eigenvalues in Weak Topology

Theorem 3.12. *Given $\gamma \in [1, \infty]$, for any admissible m, the following functionals*

$$\left(\mathcal{W}^\gamma_+, w_\gamma \times w_\gamma\right) \longrightarrow \mathbb{R}, \qquad (a,b) \longmapsto \underline{\lambda}^L_m(a,b)$$
$$\left(\mathcal{W}^\gamma_+, w_\gamma \times w_\gamma\right) \longrightarrow \mathbb{R}, \qquad (a,b) \longmapsto \overline{\lambda}^R_m(a,b) \tag{3.61}$$

are continuous. Here w_γ denotes the weak topology in $\mathcal{L}^\gamma$.

Proof. We only prove the result for $\underline{\lambda}^L_m(a,b)$. The result for $\overline{\lambda}^R_m(a,b)$ can be attained analogously. Suppose $(a_n, b_n) \to (a,b)$ in $(\mathcal{W}^\gamma_+, w_\gamma \times w_\gamma)$ as $n \to \infty$. Then $a_n \to a$ and $b_n \to b$ in $(\mathcal{L}^\gamma, w_\gamma)$. Denote $\nu_n := \underline{\lambda}^L_m(a_n, b_n)$. We aim to prove $\nu_n \to \underline{\lambda}^L_m(a,b)$.

Firstly, we show that the sequence $\{\nu_n\}_{n\in\mathbb{N}}$ is bounded. If this is false, without loss of generality, we may assume that $\nu_n \to \infty$ as $n \to \infty$. Then given any $\Lambda > 0$, there exists $N \in \mathbb{N}$ such that

$$\nu_n > \Lambda, \quad \forall n > N. \tag{3.62}$$

It follows from (3.26) that

$$\underline{\Theta}(\Lambda a_n, \Lambda b_n) < m\pi_p, \quad \forall n > N. \tag{3.63}$$

Let $n \to \infty$, we obtain by Lemma 3.1

$$\underline{\Theta}(\Lambda a, \Lambda b) \leq m\pi_p, \tag{3.64}$$

which contradicts (3.5) since $\Lambda > 0$ can be chosen arbitrarily large. Thus $\{\nu_n\}_{n\in\mathbb{N}}$ is bounded.

Since $\{\nu_n\}_{n\in\mathbb{N}}$ is bounded, passing to a subsequence, we assume that $\nu_n \to \nu_0$ as $n \to \infty$. By the definition of ν_n, we have

$$\underline{\Theta}(\nu_n a_n, \nu_n b_n) = m\pi_p. \tag{3.65}$$

Let $n \to \infty$, by Lemma 3.1 again, we get

$$\underline{\Theta}(\nu_0 a, \nu_0 b) = m\pi_p. \tag{3.66}$$

Now it suffices to prove $\nu_0 = \underline{\lambda}_m^L(a,b)$.

If this is not true, by the definition in (3.22) there holds $\nu_0 > \underline{\lambda}_m^L(a,b)$. Then it follows from Lemma 3.3(i) that there exists

$$\lambda \in \left(\underline{\lambda}_m^L(a,b), \nu_0\right) \tag{3.67}$$

such that

$$\underline{\Theta}(\lambda a, \lambda b) > m\pi_p. \tag{3.68}$$

Since $\underline{\Theta}(\lambda a_n, \lambda b_n) \to \underline{\Theta}(\lambda a, \lambda b)$ as $n \to \infty$, we may assume, without loss of generality, that

$$\underline{\Theta}(\lambda a_n, \lambda b_n) > m\pi_p, \quad \forall n \in \mathbb{N}. \tag{3.69}$$

Therefore, by (3.26), we have

$$\lambda > \underline{\lambda}_m^L(a_n, b_n) = \nu_n, \quad \forall n \in \mathbb{N}. \tag{3.70}$$

Let $n \to \infty$, one has $\lambda \geq \nu_0$, which contradicts the choice of λ in (3.67). □

4. The Dirichlet Spectrum

4.1. Structure of Dirichlet Half-Eigenvalues

Given $(a,b) \in \mathcal{W}_+^\gamma$, $1 \leq \gamma \leq \infty$, in this section we use the following notations for simplicity if there is no ambiguity

$$\begin{gathered}\Theta(\lambda) = \Theta(\lambda a, \lambda b) = \Theta\left(-\frac{\pi_p}{2}, \lambda a, \lambda b\right), \\ X(t) = X(t; a, b) = X\left(t; -\frac{\pi_p}{2}, a, b\right).\end{gathered} \tag{4.1}$$

For $a = b = 0$, (2.7) is $\theta' = (p-1)|S_p(\theta)|^{p^*}$, which has equilibria $\theta = k\pi_p$, $k \in \mathbb{Z}$. Hence, we have

$$\Theta(0) \in \left(-\frac{\pi_p}{2}, 0\right). \tag{4.2}$$

Therefore, $\lambda = 0$ is not a Dirichlet half-eigenvalue of (1.1), namely, $0 \notin \Sigma_D$. Due to the jumping nonlinearity of (1.1), $\lambda \in \Sigma_D^+ := \Sigma_D \cap \mathbb{R}^+$ if and only if $\lambda > 0$ is determined by one of the following two equations

$$\Theta\left(-\frac{\pi_p}{2}, \lambda a, \lambda b\right) = -\frac{\pi_p}{2} + m\pi_p, \tag{4.3}$$

$$\Theta\left(\frac{\pi_p}{2}, \lambda a, \lambda b\right) = \frac{\pi_p}{2} + m\pi_p, \tag{4.4}$$

for some $m \in \mathbb{Z}$. The range of m will be proved to be $\mathbb{N}$.

The following lemma shows that $\Theta(\lambda)$ is quasimonotone in $\lambda \in \mathbb{R}^+$.

Lemma 4.1. *Given $(a,b) \in \mathcal{W}_+^\gamma$, $1 \le \gamma \le \infty$, if there exist $\mu > 0$ and $m \in \mathbb{Z}$ such that*

$$\Theta(\mu) = -\frac{\pi_p}{2} + m\pi_p, \tag{4.5}$$

then

$$\left.\frac{d}{d\lambda}\Theta(\lambda)\right|_{\lambda=\mu} > 0, \tag{4.6}$$

$$\Theta(\lambda) > (<) -\frac{\pi_p}{2} + m\pi_p \quad \forall \lambda > (<)\mu. \tag{4.7}$$

Proof. If $\mu > 0$ satisfies (4.5), then μ is a Dirichlet half-eigenvalue of (1.1) and the corresponding half-function $X(t) = X(t; \lambda a, \lambda b)$ satisfies the boundary condition (1.4). By (2.15) and (2.16),

$$\begin{aligned}
\left.\frac{d}{d\lambda}\Theta(\lambda)\right|_{\lambda=u} &= \int_0^1 \left(aX_+^p + bX_-^p\right) dt \\
&= \frac{1}{\mu}\left(\int_0^1 |X'(t)|^p dt - X(t)\phi_p(X'(t))\Big|_{t=0}^1\right) \\
&= \frac{1}{\mu}\int_0^1 |X'(t)|^p dt \\
&> 0,
\end{aligned} \tag{4.8}$$

proving (4.6), and, therefore, (4.7) holds. □

It follows from (4.2), Lemmas 2.3 and 4.1 that (4.3) has a positive solution if and only if $m \in \mathbb{N}$. For each $m \in \mathbb{N}$, (4.3) has the unique positive solution denoted by $\lambda_m^+(a,b)$. Moreover, one has

$$(0 <)\ \lambda_1^+(a,b) < \lambda_2^+(a,b) < \cdots < \lambda_m^+(a,b) \cdots (\longrightarrow \infty). \tag{4.9}$$

One can prove similarly that (4.4) has a positive solution if and only if $m \in \mathbb{N}$, and for each $m \in \mathbb{N}$, (4.4) has the unique positive solution denoted by $\lambda_m^-(a,b)$. Moreover, one has

$$(0 <)\ \lambda_1^-(a,b) < \lambda_2^-(a,b) < \cdots < \lambda_m^-(a,b) \cdots (\longrightarrow \infty). \tag{4.10}$$

Let $z = -x$ in (1.1). Then $z_+ = x_-$, $z_- = x_+$, and

$$\left(\phi_p(z')\right)' + \lambda b(t)\phi_p(z_+) - \lambda a(t)\phi_p(z_-) = 0. \tag{4.11}$$

Consequently, we have

$$\lambda_m^-(a,b) = \lambda_m^+(b,a) \quad \forall m \in \mathbb{N}. \tag{4.12}$$

In summary, we can completely characterize the structure of Σ_D^+.

Theorem 4.2. *Given $(a,b) \in \mathcal{W}_+^\gamma$, $1 \le \gamma \le \infty$, then Σ_D^+ consists of two sequences* (4.9) *and* (4.10), *which are related to each other by* (4.12). *Moreover, the half-eigenfunctions corresponding to $\lambda_m^\pm(a,b)$ have precisely $m-1$ zeroes in the interval $(0,1)$.*

Proof. We need only prove the nodal property, while this can be obtained by (2.11) as done in Lemma 3.2. □

4.2. Dependence of Dirichlet Half-Eigenvalues on Weights

By (4.12), we only discuss the dependence of $\lambda_m^+(a,b)$ on weights (a,b). Analogous results hold for $\lambda_m^-(a,b)$.

Theorem 4.3. *Given $\gamma \in [1,\infty]$ and $m \in \mathbb{N}$, the functional*

$$\left(\mathcal{W}_+^\gamma, w_\gamma \times w_\gamma\right) \longrightarrow \mathbb{R}, \qquad (a,b) \longmapsto \lambda_m^+(a,b), \tag{4.13}$$

is continuous. Here w_γ denotes the weak topology in $\mathcal{L}^\gamma$.

Proof. Suppose $(a_n,b_n) \to (a,b)$ in $(\mathcal{W}_+^\gamma, w_\gamma \times w_\gamma)$. Denote $\nu_n := \lambda_m^+(a_n,b_n)$. We aim to prove $\nu_n \to \lambda_m^+(a,b)$ as $n \to \infty$.

We claim that the sequence $\{\nu_n\}_{n\in\mathbb{N}}$ is bounded. If this is false, we may assume that $\nu_n \to \infty$. Then for any $\Lambda > 0$, there exists $N \in \mathbb{N}$ such that $\nu_n > \Lambda$ for any $n > N$. By the definition of ν_n and Lemma 4.1, we have

$$\Theta(\Lambda a_n, \Lambda b_n) < -\frac{\pi_p}{2} + m\pi_p, \quad \forall n \in \mathbb{N}. \tag{4.14}$$

Let $n \to \infty$, it follows from Theorem 2.2(i) that

$$\Theta(\Lambda a, \Lambda b) \le -\frac{\pi_p}{2} + m\pi_p, \tag{4.15}$$

which contradicts with Lemma 2.3, because Λ can be chosen arbitrarily large. Thus, the claim is true.

Since $\{\nu_n\}_{n\in\mathbb{N}}$ is bounded and $\nu_n > 0$, without loss of generality, we may assume that $\nu_n \to \nu_0 \ge 0$. By the definition of ν_n, one has

$$\Theta(\nu_n a_n, \nu_n b_n) = -\frac{\pi_p}{2} + m\pi_p, \quad \forall n \in \mathbb{N}. \tag{4.16}$$

Let $n \to \infty$, by Theorem 2.2(i) again, one has

$$\Theta(\nu_0 a, \nu_0 b) = -\frac{\pi_p}{2} + m\pi_p. \tag{4.17}$$

Therefore, $\nu_0 = \lambda_m^+(a,b)(> 0)$, proving the desired result. □

Given $(a,b) \in \mathcal{W}^\gamma$, $1 \le \gamma \le \infty$, and $m \in \mathbb{N}$, denote

$$E_m(t) = E_m(t;a,b) := \frac{X(t)}{\left\{\int_0^1 \left(aX_+^p + bX_-^p\right)dt\right\}^{1/p}}, \tag{4.18}$$

where $X(t) = X(t;\lambda_m^+ a, \lambda_m^+ b)$ (see (2.17)) is the half-eigenfunction of (1.1) associated with the half-eigenvalue $\lambda_m^+ = \lambda_m^+(a,b)$. Then E_m is the half-eigenfunction associated with λ_m^+ and satisfying the normalized condition

$$\int_0^1 \left(a(E_m)_+^p + b(E_m)_-^p\right)dt = 1. \tag{4.19}$$

Theorem 4.4. *Given $\gamma \in [1,\infty]$ and $m \in \mathbb{N}$, the functional*

$$\left(\mathcal{W}_+^\gamma, \|\cdot\|_\gamma \times \|\cdot\|_\gamma\right) \longrightarrow \mathbb{R}, \qquad (a,b) \longmapsto \lambda_m^+ = \lambda_m^+(a,b) \tag{4.20}$$

is continuously differentiable. Moreover, the Fréchet derivatives of λ_m^+ at a and at b, denoted by $\partial_a\lambda_m^+$ and $\partial_b\lambda_m^+$, respectively, are given by the following functionals:

$$\partial_a\lambda_m^+ = -\lambda_m^+(E_m)_+^p \in \mathcal{C}^0 \subset \left(\mathcal{L}^\gamma, \|\cdot\|_\gamma\right)^*, \tag{4.21}$$

$$\partial_b\lambda_m^+ = -\lambda_m^+(E_m)_-^p \in \mathcal{C}^0 \subset \left(\mathcal{L}^\gamma, \|\cdot\|_\gamma\right)^*, \tag{4.22}$$

where $E_m(\cdot) = E_m(\cdot; a, b)$ is the normalized half-eigenfunction associated with $\lambda_m^+(a,b)$ as defined in (4.18).

Proof. Consider (4.3) and let $F(\lambda, a, b) := \Theta(-\pi_p/2, \lambda a, \lambda b)$. By Theorem 2.2(ii), F is continuously differentiable in $(\lambda, a, b) \in \mathbb{R} \times \mathcal{W}_+^\gamma$. We have proved in Lemma 4.1 that

$$\frac{\partial F}{\partial \lambda}(\lambda_m^+(a,b), a, b) > 0. \tag{4.23}$$

Thus the Implicit Function Theorem implies that the functional $\lambda = \lambda_m^+ = \lambda_m^+(a,b)(> 0)$ determined by (4.3) is continuously differentiable in $(a,b) \in \mathcal{W}_+^\gamma$. Moreover, for any $h \in \mathcal{L}^\gamma$, one has

$$\partial_a\Theta \circ (\lambda + a\partial_a\lambda) \circ h + \partial_b\Theta \circ (b\partial_a\lambda) \circ h = 0. \tag{4.24}$$

By (2.15) and (2.16), this reads

$$\int_0^1 \left\{ X_+^p[\lambda h + (\partial_a\lambda)ha] + X_-^p[(\partial_a\lambda)hb] \right\} dt = 0, \tag{4.25}$$

where $X(\cdot) = X(\cdot; \lambda_m^+ a, \lambda_m^+ b)$ is the half-eigenfunction of (1.1) associated with the half-eigenvalue λ_m^+. Since $\partial_a\lambda : \mathcal{L}^\gamma \to \mathbb{R}$ and $(\partial_a\lambda)h \in \mathbb{R}$, one has

$$\begin{gathered} (\partial_a\lambda)h\int_0^1 \left(aX_+^p + bX_-^p\right) dt = -\lambda\int_0^1 X_+^p h\, dt, \\ (\partial_a\lambda)h = \frac{-\lambda\int_0^1 X_+^p h\, dt}{\int_0^1 \left(aX_+^p + bX_-^p\right) dt} = -\lambda\int_0^1 (E_m)_+^p h\, dt, \end{gathered} \tag{4.26}$$

which proves (4.21). One can obtain (4.22) analogously. □

Corollary 4.5. *Given $\gamma \in [1, \infty]$ and $(a_i, b_i) \in \mathcal{W}_+^\gamma$, $i = 0, 1$, one has*

$$(a_0, b_0) \succ (\ge)(a_1, b_1) \Longrightarrow \lambda_m^+(a_0, b_0) < (\le)\, \lambda_m^+(a_1, b_1), \quad \forall m \in \mathbb{N}. \tag{4.27}$$

Proof. Let $a_\tau = \tau a_0 + (1-\tau)a_1$ and $b_\tau = \tau b_0 + (1-\tau)b_1$ for any $\tau \in [0,1]$. Although $\mathcal{W}_+^\gamma$ is not convex, if $(a_0, b_0) \geq (a_1, b_1)$, we still have $(a_\tau, b_\tau) \in \mathcal{W}_+^\gamma$ for any $\tau \in [0,1]$. Therefore, for any given $m \in \mathbb{N}$, the function

$$M : [0,1] \longrightarrow (0,\infty), \qquad \tau \longmapsto M(\tau) = \lambda_m^+(a_\tau, b_\tau), \tag{4.28}$$

is well-defined (by Theorem 4.2) and is continuously differentiable (by Theorem 4.4). Moreover, by (4.21) and (4.22), one has

$$M'(\tau) = -M(\tau)\int_0^1 \left\{ (E_m)_+^p (a_0 - a_1) + (E_m)_-^p (b_0 - b_1) \right\} dt, \quad \forall \tau \in [0,1], \tag{4.29}$$

where $E_m(\cdot) = E_m(\cdot; a_\tau, b_\tau)$ is the normalized half-eigenfunction of (1.1) associated with $\lambda_m^+(a_\tau, b_\tau)$ (cf. (4.18)). Consequently, one has $M(\tau) < (\leq)\, 0$ if only $(a_0, b_0) \succ (\geq)(a_1, b_1)$, proving the desired results. □

5. The Neumann Spectrum and Positive Principal Eigenvalues

5.1. Structure of Neumann Half-Eigenvalues

Given $(a,b) \in \mathcal{W}_+^\gamma$, $1 \leq \gamma \leq \infty$, in this section we use the following notations for simplicity if there is no ambiguity

$$\begin{aligned} \widetilde{\Theta}(\lambda) = \widetilde{\Theta}(\lambda a, \lambda b) &:= \Theta(0, \lambda a, \lambda b), \\ \widetilde{X}(t) = \widetilde{X}(t; a, b) &:= X(t; 0, a, b). \end{aligned} \tag{5.1}$$

Notice that $0 \in \Sigma_N$ with the constant half-eigenfunctions. Due to the jumping nonlinearity of (1.1), $\lambda \in \Sigma_N^+ := \Sigma_N \cap \mathbb{R}^+$ if and only if $\lambda \geq 0$ is determined by one of the following two equations:

$$\Theta(0, \lambda a, \lambda b) = m\pi_p, \tag{5.2}$$

$$\Theta(\pi_p, \lambda a, \lambda b) = \pi_p + m\pi_p, \tag{5.3}$$

for some $m \in \mathbb{Z}$. We still need to specify the range of m.

The following lemma shows that $\widetilde{\Theta}(\lambda)$ is quasimonotone in $\lambda \in \mathbb{R}^+$. We omit the proof because it is similar to that of Lemma 4.1.

Lemma 5.1. *Given $(a,b) \in \mathcal{W}_+^\gamma$, $1 \leq \gamma \leq \infty$, if there exist $\mu > 0$ and $m \in \mathbb{Z}$ such that*

$$\widetilde{\Theta}(\mu) = m\pi_p, \tag{5.4}$$

then

$$\frac{d}{d\lambda}\widetilde{\Theta}(\lambda)\Big|_{\lambda=\mu} > 0,$$
$$\widetilde{\Theta}(\lambda) > (<) -\frac{\pi_p}{2} + m\pi_p \quad \forall\lambda > (<)\,\mu. \tag{5.5}$$

Notice that $\widetilde{\Theta}(0) = 0$. As a consequence of Lemmas 2.3 and 5.1, (5.2) has a solution if and only if $m \in \mathbb{Z}^+$. More precisely, for each $m \in \mathbb{N}$, (5.2) has the unique solution $\lambda = \widetilde{\lambda}_m^+(a,b) > 0$, and for $m = 0$, (5.2) has the zero solution $\lambda = 0$ and has at most one additional positive solution (called positive principal half-eigenvalue) $\lambda = \widetilde{\lambda}_0^+(a,b)$. Moreover, one has

$$0 < \widetilde{\lambda}_0^+(a,b) < \widetilde{\lambda}_1^+(a,b) < \widetilde{\lambda}_2^+(a,b) < \cdots < \widetilde{\lambda}_m^+(a,b)\cdots(\longrightarrow \infty). \tag{5.6}$$

Similar arguments as in the proof of Theorem 3.10 show that

$$\widetilde{\lambda}_0^+(a,b)(>0) \text{ exists} \Longleftrightarrow \int_0^1 a(t)dt < 0. \tag{5.7}$$

By Lemma 5.1, if the principal half-eigenvalue $\widetilde{\lambda}_0^+$ exists, then

$$\widetilde{\Theta}(\lambda) < 0, \quad \forall\lambda \in \left(0, \widetilde{\lambda}_0^+\right). \tag{5.8}$$

Similar results hold for (5.3). Denote the positive solution of (5.3) by $\widetilde{\lambda}_m^-(a,b)$ for any $m \in \mathbb{Z}^+$. One has

$$0 < \widetilde{\lambda}_0^-(a,b) < \widetilde{\lambda}_1^-(a,b) < \widetilde{\lambda}_2^-(a,b) < \cdots < \widetilde{\lambda}_m^-(a,b)\cdots(\longrightarrow \infty), \tag{5.9}$$

where $\widetilde{\lambda}_0^-(a,b)$ is also called positive principal half-eigenvalue and

$$\widetilde{\lambda}_0^-(a,b)(>0) \text{ exists} \Longleftrightarrow \int_0^1 b(t)dt < 0. \tag{5.10}$$

It can also be proved that

$$\widetilde{\lambda}_m^-(a,b) = \widetilde{\lambda}_m^+(b,a), \quad \forall m \in \mathbb{Z}^+ \tag{5.11}$$

if only both $\widetilde{\lambda}_m^+(b,a)$ and $\widetilde{\lambda}_m^-(a,b)$ exist.

In summary, we can completely characterize the structure of Σ_N^+.

Theorem 5.2. *Given* $(a,b) \in \mathcal{W}_+^\gamma$, $1 \le \gamma \le \infty$, *then* Σ_N^+ *consists of two sequences* (5.6) *and* (5.9), *which satisfy* (5.7), (5.10), *and* (5.11). *The half-eigenfunctions corresponding to* $\widetilde{\lambda}_m^\pm(a,b)$ *have precisely* $m-1$ *zeroes in the interval* $(0,1)$.

5.2. Dependence of Neumann Half-Eigenvalues on Weights

By (5.11), we only discuss the dependence of $\tilde{\lambda}_m^+(a,b)$ on weights (a,b). Analogous results hold for $\tilde{\lambda}_m^-(a,b)$. Given $(a,b) \in \mathcal{W}_+^\gamma$ and $m \in \mathbb{Z}^+$, denote

$$\tilde{E}_m(t) = \tilde{E}_m(t;a,b) := \frac{\tilde{X}(t)}{\left\{\int_0^1 \left(a\tilde{X}_+^p + b\tilde{X}_-^p\right)dt\right\}^{1/p}}, \tag{5.12}$$

where $\tilde{X}(t) = \tilde{X}(t;\tilde{\lambda}_m^+ a, \tilde{\lambda}_m^+ b)$ is the half-eigenfunction of (1.1) associated with the half-eigenvalue $\tilde{\lambda}_m^+ = \tilde{\lambda}_m^+(a,b)$. Then $\tilde{E}_m$ is also a half-eigenfunction, and it satisfies the normalized condition

$$\int_0^1 \left(a\left(\tilde{E}_m\right)_+^p + b\left(\tilde{E}_m\right)_-^p\right)dt = 1. \tag{5.13}$$

Denote

$$\mathcal{W}_{+-}^\gamma := \left\{(a,b) \in \mathcal{W}_+^\gamma \mid \int_0^1 a(t)dt < 0\right\}. \tag{5.14}$$

Theorem 5.3. *Given $\gamma \in [1,\infty]$, the functionals*

$$\begin{aligned} \mathcal{W}_+^\gamma \longrightarrow \mathbb{R}, \qquad (a,b) \longmapsto \tilde{\lambda}_m^+ = \tilde{\lambda}_m^+(a,b) \quad (m \in \mathbb{N}), \\ \mathcal{W}_{+-}^\gamma \longrightarrow \mathbb{R}, \qquad (a,b) \longmapsto \tilde{\lambda}_0^+ = \tilde{\lambda}_0^+(a,b), \end{aligned} \tag{5.15}$$

are continuous with respect to the weak topology $w_\gamma \times w_\gamma$, and are continuously Fréchet differentiable with respect to $\|\cdot\|_\gamma \times \|\cdot\|_\gamma$. Moreover, for any $m \in \mathbb{Z}^+$, the derivatives of $\tilde{\lambda}_m^+$ at a and at b, denoted by $\partial_a\tilde{\lambda}_m^+$ and $\partial_b\tilde{\lambda}_m^+$, respectively, are given by

$$\begin{aligned} \partial_a\tilde{\lambda}_m^+ = -\tilde{\lambda}_m^+\left(\tilde{E}_m\right)_+^p \in \mathcal{C}^0 \subset \left(\mathcal{L}^\gamma, \|\cdot\|_\gamma\right)^*, \\ \partial_b\tilde{\lambda}_m^+ = -\tilde{\lambda}_m^+\left(\tilde{E}_m\right)_-^p \in \mathcal{C}^0 \subset \left(\mathcal{L}^\gamma, \|\cdot\|_\gamma\right)^*, \end{aligned} \tag{5.16}$$

where $\tilde{E}_m = \tilde{E}_m(t;a,b)$ is the normalized half-eigenfunction associated with $\tilde{\lambda}_m^+(a,b)$ as defined in (5.12).

Proof. By checking the proof of Theorems 4.3 and 4.4, the only matter to be proved is the continuity of

$$\tilde{\lambda}_0^+ : \left(\mathcal{W}_{+-}^\gamma, w_\gamma\right) \longrightarrow (0,\infty), \tag{5.17}$$

while this can be attained by using (5.8) and similar arguments as in the proof of Theorem 3.12. □

Corollary 5.4. (i) *If* $(a_i, b_i) \in \mathcal{W}^{\gamma}_{+}$, $i = 0, 1$, *and* $(a_0, b_0) \succ (\succeq)\ (a_1, b_1)$, *then*

$$\widetilde{\lambda}^{+}_{m}(a_0, b_0) < (\leq)\ \widetilde{\lambda}^{+}_{m}(a_1, b_1), \quad \forall m \in \mathbb{N}. \tag{5.18}$$

(ii) *If* $(a_i, b_i) \in \mathcal{W}^{\gamma}_{+-}$, $i = 0, 1$, *and* $(a_0, b_0) \succ (\succeq)\ (a_1, b_1)$, *then*

$$\widetilde{\lambda}^{+}_{m}(a_0, b_0) < (\leq)\ \widetilde{\lambda}^{+}_{m}(a_1, b_1), \quad \forall m \in \mathbb{Z}^{+}. \tag{5.19}$$

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