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Research Article

Spectral Singularities of Sturm-Liouville Problems with Eigenvalue-Dependent Boundary Conditions

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Let L denote the operator generated in $L_2(R_+)$ by Sturm-Liouville equation $-y'' + q(x)y = \lambda^2 y$, $x \in R_+ = [0,\infty)$, $y'(0)/y(0) = \alpha_0 + \alpha_1\lambda + \alpha_2\lambda^2$, where q is a complex-valued function and $\alpha_i \in \mathbb{C}$, $i = 0,1,2$ with $\alpha_2 \neq 0$. In this article, we investigate the eigenvalues and the spectral singularities of L and obtain analogs of Naimark and Pavlov conditions for L.

1. Introduction

Let L_0 denote Sturm-Liouville operator generated in $L_2(\mathbb{R}_+)$ by the differential expression

$$l_0(y) := -y'' + q(x)y, \quad x \in \mathbb{R}_+, \tag{1.1}$$

and the boundary condition $y(0) = 0$, where $q : \mathbb{R}_+ \to \mathbb{C}$. Since$q$ is a complex-valued function, the operator L_0 is a non-selfadjoint. The spectral analysis of L_0 has been investigated byNaĭmark [1]. He proved that some of the poles of the kernel of resolvent of L_0 are not the eigenvalues of the operator. He also showed that those poles (which are called spectral singularities by Schwartz [2]) are on the continuous spectrum. Moreover, he has shown the spectral singularities play an important role in the spectral analysis of L_0 , and if

$$\int_0^\infty e^{\varepsilon x}|q(x)|dx < \infty, \quad \varepsilon > 0, \tag{N}$$

then the eigenvalues and the spectral singularities are of a finite number and each of them is of a finite multiplicity.

One very important step in the spectral analysis of L_0 was taken by Pavlov [3]. He studied the dependence of the structure of the eigenvalues and the spectral singularities of L_0 on the behavior of potential function at infinity. He also proved that if

$$\sup_{x\in R_+}\left[e^{\varepsilon\sqrt{x}}|q(x)|\right]<\infty,\quad \varepsilon>0, \tag{P}$$

then the eigenvalues and the spectral singularities are of a finite number and each of them is of a finite multiplicity.

Conditions (N) and (P) are called Naimark and Pavlov conditions for L_0, respectively.

Lyance showed that the spectral singularities play an important role in the spectral analysis of L_0 [4, 5]. He also investigated the effect of the spectral singularities in the spectral expansion.

The spectral singularities of non-selfadjoint operator generated in $L_2(\mathbb{R}_+)$ by (1.1) and the boundary condition

$$\int_0^\infty K(x)y(x)dx+\alpha y'(0)-\beta y(0)=0 \tag{1.2}$$

was investigated in detail by Krall [6, 7].

Some problems of spectral theory of differential operator and some other types of operators with spectral singularities were studied by some authors [8–14]. Note that in all papers the boundary conditions are not depending on the spectral parameter.

In a recent series of papers, Bindinget al. and Browne[15–18] have studied the spectral theory of regular Sturm-Liouville operators with boundary conditions depending on the spectral parameter.

Let L denote the operator generated in $L_2(\mathbb{R}_+)$ by

$$-y''+q(x)y=\lambda^2 y,\quad x\in R_+, \tag{1.3}$$

$$\frac{y'(0)}{y(0)}=\alpha_0+\alpha_1\lambda+\alpha_2\lambda^2, \tag{1.4}$$

where$q$ is a complex-valued function, $\alpha_i\in\mathbb{C}, i=0,1,2$, with $\alpha_2\neq 0$. In this paper, we investigate the eigenvalues and the spectral singularities of L. In particular, we show that the analogs of Naimark and Pavlov conditions for L are

$$\begin{gathered} q\in AC(\mathbb{R}_+),\quad \lim_{x\to\infty}q(x)=0,\quad \int_0^\infty e^{\varepsilon x}|q'(x)|dx<\infty,\quad \varepsilon>0,\\ q\in AC(\mathbb{R}_+),\quad \lim_{x\to\infty}q(x)=0,\quad \sup_{x\in R_+}\left[e^{\varepsilon\sqrt{x}}|q'(x)|\right]<\infty,\quad \varepsilon>0, \end{gathered} \tag{1.5}$$

respectively, where $AC(\mathbb{R}_+)$denotes the class of complex-valued absolutely continuous functions on $\mathbb{R}_+$.

2. Jost Functions of (1.3)-(1.4)

Under the condition

$$\int_0^\infty x|q(x)|dx < \infty, \tag{2.1}$$

(1.3) has a solution $e(x,\lambda)$satisfying

$$\lim_{x\to\infty} e(x,\lambda)e^{-i\lambda x} = 1, \quad \lambda \in \overline{\mathbb{C}}_+, \tag{2.2}$$

where $\overline{\mathbb{C}}_+ = \{\lambda : \lambda \in \mathbb{C}, \operatorname{Im}\lambda \geq 0\}$. The solution $e(x,\lambda)$ is called Jost solution of (1.3). Note that Jost solution has a representation [19]

$$e(x,\lambda) = e^{i\lambda x} + \int_x^\infty K(x,t)e^{i\lambda t}dt, \quad \lambda \in \overline{\mathbb{C}}_+, \tag{2.3}$$

where $K(x,t)$ is the solution of the integral equation

$$K(x,t) = \frac{1}{2}\int_{(x+t)/2}^\infty q(s)ds + \frac{1}{2}\int_x^{(x+t)/2}\int_{t+x-s}^{t+s-x} q(s)K(s,u)du\,ds + \frac{1}{2}\int_{(x+t)/2}^\infty\int_s^{t+s-x} q(s)K(s,u)du\,ds, \tag{2.4}$$

and $K(x,t)$ are continuously differentiable with respect to their arguments. We also have

$$\begin{gathered} |K(x,t)| \leqslant cw\left(\frac{x+t}{2}\right), \\ |K_x(x,t)|, |K_t(x,t)| \leq \frac{1}{4}\left|q\left(\frac{x+t}{2}\right)\right| + cw\left(\frac{x+t}{2}\right), \end{gathered} \tag{2.5}$$

where $w(x) = \int_x^\infty |q(s)|\,ds$ and $c > 0$ is a constant.

Let

$$\begin{gathered} E^+(\lambda) := e'(0,\lambda) - \left(\alpha_0 + \alpha_1\lambda + \alpha_2\lambda^2\right)e(0,\lambda), \quad \lambda \in \overline{\mathbb{C}}_+, \\ E^-(\lambda) := e'(0,-\lambda) - \left(\alpha_0 + \alpha_1\lambda + \alpha_2\lambda^2\right)e(0,-\lambda), \quad \lambda \in \overline{\mathbb{C}}_-, \end{gathered} \tag{2.6}$$

where $\overline{\mathbb{C}}_- = \{\lambda : \lambda \in \mathbb{C}, \operatorname{Im}\lambda \leq 0\}$. Therefore, $E^+$and E^- are analytic in $\mathbb{C}_+ = \{\lambda : \lambda \in \mathbb{C}, \operatorname{Im}\lambda > 0\}$ and $\mathbb{C}_- = \{\lambda : \lambda \in \mathbb{C}, \operatorname{Im}\lambda < 0\}$, respectively, and continuous up to real axis. The functions E^+ and E^- are called Jost functions of L.

Let us denote the eigenvalues and the spectral singularities of L by $\sigma_d(L)$ and $\sigma_{ss}(L)$, respectively. It is evident that

$$\sigma_d(L) = \{\lambda : \lambda \in \mathbb{C}_+, E^+(\lambda) = 0\} \cup \{\lambda : \lambda \in \mathbb{C}_-, E^-(\lambda) = 0\},$$
$$\sigma_{ss}(L) = \{\lambda : \lambda \in \mathbb{R}^*, E^+(\lambda) = 0\} \cup \{\lambda : \lambda \in \mathbb{R}^*, E^-(\lambda) = 0\}, \tag{2.7}$$

$$\{\lambda : \lambda \in \mathbb{R}^*, E^+(\lambda) = 0\} \cap \{\lambda : \lambda \in \mathbb{R}^*, E^-(\lambda) = 0\} = \emptyset, \tag{2.8}$$

where $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$.

Definition 2.1. The multiplicity of a zero E^+(or E^-) in $\overline{\mathbb{C}}_+$ (or $\overline{\mathbb{C}}_-$) is defined as the multiplicity of the corresponding eigenvalue and spectral singularity of L.

In order to investigate the quantitative properties of the eigenvalues and the spectral singularities of L, we need to discuss the quantitative properties of the zeros of E^+ and E^- in $\overline{\mathbb{C}}_+$ and $\overline{\mathbb{C}}_-$, respectively.

Define

$$M_1^{\pm} = \{\lambda : \lambda \in \mathbb{C}_{\pm}, E^{\pm}(\lambda) = 0\}, \qquad M_2^{\pm} = \{\lambda : \lambda \in \mathbb{R}^*, E^{\pm}(\lambda) = 0\}, \tag{2.9}$$

then by (2.7), we have

$$\sigma_d(L) = M_1^+ \cup M_1^-, \qquad \sigma_{ss}(L) = M_2^+ \cup M_2^-. \tag{2.10}$$

Now, let us assume that

$$q \in AC(R_+), \quad \lim_{x\to\infty} q(x) = 0, \quad \int_0^\infty x^3 |q'(x)| dx < \infty. \tag{2.11}$$

Theorem 2.2. *Under condition* (2.11), *the functions E^+ and E^- have the representations*

$$E^+(\lambda) = -\alpha_2\lambda^2 + \beta^+\lambda + \delta^+ + \int_0^\infty f^+(t) e^{i\lambda t} dt, \quad \lambda \in \overline{\mathbb{C}}_+, \tag{2.12}$$

$$E^-(\lambda) = -\alpha_2\lambda^2 + \beta^-\lambda + \delta^- + \int_0^\infty f^-(t) e^{-i\lambda t} dt, \quad \lambda \in \overline{\mathbb{C}}_-, \tag{2.13}$$

where $\beta^{\pm}, \delta^{\pm} \in \mathbb{C}$, and $f^{\pm} \in L_1(\mathbb{R}_+)$.

Proof. Using (2.3),(2.4), and (2.6), we get (2.12), where

$$\beta^+ = i - \alpha_1 - i\alpha_2 K(0,0),$$
$$\delta^+ = -K(0,0) - \alpha_0 - i\alpha_1 K(0,0) + \alpha_2 K_t(0,0), \tag{2.14}$$
$$f^+(t) = K_x(0,t) - \alpha_0 K(0,t) - i\alpha_1 K_t(0,t) + \alpha_2 K_{tt}(0,t).$$

From (2.4), we see that

$$|K_{tt}(0,t)| \le c\left[t\left|q\left(\frac{t}{2}\right)\right| + \left|q'\left(\frac{t}{2}\right)\right| + tw\left(\frac{t}{2}\right) + w_1\left(\frac{t}{2}\right)\right] \tag{2.15}$$

holds, where $w_1(t) = \int_t^\infty w(s)ds$ and $c > 0$ is a constant. It follows from (2.5), (2.14), and (2.15) that $f^+ \in L_1(\mathbb{R}_+)$. In a similar way, we obtain (2.13). □

Theorem 2.3. *Under condition* (2.11)*, we have the following.*

(i) *The set of $\sigma_d(L)$ is bounded and has at most a countable number of elements, and its limit points can lie only in a bounded subinterval of the real axis.*

(ii) *The set of $\sigma_{ss}(L)$ is bounded and its linear Lebesgue measure is zero.*

Proof. From (2.14) and (2.15), we see that

$$\begin{aligned} E^+(\lambda) &= -\alpha_2\lambda^2 + \beta^+\lambda + \delta^+ + o(1), \quad \lambda \in \overline{\mathbb{C}}_+,\ |\lambda| \longrightarrow \infty, \\ E^-(\lambda) &= -\alpha_2\lambda^2 + \beta^-\lambda + \delta^- + o(1), \quad \lambda \in \overline{\mathbb{C}}_-,\ |\lambda| \longrightarrow \infty. \end{aligned} \tag{2.16}$$

Using (2.10), (2.16), and the uniqueness theorem of analytic functions [20], we get (i) and (ii). □

3. Naĭmark and Pavlov Conditions for L

We will denote the set of all limit points of M_1^+ and M_1^- by M_3^+ and M_3^-, respectively, and the set of all zeros of E^+ and E^- with infinity multiplicity in $\overline{\mathbb{C}}_+$ and $\overline{\mathbb{C}}_-$, by M_4^+ and M_4^-, respectively. It is obvious that

$$M_3^\pm \subset M_2^\pm, \quad M_4^\pm \subset M_2^\pm, \quad M_3^\pm \subset M_4^\pm, \tag{3.1}$$

and the linear Lebesgue measures of $M_3^\pm$ and $M_4^\pm$ are zero.

Theorem 3.1. *If*

$$q \in AC(\mathbb{R}_+), \quad \lim_{x\to\infty} q(x) = 0, \quad \int_0^\infty e^{\varepsilon x}\left|q'(x)\right|dx < \infty, \quad \varepsilon > 0, \tag{3.2}$$

then the operator L has a finite number of eigenvalues and spectral singularities, and each of them is of a finite multiplicity.

Proof. From (2.5), (2.14), (2.15), and (3.2), we find that

$$\left|f^+(t)\right| \le ce^{-(\varepsilon/2)t}, \tag{3.3}$$

where $c > 0$ is a constant. By (2.12) and (3.3), we observe that the function E^+ has an anlytic continuation to the half-plane Im $\lambda > -\varepsilon/4$. So we get that $M_4^+ = \emptyset$. It follows from (3.1) that $M_3^+ = \emptyset$. Therefore the sets M_1^+ and M_2^+ have a finite number of elements with a finite multiplicity. We obtain similar results for the sets M_1^- and M_2^-. By (2.10) we have the proof of the theorem. □

Now let us assume that

$$q \in AC(\mathbb{R}_+), \quad \lim_{x\to\infty} q(x) = 0, \quad \sup_{x\in R_+}\left[e^{\varepsilon\sqrt{x}}\left|q'(x)\right|\right] < \infty, \quad \varepsilon > 0. \tag{3.4}$$

Hence, we have the following lemma.

Lemma 3.2. *It holds that* $M_4^+ = M_4^- = \emptyset$.

Proof. From (2.12) and (3.4), we find that the function E^+ is analytic in $\mathbb{C}_+$, and all of its derivatives are continuous in $\overline{\mathbb{C}}_+$. For a sufficiently large $T > 0$, we have

$$\left|\frac{d^k}{d\lambda^k}E^+(\lambda)\right| \le A_k, \quad \lambda \in \overline{\mathbb{C}}_+,\ |\lambda| \le T,\ k = 0,1,2,\ldots, \tag{3.5}$$

where

$$A_k = 2^k c\int_0^\infty t^k e^{-(\varepsilon/2)\sqrt{t}}dt, \quad k = 0,1,2,\ldots, \tag{3.6}$$

and $c > 0$ is a constant. Since the function E^+ is not equal to zero identically, then by Pavlov's theorem, M_4^+ satisfies

$$\int_0^h \ln A(s)d\mu(M_4^+, s) > -\infty, \tag{3.7}$$

where $A(s) = \inf_k (A_k s^k/k!), \mu(M_4^+, s)$ is the linear Lebesgue measure of s-neighborhood of M_4^+,[3]. Now, we obtain the following estimates for A_k :

$$A_k \le Bb^k k^k k!, \tag{3.8}$$

where B and b are constants depending on c and ε. From (3.8), we get that

$$A(s) \le B\inf_k\left(b^k s^k k^k\right) \le B\exp\left(-b^{-1}e^{-1}s^{-1}\right). \tag{3.9}$$

Now, (3.7) yields that

$$\int_0^h \frac{1}{s}d\mu(M_4^+, s) < \infty. \tag{3.10}$$

However, (3.10) holds for an arbitrary s, if and only if $\mu(M_4^+, s) = 0$ or $M_4^+ = \emptyset$. In a similar way we can prove that $M_4^- = \emptyset$. □

Theorem 3.3. *Under condition* (3.4), *the operator L has a finite number of eigenvalues and spectral singularities, and each of them is of a finite multiplicity.*

Proof. To be able to prove the theorem, we have to show that the functions E^+ and E^- have a finite number of zeros with finite multiplicities in $\overline{\mathbb{C}}_+$ and $\overline{\mathbb{C}}_-$, respectively. We give the proof for E^+.

From Lemma 3.2 and (3.1), we find that $M_3^+ = \emptyset$. So the bounded sets M_1^+ and M_2^+ have no limit points, that is, the function E^+ has only a finite number of zeros in $\overline{\mathbb{C}}_+$. Since $M_4^+ = \emptyset$, these zeros are of finite multiplicity. □

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