Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2009, Article ID 723236, 27 pages
doi:10.1155/2009/723236

Research Article

On the Stochastic 3D Navier-Stokes-α Model of Fluids Turbulence

Gabriel Deugoue and Mamadou Sango

Department of Mathematics and Applied Mathematics, University of Pretoria, Pretoria 002, South Africa

Correspondence should be addressed to Mamadou Sango, mamadou.sango@up.ac.za

Received 24 June 2009; Accepted 21 October 2009

Recommended by Ruediger Landes

We investigate the stochastic 3D Navier-Stokes-α model which arises in the modelling of turbulent flows of fluids. Our model contains nonlinear forcing terms which do not satisfy the Lipschitz conditions. The adequate notion of solutions is that of probabilistic weak solution. We establish the existence of a such of solution. We also discuss the uniqueness.

1. Introduction

In this paper, we are interested in the study of probabilistic weak solutions of the 3D Navier-Stokes-α(NS $-\alpha$) model (also known as the Lagrangien averaged Navier-Stokes-alpha model or the viscous Camassa-Holm equations) with homogeneous Dirichlet boundary conditions in a bounded domains in the case in which random perturbations appear. To be more precise, let D be a connected and bounded open subset of R^3 with C^2 boundary ∂D and a final time $T > 0$. We denote by A the Stokes operator and consider the system

$$\begin{aligned}
&\partial_t(u-\alpha\Delta u)+\nu(Au-\alpha\Delta(Au))+(u\cdot\nabla)(u-\alpha\Delta u)-\alpha\nabla u^*\cdot\Delta u+\nabla p\\
&\qquad = F(t,u)+G(t,u)\frac{dW}{dt}, \quad \text{in } D\times(0,T),\\
&\nabla\cdot u=0, \quad \text{in } D\times(0,T),\\
&u=0, \qquad Au=0, \quad \text{on } \partial\mathrm{D}\times(0,\mathrm{T}),\\
&u(0)=u_0, \quad \text{in } D,
\end{aligned} \tag{1.1}$$

where $u = (u_1, u_2, u_3)$ and p are unknown random fields on $D \times (0,T)$, representing, respectively, the large-scale velocity and the pressure, in each point of $D\times(0,T)$. The constant

$\nu > 0$ and $\alpha > 0$ are given, and represent, respectively, the kinematic viscosity of the fluid, and the square of the spatial scale at which fluid motion is filtered. The terms $F(t,u)$ and $G(t,u)(dW/dt)$ are external forces depending on u, where W is an R^m-valued standard Wiener process. Finally, u_0 is a given nonrandom velocity field.

The deterministic version of (1.1), that is, when $G = 0$ has been the object of intense investigations over the last years [1–5] and the initial motivation was to find a closure model for the 3D turbulence-averaged Reynolds model. A key interest in the model is the fact that it serves as a good approximation to the 3D Navier-Stokes equations. One of the main reasons justifying its use is the high computational cost that the Navier-Stokes model requires. Many important results have been obtained in the deterministic case. More precisely, the global well posedness of weak solutions for the NS-α model on bounded domains has been established in [6, 7] amongst others, and the asymptotic behavior can be found in [6]. Similar results have been proved by Foias et al. [8] in the case of periodic boundary conditions.

However, in order to consider a more realistic model our problem, it is sensible to introduce some kind of noise in the equations. This may reflect, some environmental effects on the phenomena, some external random forces, and so forth. To the best of our knowledge, the existence and uniqueness of solutions of the stochastic version (1.1) which we consider in this paper has only been analyzed in [9] (see also [10]) in the case of Lipschitz assumptions on F and G. The case of non-Lipschitz assumptions on the coefficients F and G is the main concern of the present paper. This question has been opened till now. The general motivation for studying weak rather than strong solutions of stochastic equations is that existence of weak solutions can be carried through under weaker regularity on the coefficients. This was pointed out, for instance, in [11].

In this paper, we will establish the existence of probabilistic weak solutions for the problem (1.1) under appropriate conditions on the data. Under the strong assumptions on F and G, we prove the uniqueness of weak solutions. The method used for the proof of our existence results is different from the method in [9]. To prove the existence, we use the Galerkin approximation method employing special bases, combined with some famous theorems of probabilistic nature due to Prokhorov [12] and Skorokhod [13].

The paper is organized as follows. In Section 2, we establish some properties of nonlinear term appearing in our equations. The rigorous statement of our problem as well as the main results are included in Section 3 and we show how our problem can be reformulated as an abstract stochastic model. Section 4 is devoted to the proof of our main results.

2. Properties of the Nonlinear Terms in (1.1)

Following [9], we establish some properties of the nonlinear term $(u\cdot\nabla)(u-\alpha\Delta u)-\alpha\nabla u^{*}\cdot\Delta u$ appearing in (1.1).

We denote by $(\cdot,\cdot)$ and $|\cdot|$, respectively, the scalar product and associated norm in $(L^2(D))^3$, and by $(\nabla u, \nabla v)$, the scalar product in $((L^2(D))^3)^3$ of the gradients of u and v. We consider the scalar product in $(H_0^1(D))^3$ defined by

$$((u,v)) = (u,v) + \alpha(\nabla u, \nabla v), \quad u,v \in \left(H_0^1(D)\right)^3, \tag{2.1}$$

where its associated norm $\|\cdot\|$ is, in fact, equivalent to the usual gradient norm. We denote by H the closure in $(L^2(D))^3$ of the set

$$\mathcal{V} = \left\{v \in (\mathfrak{D}(D))^3 : \nabla \cdot v = 0 \text{ in } D\right\}, \tag{2.2}$$

and by V the closure of $\mathcal{V}$ in $(H_0^1(D))^3$. Then H is a Hilbert space equipped with the inner product of $(L^2(D))^3$, and V is a Hilbert subspace of $(H_0^1(D))^3$.

Denote by A the Stokes operator, with domain $D(A) = (H^2(D))^3 \cap V$, defined by

$$Aw = -\mathcal{P}(\Delta w), \quad w \in D(A), \tag{2.3}$$

where $\mathcal{P}$ is the projection operator from $(L^2(D))^3$ onto H. Recall that as ∂D is C^2, $|Aw|$ defines in $D(A)$ a norm which is equivalent to the $(H^2(D))^3$ norm, that is, there exists a constant $c_1(D)$, depending only on D, such that

$$\|w\|_{(H^2(D))^3} \leq c_1(D)|Aw|, \quad \forall w \in D(A), \tag{2.4}$$

and so $D(A)$ is a Hilbert space with respect to the scalar product

$$(v,w)_{D(A)} = (Av, Aw). \tag{2.5}$$

For $u \in D(A)$ and $v \in (L^2(D))^3$, we define $(u \cdot \nabla)v$ as the element of $(H^{-1}(D))^3$ given by

$$\langle (u \cdot \nabla)v, w\rangle_{-1} = \sum_{i,j=1}^{3} \langle \partial_i v_j, u_i w_j\rangle_{-1}, \quad \forall w \in \left(H_0^1(D)\right)^3, \tag{2.6}$$

where by $\langle u, v\rangle_{-1}$, we denote either the duality product between $(H^{-1}(D))^3$ and $(H_0^1(D))^3$ or between $H^{-1}(D)$ and $H_0^1(D)$.

Observe that (2.6) is meaningful, since $H^2(D) \subset L^\infty(D)$ and $H_0^1(D) \subset L^6(D)$ with continuous injections. This implies that $u_i w_j \in H_0^1(D)$, and there exists a constant $c_2(D) > 0$, depending only on D, such that

$$|\langle (u \cdot \nabla)v, w\rangle_{-1}| \leq c_2(D)|Au|\|v\|\|w\|, \quad \forall (u,v,w) \in D(A) \times \left(L^2(D)\right)^3 \times \left(H_0^1(D)\right)^3. \tag{2.7}$$

Observe also that if $v \in (H^1(D))^3$, then the definition above coincides with the definition of $(u \cdot \nabla)v$ as the vector function whose components are $\sum_{i=1}^{3} u_i \partial_i v_j$, for $j = 1, 2, 3$. However, as it not known whether the solutions of the stochastic problem (1.1) have the same regularity as the deterministic case (we only can ensure H^2 instead of H^3), the present extension is necessary.

Now, if $u \in D(A)$, then $\nabla u^* \in (H^1(D))^{3\times 3} \subset (L^6(D))^{3\times 3}$, and consequently, for $v \in (L^2(D))^3$, we have that $\nabla u^* \cdot v \in (L^{3/2}(D))^3 \subset (H^{-1}(D))^3$, with

$$\langle \nabla u^* \cdot v, w \rangle_{-1} = \sum_{i,j=1}^{3} \int_D (\partial_j u_i) v_i w_j \, dx, \quad \forall w \in \left(H_0^1(D)\right)^3. \tag{2.8}$$

It follows that there exists a constant $c_3(D)$, depending only on D, such that

$$|\langle \nabla u^* \cdot v, w \rangle_{-1}| \le c_3(D) |Au| |v| \|w\|, \quad \forall (u, v, w) \in D(A) \times \left(L^2(D)\right)^3 \times \left(H_0^1(D)\right)^3. \tag{2.9}$$

We have the following results.

Proposition 2.1. *For all $(u, w) \in D(A) \times D(A)$ and for all $v \in (L^2(D))^3$, it follows that*

$$\langle (u \cdot \nabla) v, w \rangle_{-1} = -\langle \nabla w^* \cdot v, u \rangle_{-1}. \tag{2.10}$$

Proof. If $(u, w) \in D(A) \times D(A)$, then for each $i, j = 1, 2, 3$, we have $u_i w_j \in H_0^1(D)$ and consequently

$$\begin{aligned} \langle \partial_i v_j, u_i w_j \rangle_{-1} &= -\int_D v_j \partial_i (u_i w_j) dx \\ &= -\int_D v_j w_j \partial_i u_i \, dx - \int_D v_j u_i \partial_i w_j \, dx, \end{aligned} \tag{2.11}$$

using $\nabla \cdot u = 0$, we have (2.10). □

Consider now the bilinear form defined by

$$\begin{aligned} &b^*(u, v, w) = \langle (u \cdot \nabla) v, w \rangle_{-1} + \langle \nabla u^* \cdot v, w \rangle_{-1}, \\ &(u, v, w) \in D(A) \times \left(L^2(D)\right)^3 \times \left(H_0^1(D)\right)^3. \end{aligned} \tag{2.12}$$

Proposition 2.2. *The bilinear form b^* satisfies*

$$b^*(u,v,w) = -b^*(w,v,u), \quad \forall (u,v,w) \in D(A) \times \left(L^2(D)\right)^3 \times D(A), \tag{2.13}$$

and consequently,

$$b^*(u,v,u) = 0, \quad \forall (u,v) \in D(A) \times \left(L^2(D)\right)^3. \tag{2.14}$$

Moreover, there exists a constant $c(D) > 0$, depending only on D, such that

$$\begin{aligned} |b^*(u,v,w)| &\le c(D)|Au||v|\|w\|, \quad \forall (u,v,w) \in D(A) \times \left(L^2(D)\right)^3 \times \left(H_0^1(D)\right)^3, \\ |b^*(u,v,w)| &\le c(D)\|u\||v||Aw|, \quad \forall (u,v,w) \in D(A) \times \left(L^2(D)\right)^3 \times D(A). \end{aligned} \tag{2.15}$$

Thus, in particular, b^* is continuous on $D(A) \times (L^2(D))^3 \times (H_0^1(D))^3$.

Proof. The proof is straightforward consequences of (2.7), (2.9), and (2.10). □

3. Statement of the Problem and the Main Results

We now introduce some probabilistic evolutions spaces.

Let $(\Omega, F, \{F_t\}_{0\le t\le T}, P)$ be a filtered probability space and let X be a Banach space. For $r, q \ge 1$, we denote by

$$L^p(\Omega, F, P; L^r(0,T;X)) \tag{3.1}$$

the space of functions $u = u(x,t,\omega)$ with values in X defined on $[0,T] \times \Omega$ and such that

(1) u is measurable with respect to (t,ω) and for almost all t, u is F_t measurable,

(2)

$$\|u\|_{L^p(\Omega,F,P;L^r(0,T;X))} = \left[E\left(\int_0^T \|u\|_X^r dt\right)^{p/r}\right]^{1/r} < \infty, \tag{3.2}$$

where E denotes the mathematical expectation with respect to the probability measure P.

The space $L^p(\Omega, F, P; L^r(0,T;X))$ so defined is a Banach space.

When $r = \infty$, the norm in $L^p(\Omega, F, P; L^\infty(0,T;X))$ is given by

$$\|u\|_{L^p(\Omega,F,P;L^\infty(0,T;X))} = \left(E \ \text{supess}_{0\le t\le T}\|u\|_X^p\right)^{1/p}. \tag{3.3}$$

We make precise our assumptions on (1.1).

We start with the nonlinear function F and G. We assume that

$$
\begin{aligned}
&F : (0,T) \times V \longrightarrow \left(H^{-1}(D)\right)^3, \text{ measurable} \\
&\text{a.e. } t, \quad u \rightarrowtail F(t,u) : \text{continuous from } V \text{ to } \left(H^{-1}(D)\right)^3 \\
&\|F(t,u)\|_{(H^{-1}(D))^3} \leq c(1 + \|u\|), \\
&G : (0,T) \times V \longrightarrow \left((L^2(D))^3\right)^m, \text{ measurable} \\
&\text{a.e. } t, \quad u \rightarrowtail G(t,u) : \text{continuous from } V \text{ to } \left(\left(L^2(D)\right)^3\right)^m \\
&|G(t,u)|_{((L^2(D))^3)^m} \leq c(1 + \|u\|).
\end{aligned}
\tag{3.4}
$$

We will define the concept of weak solution of the problem (1.1), namely, the following.

Definition 3.1. A weak solution of (1.1) means a system $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\leq t\leq T}, \mathcal{P}, W, u)$ such that

(1) $(\Omega, \mathcal{F}, \mathcal{P})$ is a probability space, $(\{\mathcal{F}_t\}, 0 \leq t \leq T)$ is a filtration,

(2) W is an m-dimensional $\{\mathcal{F}_t\}$ standard Wiener process,

(3) $u(t)$ is $\mathcal{F}_t$ adapted for all $t \in [0,T]$:

$$
u \in L^p\left(\Omega, \mathcal{F}, \mathcal{P}; L^2(0,T,D(A))\right) \cap L^p(\Omega, \mathcal{F}, \mathcal{P}; L^\infty(0,T,V)), \quad \forall 1 \leq p < \infty,
\tag{3.5}
$$

(4) for almost all $t \in (0,T)$, the following equation holds $\mathcal{P}$-a.s.

$$
\begin{aligned}
((u(t), \Phi)) + \nu \int_0^t (u(s) + \alpha A u(s), A\Phi) ds + \int_0^t b^*(u(s), u(s) - \alpha \Delta u(s), \Phi) ds \\
= ((u_0, \Phi)) + \int_0^t \langle F(s, u(s)), \Phi \rangle_{-1} ds + \left(\int_0^t G(s, u(s)) dW(s), \Phi \right)
\end{aligned}
\tag{3.6}
$$

for all $\Phi \in D(A)$.

Our two major results are as follows.

Theorem 3.2 (Existence). *Assume* (3.4) *and* $u_0 \in V$. *Then there exists a weak solution* $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\leq t\leq T}, \mathcal{P}, W, u)$ *of* (1.1) *in the sense of Definition 3.1.*

Moreover $u \in L^p(\Omega, \mathcal{F}, \mathcal{P}; C([0,T]; V))$*, and there exists a unique* $\widetilde{p} \in L^2(\Omega, \mathcal{F}_t, \mathcal{P}; H^{-1}(0,t; H^{-1}(D))$*, for all* $t \in [0,T]$*, such that* $\mathcal{P}$*-a.s.*

$$\begin{aligned}
&\partial_t(u - \alpha\Delta u) + \nu(Au - \alpha\Delta(Au)) + (u\cdot\nabla)(u - \alpha\Delta u) - \alpha\nabla u^* \cdot \Delta u + \nabla\widetilde{p} \\
&\qquad = F(t,u) + G(t,u)\frac{dW}{dt}, \quad \text{in } \left(\mathcal{D}'((0,T)\times D)\right)^3, \\
&\int_D \widetilde{p}\,dx = 0, \quad \text{in } \mathcal{D}'(0,T),
\end{aligned} \tag{3.7}$$

where $G(t,u)(dW/dt)$ *denotes the time derivative of* $\int_0^t G(s,u(s))dW_s$*, that is, by definition*

$$G(t,u)\frac{dW}{dt} = \partial_t\left(\int_0^{\cdot} G(s,u(s))dW_s\right), \quad \text{in } \mathcal{D}'\left(0,T; \left(L^2(D)\right)^3\right), \ \mathcal{P}\text{-a.s.} \tag{3.8}$$

Corollary 3.3 (Uniqueness). *Assume that* F *and* G *are Lipschitz with respect to the second variable* $u_0 \in V$*. Then there exists a unique weak solution of problem* (1.1) *in the sense of Definition 3.1.*

Moreover, two strong solutions on the same Brownian stochastic basis coincide a.s.

3.1. Formulation of Problem (1.1) *as an Abstract Problem*

We will rewrite our model as an abstract problem.

We identify V with its topological dual V' and we have the Gelfand triple $D(A) \subset V \subset D(A)'$.

We denote by $\langle \cdot, \cdot \rangle$ the duality product between $D(A)'$ and $D(A)$. We define

$$\langle \widetilde{A}u, v\rangle = \nu(Au, v) + \nu\alpha(Au, Av), \quad u, v \in D(A). \tag{3.9}$$

It is clear that for all $v \in D(A)$,

$$2\langle \widetilde{A}u, v\rangle = 2\nu(Av, v) + 2\nu\alpha(Av, Av) \geq 2\nu\alpha|Av|^2, \tag{3.10}$$

and, if we denote by λ_k and $w_k, k \geq 1$, the eigenvalues, and their corresponding eigenvalues associated to A, then

$$\langle \widetilde{A}w_k, v\rangle = \nu\lambda_k((w_k, v)). \tag{3.11}$$

Thus, taking

$$\tilde{\alpha} = 2\nu\alpha, \tag{3.12}$$

we have

(a) $\tilde{A}$ is a linear continuous operator $\tilde{A} \in \mathcal{L}(D(A), D(A)')$, such that

$$\begin{aligned} &\text{(a1)} \quad \tilde{A} \text{ is self-adjoint} \\ &\text{(a2)} \quad \text{there exists } \tilde{\alpha} > 0, \text{ such that} \\ &2\left\langle \tilde{A}v, v \right\rangle \geq \tilde{\alpha}\|v\|_{D(A)}^2, \quad \forall v \in D(A). \end{aligned} \tag{3.13}$$

On the other hand, denote

$$\begin{gathered} \langle \tilde{B}(u,v), w\rangle = b^*(u, v - \alpha\Delta v, w), \qquad (u,v,w) \in D(A) \times D(A) \times D(A), \\ \left(\left(\tilde{F}(t,u), w\right)\right) = \langle F(t,u), w\rangle_{-1}, \quad (u,w) \in V \times V. \end{gathered} \tag{3.14}$$

Thus it is straightforward to check that if we take

$$c_1 = (1+\alpha)c_1(D)c(D), \tag{3.15}$$

then we obtain that

(b) $\tilde{B} : D(A) \times D(A) \longrightarrow D(A)'$ is a bilinear mapping such that

$$\text{(b1)} \quad \langle \tilde{B}(u,v), u\rangle = 0, \quad \forall u, v \in D(A), \tag{3.16}$$

$$\text{(b2)} \quad \left\|\tilde{B}(u,v)\right\|_{D(A)'} \leq c_1\|u\|\|v\|_{D(A)}, \quad \forall u, v \in D(A) \times D(A), \tag{3.17}$$

$$\text{(b3)} \quad \left|\langle \tilde{B}(u,v), w\rangle\right| \leq c_1\|u\|_{D(A)}\|v\|_{D(A)}\|w\|, \quad \forall u, v, w \in D(A). \tag{3.18}$$

(c) $\tilde{F} : (0,T) \times V \longrightarrow V$, measurable such that

$$\begin{aligned} &\text{(c1)} \quad \text{a.e. } t, \quad u \rightarrowtail \tilde{F}(t,u) : \text{continuous from } V \text{ to } V \\ &\text{(c2)} \quad \left\|\tilde{F}(t,u)\right\| \leq c(1 + \|u\|). \end{aligned} \tag{3.19}$$

Now, let I denote the identity operator in H, and define $\widetilde{G}(t,u)$ as

$$\widetilde{G}(t,u) = (I+\alpha A)^{-1} \circ \wp \circ G(t,u), \quad u \in V. \tag{3.20}$$

$I+\alpha A$ is bijective from $D(A)$ onto H, and

$$\left(\left((I+\alpha A)^{-1} f, w\right)\right) = (f,w), \quad \forall f \in H,\ w \in V. \tag{3.21}$$

Thus, for each $f \in H$,

$$\left\|(I+\alpha A)^{-1} f\right\|^2 = (f,u) \le |f||u|, \tag{3.22}$$

where $u = (I+\alpha A)^{-1} f$, that is, $(u,w_k) + \alpha(Au,w_k) = (f,w_k)$, for all $k \ge 1$, so $(1+\alpha\lambda_k)(u,w_k) = (f,w_k)$, which implies

$$\begin{gathered}(u,w_k) = \frac{1}{(1+\alpha\lambda_k)}(f,w_k) \le \frac{1}{1+\alpha\lambda_1}(f,w_k), \\ |u|^2 = \sum_{k=1}^{\infty}(u,w_k)^2 \le \frac{1}{(1+\alpha\lambda_1)^2}\sum_{k=1}^{\infty}(f,w_k)^2 = \frac{1}{(1+\alpha\lambda_1)^2}|f|^2.\end{gathered} \tag{3.23}$$

Therefore,

$$\left\|(I+\alpha A)^{-1} f\right\|^2 \le \frac{1}{1+\alpha\lambda_1}|f|^2, \tag{3.24}$$

and, consequently, taking

$$\widetilde{c} = \frac{c}{\sqrt{1+\alpha\lambda_1}}, \tag{3.25}$$

we obtain that

(d) $\widetilde{G} : (0,T) \times V \longrightarrow V^{\otimes m}$, measurable such that

$$\begin{aligned}&\text{(d1)} \quad \text{a.e. } t, \quad u \rightarrowtail \widetilde{G}(t,u) : \text{continuous from } V \text{ to } V^{\otimes m} \\ &\text{(d2)} \quad \left\|\widetilde{G}(t,u)\right\|_{V^{\otimes m}} \le \widetilde{c}(1+\|u\|),\end{aligned} \tag{3.26}$$

where $V^{\otimes m}$ *is the product of* m *copies of* V. Next, for each $j \geq 1$, and all $(t, u, \Phi) \in (0, T) \times V \times D(A)$, we have

$$(G(t,u),\Phi) = \left((I+\alpha A)\widetilde{G}(t,u),\Phi\right) = \left(\left(\widetilde{G}(t,u),\Phi\right)\right), \tag{3.27}$$

and, for all $u \in L^2(\Omega, \mathcal{F}, \mathcal{P}; L^\infty(0,T;V))$, $(t,\Phi) \in (0,T) \times D(A)$, it follows that

$$\begin{aligned}\left(\int_0^t G(s,u(s))dW(s),\Phi\right) &= \sum_{j=1}^{d}\int_0^t (G_j(s,u(s)),\Phi)dW_j(s)\\ &= \sum_{j=1}^{d}\int_0^t \left(\widetilde{G}_j(s,u(s),\Phi)\right)dW_j(s)\\ &= \left(\left(\int_0^t \widetilde{G}(s,u(s))dW(s),\Phi\right)\right).\end{aligned} \tag{3.28}$$

Consequently, in this abstract framework, a weak solution $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\leq t\leq T}, \mathcal{P}, W, u)$ of (1.1) is equivalently as follows.

Definition 3.4. It holds that

(1) $(\Omega, \mathcal{F}, \mathcal{P})$ is a probability space, $(\{\mathcal{F}_t\}, 0 \leq t \leq T)$ is a filtration,

(2) W is a m-dimensional $\{\mathcal{F}_t\}$ standard Wiener process,

(3) $u(t)$ is $\mathcal{F}_t$ adapted for all $t \in [0,T]$

$$u \in L^p\left(\Omega,\mathcal{F},\mathcal{P};L^2(0,T,D(A))\right) \cap L^p(\Omega,\mathcal{F},\mathcal{P};L^\infty(0,T,V)), \quad \forall 1 \leq p < \infty, \tag{3.29}$$

(4) for almost all $t \in (0,T)$, the following equation holds $\mathcal{P}$-a.s.

$$\begin{aligned}u(t) + \int_0^t \widetilde{A}u(s)ds + \int_0^t \widetilde{B}(u(s),u(s))ds\\ = u_0 + \int_0^t \widetilde{F}(s,u(s))ds + \int_0^t \widetilde{G}(s,u(s))dW(s),\end{aligned} \tag{3.30}$$

as an equality in $D(A)'$.

Remark 3.5. However, (3.30) implies that $u \in \mathcal{C}(0,T;D(A)')$, then u is weakly continuous in V [14, page 263] and the initial condition is meaningful.

4. Proofs of the Main Results

4.1. Proof of Theorem 3.2

We make use of the Galerkin approximation combined with the method of compactness.

We will split the proof into six steps.

4.1.1. Step 1. Construction of an Approximating Sequence

As the injection $D(A) \hookrightarrow V$ is compact, consider an orthonormal basis $\{e_j\}_{j=1,2,\ldots}$ in $D(A)$ which is orthogonal in V such that e_j are eigenfunctions of the spectral problem

$$(e_j, v)_{D(A)} = \lambda_j((e_j, v)), \quad \forall v \in D(A), \tag{4.1}$$

where $(.,.)_{D(A)}$ denotes the scalar product in $D(A)$. For each $N \in \mathbb{N}$, let V_N be the span of $\{e_1, \ldots, e_N\}$.

Consider the probabilistic system

$$\left(\overline{\Omega}, \overline{F}, \left\{\overline{F}_t\right\}_{0 \le t \le T}, \overline{P}, \overline{W}\right). \tag{4.2}$$

We denote by $\overline{E}$ the mathematical expectation with respect to $(\overline{\Omega}, \overline{F}, \overline{P})$.

We look for a sequence of functions $u^N(t)$ in V_N, that is,

$$u^N(t) = \sum_{j=1}^{N} c_{Nj}(t, \overline{\omega}) e_j(x), \tag{4.3}$$

solutions of the following stochastic ordinary differential equations in V_N :

$$\begin{aligned}
&d\left(\left(u^N, e_j\right)\right) + \left(\langle \widetilde{A} u^N(t), e_j\rangle + \langle \widetilde{B}\left(u^N(t), u^N(t)\right), e_j\rangle\right) dt \\
&\quad = \left(\left(\widetilde{F}\left(t, u^N(t)\right), e_j\right)\right) dt + \left(\left(\widetilde{G}\left(t, u^N(t)\right), e_j\right)\right) d\overline{W}, \quad j = 1, 2, \ldots, N \\
&u^N(0) = u_0^N,
\end{aligned} \tag{4.4}$$

where $u_0^N \in V_N$ and is chosen with the requirements that

$$u_0^N \longrightarrow u_0 \quad \text{in } V \text{ as } N \longrightarrow \infty. \tag{4.5}$$

There exists a a maximal solution to (4.4), that is, a stopping time $T_N \leq T$ such that (4.4) holds for $t < T_N$ [11]. Solvability over $(0,T)$ will follow from a priori estimates for u^N that we derive in the following section.

We have the following Fourier expansion:

$$u^N(t) = \sum_{j=1}^{N} \left(u^N(t), e_j\right)_{D(A)} e_j = \sum_{j=1}^{N} \lambda_j \left(\left(u^N(t), e_j\right)\right) e_j,$$
$$\left\|u^N(t)\right\|^2 = \sum_{j=1}^{N} \lambda_j \left(\left(u^N(t), e_j\right)\right)^2. \tag{4.6}$$

4.1.2. Step 2. A Priori Estimates

Throughout C and C_i $(i = 1, \ldots)$ denotes a positive constant independent of N.

We have the following Lemma.

Lemma 4.1. *It holds that u^N satisfies the following a priori estimates:*

$$\overline{E} \sup_{0 \leq t \leq T} \left\|u^N(s)\right\|^2 + 2\tilde{\alpha}\overline{E} \int_0^T \left\|u^N(s)\right\|_{D(A)}^2 ds \leq C_1, \tag{4.7}$$

where C_1 is a constant independent of N.

Proof. By Ito's formula, we obtain from (3.16) and (4.4) that

$$d\left\|u^N(t)\right\|^2 + 2\langle \tilde{A}u^N(t), u^N(t)\rangle dt = \left[2\left(\left(\tilde{F}\left(t, u^N(t)\right), u^N(t)\right)\right) + \sum_{j=1}^{N} \lambda_j \left(\left(\tilde{G}\left(t, u^N(t)\right), e_j\right)\right)^2\right] dt$$
$$+ 2\left(\left(\tilde{G}\left(t, u^N(t)\right), u^N(t)\right)\right) d\overline{W}. \tag{4.8}$$

Integrating (4.8) with respect to t, and using (3.13) and (3.19), we have

$$\left\|u^N(t)\right\|^2 + \tilde{\alpha} \int_0^t \left\|u^N(s)\right\|_{D(A)}^2 ds \leq \left\|u_0^N\right\|^2 + C + C\int_0^t \left\|u^N(s)\right\|^2 ds$$
$$+ 2\int_0^t \left(\left(\tilde{G}\left(s, u^N(s)\right), u^N(s)\right)\right) d\overline{W}(s). \tag{4.9}$$

Let us estimate the stochastic integral in this inequality. By Burkholder-Davis Gundy's inequality [15], we have

$$\overline{E}\sup_{0\le s\le t}\left|\int_0^s\left(\left(\widetilde{G}\left(s,u^N(s)\right),u^N(s)\right)\right)d\overline{W}(s)\right| \le C\overline{E}\left(\int_0^t\left(\left(\widetilde{G}\left(s,u^N(s)\right),u^N(s)\right)\right)^2 ds\right)^{1/2}$$
$$\le \epsilon\overline{E}\sup_{0\le s\le t}\left\|u^N(s)\right\|^2 + C_\epsilon\int_0^t\left(1+\left\|u^N(s)\right\|^2\right)ds, \tag{4.10}$$

here we have used $H\ddot{o}lder's$ and Young's inequalities; ϵ is an arbitrary positive number.

Using (4.10) and (4.9) together with appropriate choice of ϵ, we obtain

$$\overline{E}\sup_{0\le s\le t}\left\|u^N(s)\right\|^2 + 2\widetilde{\alpha}\overline{E}\int_0^t\left\|u^N(s)\right\|^2_{D(A)}ds \le C + C\overline{E}\int_0^t\left\|u^N(s)\right\|^2 ds. \tag{4.11}$$

By Gronwall's lemma, we obtain the sought estimate (4.7). □

The following result is related to the higher integrability of u^N.

Lemma 4.2. *It holds that*

$$\overline{E}\sup_{0\le s\le T}\left\|u^N(s)\right\|^p \le C_p \quad \forall 1\le p<\infty. \tag{4.12}$$

Proof. By Ito's formula, it follows from (4.4) that for $p\ge 4$, we have

$$d\left\|u^N(t)\right\|^{p/2} = \frac{p}{2}\left\|u^N(t)\right\|^{p/2-2}\Bigg[-\left\langle \widetilde{A}u^N(t),u^N(t)\right\rangle - 2\left\langle\widetilde{B}\left(u^N(t),u^N(t)\right),u^N(t)\right\rangle$$
$$+2\left(\left(\widetilde{F}\left(t,u^N(t)\right),u^N(t)\right)\right) + \frac{1}{2}\sum_{i=1}^N\lambda_i\left(\left(\widetilde{G}\left(t,u^N(t)\right),e_i\right)\right)^2$$
$$+\frac{p-4}{4}\frac{\left(\left(\widetilde{G}(u^N(t),u^N(t))\right)\right)^2}{\|u^N(t)\|^2}\Bigg]dt$$
$$+\frac{p}{2}\left\|u^N(t)\right\|^{p/2-2}\left(\left(\widetilde{G}\left(t,u^N(t)\right),u^N(t)\right)\right)d\overline{W}. \tag{4.13}$$

Using the assumptions (3.16), (3.19), (3.26), it follows that

$$\begin{aligned}\sup_{0\le s\le t}\left\|u^N(s)\right\|^{p/2} &\le \left\|u_0^N\right\|^{p/2} + C\int_0^t\left(1+\left\|u^N(s)\right\|^{p/2}\right)ds \\ &\quad + \frac{p}{2}\sup_{0\le s\le t}\left|\int_0^s\left\|u^N(s)\right\|^{p/2-2}\left(\left(\widetilde{G}\left(s,u^N(s)\right),u^N(s)\right)\right)d\overline{W}\right|.\end{aligned} \tag{4.14}$$

Squaring the both sides of this inequality and passing to mathematical expectation, we deduce from the Martingale inequality, that is,

$$\overline{E}\sup_{0\le s\le t}\left\|u^N(s)\right\|^{p} \le C\left(\left\|u_0^N\right\|^{p} + T + \overline{E}\int_0^t\left\|u^N(s)\right\|^{p}ds\right). \tag{4.15}$$

From Gronwall's inequality, we deduce that

$$\overline{E}\sup_{0\le s\le t}\left\|u^N(s)\right\|^{p} \le C_p \tag{4.16}$$

for all $1\le p<\infty$. □

We also have the following lemma.

Lemma 4.3. *It holds that u^N satisfies*

$$\overline{E}\left(\int_0^T\left\|u^N(s)\right\|^2_{D(A)}ds\right)^p \le C_p \tag{4.17}$$

for all $1\le p<\infty$.

Proof. Using (4.9), we have

$$\begin{aligned}\widetilde{\alpha}^p\left(\int_0^t\left\|u^N(s)\right\|^2_{D(A)}\right)^p &\le C\left\|u_0^N\right\|^{2p} + C + C\left(\int_0^t\left\|u^N(s)\right\|^2 ds\right)^p \\ &\quad + C\left|\int_0^t((\widetilde{G}(s,u^N(s)),u^N(s)))d\overline{W}\right|^p.\end{aligned} \tag{4.18}$$

Taking the mathematical expectation and use the Burkholder-Gundy's inequality, the proof of the lemma follows from Lemma 4.2. □

Lemma 4.4. *It holds that*

$$\overline{E}\sup_{0\le|\theta|\le\delta\le 1}\int_0^T \left\|u^N(t+\theta)-u^N(t)\right\|_{D(A)'}^2 dt \le C\delta. \tag{4.19}$$

Proof. We note that the functions $\{\lambda_j e_j\}_{j=1,2,\dots}$ form an orthonormal basis in the dual $D(A)'$ of $D(A)$. Let P^N be the orthogonal projection of $D(A)'$ onto the span $\{\lambda_1 e_1,\dots,\lambda_N e_N\}$, that is,

$$P^N h = \sum_{j=1}^{N}\lambda_j\langle h, e_j\rangle e_j. \tag{4.20}$$

Thus (4.4) can be rewritten in an integral form as the equality between random variables with values in $D(A)'$ as

$$\begin{aligned} u^N(t) &+ \int_0^t P^N\left(\widetilde{A}u^N(s)+\widetilde{B}\left(u^N(s),u^N(s)\right)-\widetilde{F}\left(s,u^N(s)\right)\right)ds \\ &= u_0^N + \int_0^t P^N\widetilde{G}\left(s,u^N(s)\right)d\overline{W}. \end{aligned} \tag{4.21}$$

For any positive θ, we have

$$\begin{aligned} &\left\|u^N(t+\theta)-u^N(t)\right\|_{D(A)'} \\ &\le \left\|\int_t^{t+\theta}\left(\widetilde{A}u^N(s)+\widetilde{B}\left(u^N(s),u^N(s)\right)-\widetilde{F}\left(s,u^N(s)\right)\right)ds\right\|_{D(A)'} + \left\|\int_t^{t+\theta}\widetilde{G}\left(s,u^N(s)\right)d\overline{W}\right\|_{D(A)'}. \end{aligned} \tag{4.22}$$

Taking the square and use the properties of $\widetilde{A}, \widetilde{B}$ and $\widetilde{F}$, we have

$$\begin{aligned} \left\|u^N(t+\theta)-u^N(t)\right\|_{D(A)'}^2 \le C\theta^2 &+ C\left(\int_t^{t+\theta}\left\|u^N(s)\right\|_{D(A)}^2 ds\right)^2 \\ &+ C\sup_{0\le t\le T}\left\|u^N(s)\right\|^2\left(\int_t^{t+\theta}\left\|u^N(s)\right\|_{D(A)} ds\right)^2 \\ &+ C\theta^2\sup_{0\le s\le T}\left\|u^N(s)\right\|^2 + \left\|\int_t^{t+\theta}\widetilde{G}\left(s,u^N(s)\right)d\overline{W}\right\|^2. \end{aligned} \tag{4.23}$$

For fixed δ, taking the supremun over $\theta \leq \delta$, integrating with respect to t, and taking the mathematical expectation, we have

$$\begin{aligned}\overline{E}\sup_{0\leq\theta\leq\delta}\int_0^T \left\|u^N(t+\theta)-u^N(t)\right\|_{D(A)'}^2 dt &\leq C\delta^2 + C\overline{E}\int_0^T\left(\int_t^{t+\delta}\left\|u^N(s)\right\|_{D(A)}^2 ds\right)^2 dt \\ &+ C\overline{E}\sup_{0\leq s\leq T}\left\|u^N(s)\right\|^2\int_0^T\left(\int_t^{t+\delta}\left\|u^N(s)\right\|_{D(A)} ds\right)^2 dt \\ &+ C\delta^2\overline{E}\sup_{0\leq s\leq T}\left\|u^N(s)\right\|^2 \\ &+ \overline{E}\int_0^T\sup_{0\leq\theta\leq\delta}\left\|\int_t^{t+\theta}\widetilde{G}\left(s,u^N(s)\right)d\overline{W}\right\|^2 dt.\end{aligned} \tag{4.24}$$

We estimate the integrals in this inequality.

We have by $H\ddot{o}lder's$ inequality

$$\begin{aligned}I_1 &= \overline{E}\sup_{0\leq s\leq T}\left\|u^N(s)\right\|^2\int_0^T\left(\int_t^{t+\delta}\left\|u^N(s)\right\|_{D(A)} ds\right)^2 dt \\ &\leq \delta^2\overline{E}\sup_{0\leq s\leq T}\left\|u^N(s)\right\|^2\int_0^T\left\|u^N(s)\right\|_{D(A)}^2 ds.\end{aligned} \tag{4.25}$$

Using the $H\ddot{o}lder's$ inequality and the estimates of Lemmas 4.2 and 4.3, we have

$$I_1 \leq C\delta^2. \tag{4.26}$$

By Martingale's inequality, we have

$$\begin{aligned}I_2 &= \overline{E}\int_0^T\sup_{0\leq\theta\leq\delta}\left\|\int_t^{t+\theta}\widetilde{G}\left(s,u^N(s)\right)d\overline{W}\right\|^2 dt \\ &\leq \overline{E}\int_0^T\left(\int_t^{t+\delta}\left\|\widetilde{G}(s,u^N(s))\right\|^2 ds\right) dt.\end{aligned} \tag{4.27}$$

Using the assumptions on $\widetilde{G}$ and the estimate of Lemma 4.2, we have

$$I_2 \leq C\delta. \tag{4.28}$$

Collecting the results and making a similar reasoning with $\theta < 0$, we obtain from (4.24) that

$$\overline{E} \sup_{0 \le |\theta| \le \delta} \int_0^T \left\| u^N(t+\theta) - u^N(t) \right\|_{D(A)'}^2 \le C\delta \tag{4.29}$$

□

The following lemma is from [16], and it is a compactness results which represents a variation of the compactness theorems in [17, Chapter I, Section 5]. It will be useful for us to prove the tightness property of Galerkin solution.

Proposition 4.5. *For any sequences of positives reals number ν_m, μ_m which tend to 0 as $m \to \infty$, the injection of*

$$Y_{\mu_n,\nu_n} = \left\{ y \in L^2(0,T;D(A)) \cap L^\infty(0,T;V) \mid \sup_m \frac{1}{\nu_m} \sup_{|\theta| \le \mu_m} \left(\int_0^T \| y(t+\theta) - y(t) \|_{D(A)'}^2 \right)^{1/2} < \infty \right\} \tag{4.30}$$

in $L^2(0,T;V)$ is compact.

Furthermore Y_{μ_n,ν_n} is a Banach space with the norm

$$\begin{aligned} \|y\|_{Y_{\mu_n,\nu_n}} = {} & \sup_{0\le t\le T} \|y(t)\| + \left(\int_0^T \|y(t)\|_{D(A)}^2 dt \right)^{1/2} \\ & + \sup_n \frac{1}{\nu_n} \sup_{|\theta| \le \mu_n} \left(\int_0^T \|y(t+\theta) - y(t)\|_{D(A)'}^2 dt \right)^{1/2}. \end{aligned} \tag{4.31}$$

Alongside with Y_{μ_n,ν_n}, we also consider the space $X_{p,\mu_n,\nu_n} (1 \le p < \infty)$ of random variables y such that

$$\begin{aligned} & \overline{E} \sup_{0\le t\le T} \|y(t)\|^p < \infty; \qquad \overline{E} \left(\int_0^T \|y(t)\|_{D(A)}^2 dt \right)^{p/2} < \infty; \\ & \overline{E} \sup_n \frac{1}{\nu_n} \sup_{|\theta|\le\mu_n} \int_0^T \|y(t+\theta) - y(t)\|_{D(A)'}^2 \, dt < \infty. \end{aligned} \tag{4.32}$$

Endowed with the norm

$$\begin{aligned} \|y\|_{X_{p,\nu_n,\mu_n}} = {} & \left(\overline{E} \sup_{0\le t\le T} \|y(t)\|^p \right)^{1/p} + \left(\overline{E} \left(\int_0^T \|y(t)\|_{D(A)}^2 dt \right)^{p/2} \right)^{p/2} \\ & + \overline{E} \sup_n \frac{1}{\nu_n} \left(\sup_{|\theta|\le\mu_n} \int_0^T \|y(t+\theta) - y(t)\|_{D(A)'}^2 \, dt \right)^{1/2}, \end{aligned} \tag{4.33}$$

X_{p,μ_n,ν_n} is a Banach space. The priori estimates of the preceding lemmas enable us to claim that for any $1 \leq p < \infty$ and for μ_n, ν_n such that the series $\sum_{n=1}^{\infty}(\sqrt{\mu_n}/\nu_n)$ converges, the sequence of Galerkin solutions $\{u^N : N \in N\}$ is bounded in X_{p,μ_n,ν_n}.

4.1.3. Step 3. Tightness Property of Galerkin Solutions

Now, we consider the set

$$S = C(0,T;R^m) \times L^2(0,T;V), \tag{4.34}$$

and $B(S)$ the σ-algebra of the Borel sets of S.

For each N, let ϕ be the map

$$\phi : \overline{\Omega} \longrightarrow S : \overline{\omega} \longmapsto \left(\overline{W}(\overline{\omega},\cdot), u^N(\overline{\omega},\cdot)\right). \tag{4.35}$$

For each N, we introduce a probability measure Π_N on $(S, B(S))$ by

$$\Pi_N(A) = \overline{P}\left(\phi^{-1}(A)\right) \tag{4.36}$$

for all $A \in B(S)$. The main result of this subsection is the following.

Theorem 4.6. *The family of probability measures $\{\Pi_N; N \in N\}$ is tight.*

Proof. For $\varepsilon > 0$, we should find the compact subsets

$$\Sigma_\varepsilon \subset C(0,T;R^m), \qquad Y_\varepsilon \subset L^2(0,T;V), \tag{4.37}$$

such that

$$\overline{P}\left(\overline{\omega} : \overline{W}(\overline{\omega},\cdot) \notin \Sigma_\varepsilon\right) \leq \frac{\varepsilon}{2}, \tag{4.38}$$

$$\overline{P}\left(\overline{\omega} : u^N(\overline{\omega},\cdot) \notin Y_\varepsilon\right) \leq \frac{\varepsilon}{2}. \tag{4.39}$$

The quest for Σ_ε is made by taking account of some fact about the Wiener process such as the formula

$$\overline{E}\left|\overline{W}(t_2) - \overline{W}(t_1)\right|^{2j} = (2j-1)!(t_2 - t_1)^j, \quad j = 1, 2, \ldots. \tag{4.40}$$

For a constant L_ε depending on ε to be chosen later and $n \in N$, we consider the set

$$\Sigma_\varepsilon = \left\{ W(\cdot) \in C(0,T;R^m) : \sup_{t_1,t_2 \in [0,T], |t_2 - t_1| \leq 1/n^6} n|W(t_2) - W(t_1)| \leq L_\varepsilon \right\} \tag{4.41}$$

Making use of Markov's inequality:

$$\overline{P}(\omega : \xi(\omega) \geq \alpha) \leq \frac{1}{\alpha^k}\overline{E}\left[|\xi(\omega)|^k\right] \tag{4.42}$$

for a random variable ξ on $(\overline{\Omega}, \overline{F}, \overline{P})$ and positives variables α and k, we get

$$\begin{aligned}
\overline{P}\left(\overline{\omega} : \overline{W}(\overline{\omega}, \cdot) \notin \Sigma_\varepsilon\right) &\leq \overline{P}\left[\bigcup_n \left\{\omega : \sup_{t_1,t_2\in[0,T]:|t_2-t_1|<1/n^6} \left|\overline{W}(t_2) - \overline{W}(t_1)\right| > \frac{L_\varepsilon}{n}\right\}\right] \\
&\leq \sum_{n=1}^{\infty}\sum_{i=0}^{n^6-1}\left(\frac{n}{L_\varepsilon}\right)^4 \overline{E} \sup_{iT/n^6\leq t\leq (i+1)T/n^6} \left|\overline{W}(t) - \overline{W}\left(iTn^{-6}\right)\right|^4 \\
&\leq c\sum_{n=1}^{\infty}\left(\frac{n}{L_\varepsilon}\right)^4 \left(Tn^{-6}\right)^2 n^6 \\
&= \frac{c}{L_\varepsilon^4}\sum_{n=1}^{\infty}\frac{1}{n^2},
\end{aligned} \tag{4.43}$$

we choose

$$L_\varepsilon^4 = 2C\varepsilon^{-1}\sum_{n=1}^{\infty}\frac{1}{n^2} \tag{4.44}$$

to get (4.38).

Next we choose Y_ε as a ball of radius M_ε in Y_{μ_n,ν_n} centered at zero and with μ_n, ν_n, independent of ε, converging to zero, and such that $\sum_n(\sqrt{\mu_n}/\nu_n)$ converges.

From Proposition 4.5, Y_ε is a compact subset of $L^2(0, T; V)$.

We have further

$$\overline{P}\left(\omega : u^N(\omega, \cdot) \notin Y_\varepsilon\right) \leq \overline{P}\left(\omega : \left\|u^N\right\|_{Y_{\mu_n,\nu_n}} > M_\varepsilon\right) \leq \frac{1}{M_\varepsilon}\overline{E}\left\|u^N\right\|_{Y_{\mu_n,\nu_n}} \leq \frac{c}{M_\varepsilon}, \tag{4.45}$$

choosing $M_\varepsilon = 2c\varepsilon^{-1}$, we get (4.39).

From (4.38) and (4.39), we have

$$\overline{P}\left(\omega : \overline{W}(\omega, \cdot) \in \Sigma_\varepsilon; u^N(\omega, \cdot) \in Y_\varepsilon\right) \geq 1 - \varepsilon, \tag{4.46}$$

this proves that

$$\Pi_N(\Sigma_\varepsilon \times Y_\varepsilon) \geq 1 - \varepsilon, \quad \forall N \in \mathbb{N}. \tag{4.47}$$

□

4.1.4. Step 4. Applications of Prokhorov and Skorokhod Results

From the tightness property of $\{\Pi_N\}$ and Prokhorov's theorem [12], we have that there exist a subsequence $\{\Pi_{N_j}\}$ and a measure Π such that $\Pi_{N_j} \to \Pi$ weakly.

By Skorokhod's theorem [13], there exist a probability space $(\Omega, \mathcal{F}, P)$ and random variables $(W_{N_j}, u^{N_j}), (W, u)$ on $(\Omega, \mathcal{F}, P)$ with values in S such that

$$\text{the law of } \left(W_{N_j}, u^{N_j}\right) \text{is } \Pi_{N_j}, \tag{4.48}$$

$$\text{the law of } (W, u) \text{ is } \Pi, \tag{4.49}$$

$$\left(W_{N_j}, u^{N_j}\right) \longrightarrow (W, u) \quad \text{in } {}^{\prime}S,\ P\text{-a.s.} \tag{4.50}$$

Hence, $\{W_{N_j}\}$ is a sequence of an m-dimensional standard Wiener process.

Let $\mathcal{F}^t = \sigma\{W(s), u(s), 0 \le s \le t\}$.

Arguing as in [16], we prove that $W(t)$ is an m-dimensional $\mathcal{F}^t$ standard Wiener process and the pair (W_{N_j}, u^{N_j}) satisfies the equation

$$\begin{aligned} u^{N_j}(t) + \nu\int_0^t P^{N_j}\tilde{A}u^{N_j}(s)ds + \int_0^t P^{N_j}\tilde{B}\left(u^{N_j}(s), u^{N_j}(s)\right)ds \\ = \int_0^t P^{N_j}\tilde{F}\left(s, u^{N_j}(s)\right)ds + \int_0^t P^{N_j}\tilde{G}\left(s, u^{N_j}(s)\right)dW_{N_j} + u_0^{N_j}. \end{aligned} \tag{4.51}$$

4.1.5. Step 5. Passage to the Limit

From (4.51), it follows that u^{N_j} satisfies the results of the Lemmas 4.2, 4.3, and 4.4. Therefore, we have for $p \ge 1$ the a priori estimates

$$\begin{aligned} &E\sup_{0\le t\le T}\left\|u^{N_j(t)}\right\|^p \le C; \\ &E\left(\int_0^T\left\|u^{N_j}(t)\right\|_{D(A)}^2 dt\right)^p \le C; \\ &E\sup_{0\le\theta\le\delta}\int_0^T\left\|u^{N_j}(t+\theta) - u^{N_j}\right\|_{D(A)'}^2 dt \le C(\alpha)\delta \end{aligned} \tag{4.52}$$

thus modulo the extraction of a subsequence denoted again by u^{N_j}, we have

$$\begin{aligned} &u^{N_j} \longrightarrow u \quad \text{weakly} * \text{in } L^p(\Omega, \mathcal{F}, P; L^\infty(0, T; V)); \\ &u^{N_j} \longrightarrow u \quad \text{weakly in } L^p\left(\Omega, \mathcal{F}, P; L^2(0, T; D(A))\right); \end{aligned}$$

$$E\sup_{0\le t\le T}\|u(t)\|^p \le C; \qquad E\left(\int_0^T \|u(t)\|_{D(A)}^2 dt\right)^p \le C;$$
$$E\sup_{0\le\theta\le\delta}\int_0^T \|u(t+\theta)-u(t)\|_{D(A)'}^2\, dt \le C\delta. \tag{4.53}$$

By (4.50) and the first estimate in (4.52) and Vitali's theorem, we have

$$u^{N_j} \longrightarrow u \quad \text{strongly in } L^2\Big(\Omega, \mathcal{F}, P; L^2(0,T,V)\Big), \tag{4.54}$$

and thus modulo the extraction of a subsequence and for almost every (ω, t) with respect to the measure $dP \otimes dt$:

$$u^{N_j} \longrightarrow u \quad \text{in } V. \tag{4.55}$$

This convergence together with the condition on $\widetilde{F}$, the first estimate in (4.52) and Vitali's theorem, give

$$\widetilde{F}\Big(\cdot, u^{N_j}(\cdot)\Big) \longrightarrow \widetilde{F}(\cdot, u(\cdot)) \quad \text{strongly in } L^2\Big(\Omega, \mathcal{F}, P; L^2(0,T,V)\Big),$$
$$\int_0^t \widetilde{F}\Big(s, u^{N_j(s)}\Big)ds \longrightarrow \int_0^t \widetilde{F}(s, u(s))ds \quad \text{strongly in } L^2\Big(\Omega, \mathcal{F}, P; L^2(0,T,V)\Big). \tag{4.56}$$

As

$$u^{N_j} \longrightarrow u \quad \text{weakly in } L^2\Big(\Omega, \mathcal{F}, P; L^2(0,T;D(A))\Big), \tag{4.57}$$

then

$$\int_0^t \widetilde{A}u^{N_j}(s)ds \longrightarrow \int_0^t \widetilde{A}u(s)ds \quad \text{weakly in } L^2\Big(\Omega, \mathcal{F}, P; L^2(0,T;D(A)')\Big). \tag{4.58}$$

We also have

$$\int_0^t \widetilde{B}\Big(u^{N_j}(s), u^{N_j}(s)\Big)ds \longrightarrow \int_0^t \widetilde{B}(u(s), u(s))ds \quad \text{weakly in } L^2\Big(\Omega, \mathcal{F}, P; L^2(0,T;D(A)')\Big). \tag{4.59}$$

In fact, since $L^\infty(\Omega \times (0,T), dP \times dt; D(A))$ is dense in $L^2(\Omega, \mathcal{F}, P; L^2(0,T; D(A)))$, and $\widetilde{B}(u^{N_j}(s), u^{N_j}(s))$ is bounded in $L^2(\Omega, \mathcal{F}, P; L^2(0,T; D(A)'))$ it suffices to prove that for all $\varphi \in L^\infty(\Omega \times (0,T), dP \times dt; D(A))$,

$$E\int_0^T \langle \widetilde{B}\left(u^{N_j}(s), u^{N_j}(s)\right), \varphi(s)\rangle_{D(A)'} ds \longrightarrow E\int_0^T \langle \widetilde{B}(u(s), u(s)), \varphi(s)\rangle_{D(A)'}\, ds. \tag{4.60}$$

Indeed, we have

$$\begin{aligned}
&E\int_0^T \langle \widetilde{B}\left(u^{N_j}(s), u^{N_j}(s)\right) - \widetilde{B}(u(s), u(s)), \varphi(s)\rangle_{D(A)'}\, ds \\
&\quad = E\int_0^T \left\langle \widetilde{B}(u^{N_j}(s) - u(s), u^{N_j}(s)), \varphi(s) \right\rangle_{D(A)'} ds \\
&\qquad + E\int_0^T \left\langle \widetilde{B}(u(s), u^{N_j}(s) - u(s)), \varphi(s) \right\rangle_{D(A)'} ds \\
&\quad = I_{1j} + I_{2j}, \\
&I_{1j} = E\int_0^T \left\langle \widetilde{B}\left(u^{N_j}(s) - u(s), u^{N_j}(s)\right), \varphi(s) \right\rangle_{D(A)'} ds
\end{aligned} \tag{4.61}$$

By the property (3.17) of $\widetilde{B}$, we have

$$I_{1j} \le CE\int_0^T \left\| u^{N_j}(s) - u(s) \right\| \left\| u^{N_j}(s) \right\|_{D(A)} \left| A\varphi(s) \right| ds, \tag{4.62}$$

applying Cauchy-Schwarz inequality

$$I_{1j} \le C_\varphi \left(E\int_0^T \left\| u^{N_j}(s) - u(s) \right\|^2 ds \right)^{1/2} \left(E\int_0^T \left\| u^{N_j}(s) \right\|_{D(A)}^2 ds \right)^{1/2}. \tag{4.63}$$

Since

$$u^{N_j} \longrightarrow u \quad \text{strongly in } L^2\left(\Omega, \mathcal{F}, P; L^2(0,T;V)\right), \tag{4.64}$$

and u^{N_j} is bounded in $L^2(\Omega, \mathcal{F}, P; L^2(0,T; D(A)))$, we conclude that

$$\begin{gathered}
I_{1j} \longrightarrow 0 \quad \text{as } j \longrightarrow \infty. \\
I_{2j} = E\int_0^T \langle \widetilde{B}\left(u(s), u^{N_j} - u(s)\right), \varphi(s)\rangle_{D(A)'}\, ds.
\end{gathered} \tag{4.65}$$

Again thanks to the property (3.18) of $\widetilde{B}$, as

$$u^{N_j} \longrightarrow u \quad \text{weakly in } L^2\Big(\Omega, \mathcal{F}, P; L^2(0,T;D(A))\Big), \tag{4.66}$$

we obtain $I_{2j} \to 0$ as $j \to \infty$ since any strongly continuous linear operator is weakly continuous. We are now left with the proof of

$$\int_0^t \widetilde{G}\Big(s, u^{N_j}(s)\Big) dW_{N_j}(s) \longrightarrow \int_0^t \widetilde{G}(s,u(s))dW(s) \quad \text{weakly} * L^2(\Omega, \mathcal{F}, P; L^\infty(0,T;D(A)')), \tag{4.67}$$

which can be prove with the same argument like in [16].

Collecting all the convergence results, we deduce that

$$\begin{aligned} u(t) + \nu \int_0^t \widetilde{A}u(s)ds + \int_0^t \widetilde{B}(u(s),u(s))ds \\ = \int_0^t \widetilde{F}(s,u(s))ds + \int_0^t \widetilde{G}(s,u(s))dW(s) + u_0, \quad P\text{-a.s.} \end{aligned} \tag{4.68}$$

as the equality in $D(A)'$.

We have $\widetilde{B}(u,u) \in L^2(\Omega,\mathcal{F},P;L^\infty(0,T;D(A)'))$, $\widetilde{A}u - \widetilde{F}(t,u) \in L^2(\Omega,\mathcal{F},P;L^\infty(0,T;D(A)'))$, $\widetilde{G}(t,u) \in L^2(\Omega,\mathcal{F},P;L^\infty(0,T;V^{\otimes m}))$.

Thus, from the classical results in [18] (see also [19]), we deduce from (4.68) that u is P-a.s. continuous with values in V.

4.1.6. Step 6. Existence of the Pressure

For the existence of the pressure, we use a generalization of the Rham's theorem processes [20, Theorem 4.1, Remark 4.3]. From (3.6), we have for all $v \in \mathcal{V}$,

$$\begin{aligned} \Big\langle -\partial_t(u - \alpha\Delta u) - \nu(Au - \alpha\Delta(Au)) - (u\cdot\nabla)(u - \alpha\Delta u) \\ + \alpha\nabla u^* \cdot \Delta u + F(\cdot,u) + G(\cdot,u)\frac{dW}{dt}, v\Big\rangle_{(\mathfrak{D}'(D))^3\times(\mathfrak{D}(D))^3} = 0. \end{aligned} \tag{4.69}$$

We denote

$$\begin{aligned} h = -\partial_t(u - \alpha\Delta u) - \nu(Au - \alpha\Delta(Au)) - (u\cdot\nabla)(u - \alpha\Delta u) \\ + \alpha\nabla u^* \cdot \Delta u + F(\cdot,u) + G(\cdot,u)\frac{dW}{dt}. \end{aligned} \tag{4.70}$$

We will prove that the regularity on u, implies that

$$h \in L^2\left(\Omega, \mathcal{F}_t, P; H^{-1}\left(0, t; \left(H^{-2}(D)\right)^3\right)\right). \tag{4.71}$$

By (2.7) and (2.9), we have as $u \in L^4(\Omega, \mathcal{F}, P; L^2(0, T; D(A)))$,

$$\begin{gathered}(u \cdot \nabla(u - \alpha\Delta u)) + \nabla u^* \cdot \Delta u \in L^2\left(\Omega, \mathcal{F}_t, P; L^1\left(0, t; \left(H^{-1}(D)\right)^3\right)\right), \\ Au - \alpha\Delta(Au) \in L^4\left(\Omega, \mathcal{F}_t, P; L^2\left(0, t; \left(H^{-2}(D)\right)^3\right)\right).\end{gathered} \tag{4.72}$$

We also have

$$\begin{gathered}u - \alpha\Delta u \in L^4\left(\Omega, \mathcal{F}_t, P; L^2\left(0, t; \left(L^2(D)\right)^3\right)\right), \\ \partial_t(u - \alpha\Delta u) \in L^4\left(\Omega, \mathcal{F}_t, P; H^{-1}\left(0, t; \left(L^2(D)\right)^3\right)\right), \quad \forall t \in [0, T].\end{gathered} \tag{4.73}$$

Again, as $u \in L^4(\Omega, \mathcal{F}, P; C([0, T]; V))$, then its follows that

$$\begin{gathered}F(t, u) \in L^4\left(\Omega, \mathcal{F}_t, P; L^2\left(0, t; \left(H^{-1}(D)\right)^3\right)\right), \\ G(t, u)\frac{dW}{dt} \in L^4\left(\Omega, \mathcal{F}_t, P; W^{-1,\infty}\left(0, t; \left(L^2(D)\right)^3\right)\right),\end{gathered} \tag{4.74}$$

for all $t \in [0, T]$.

Then $h \in L^2(\Omega, \mathcal{F}_t, P; H^{-1}(0, t; (H^{-2}(D))^3)$, and

$$\langle h, v \rangle_{(\mathcal{D}'(D))^3 \times (\mathcal{D}(D))^3} = 0, \quad \forall v \in \mathcal{V}. \tag{4.75}$$

Therefore, by a generalization of the Rham theorem processes [20], there exists a unique $\tilde{p} \in L^2(\Omega, \mathcal{F}_t, P; H^{-1}(0, t; (H^{-1}(D))^3)$ such that P-a.s.

$$\nabla\tilde{p} = h, \quad \int_D \tilde{p}\, dx = 0, \text{ that is, } (3.7). \tag{4.76}$$

Theorem 3.2 is proved.

4.2. Proof of Corollary 3.3

Proof. We will prove the pathwise uniqueness which implies uniqueness of weak solutions. Let L_F and L_G be two real such that

$$\|F(t,u)-F(t,v)\|_{(H^{-1}(D))^3} \le L_F\|u-v\|,$$
$$\|G(t,u)-G(t,v)\|_{((L^2(D))^3)^m} \le L_G\|u-v\|. \tag{4.77}$$

Then $\widetilde{F}$ and $\widetilde{G}$ are defined, respectively, by (3.14) and (3.20) satisfying

$$\left\|\widetilde{F}(t,u)-\widetilde{F}(t,v)\right\|_V \le L_{\widetilde{F}}\|u-v\|,$$
$$\left\|\widetilde{G}(t,u)-\widetilde{G}(t,u)\right\|_{V^{\otimes m}} \le L_{\widetilde{G}}\|u-v\|. \tag{4.78}$$

Let u_1 and u_2 two weak solutions of problem (1.1) defined on the same probability space together with the same Wiener process and starting from the same initial value u_0.

We denote $\overline{u} = u_1 - u_2$. Take $\mu > 0$ to be defined later and $\rho(t) = \exp(-\mu\int_0^t \|u_2(s)\|^2_{D(A)}\,ds)$, $0 \le t \le T$.

Applying Ito's formula to the real process $\rho(t)\|\overline{u}(t)\|^2$, we obtain from (3.13), (3.18), (3.19), and (3.26) that

$$\begin{aligned}\rho(t)\|\overline{u}(t)\|^2 + \widetilde{\alpha}\int_0^t \rho(s)\|\overline{u}(s)\|^2_{D(A)}\,ds \le{}& L^2_{\widetilde{G}}\int_0^t \rho(s)\|\overline{u}(s)\|^2 ds\\
&+ 2\widetilde{c}\int_0^t \rho(s)\|u_2(s)\|_{D(A)}\|\overline{u}(s)\|_{D(A)}\|\overline{u}(s)\|ds\\
&+ 2L_{\widetilde{F}}\int_0^t \rho(s)\|\overline{u}(s)\|_{D(A)}\|\overline{u}(s)\|ds\\
&+ 2\int_0^t \left(\left(\rho(s)\left(\widetilde{G}(s,u_1(s))-\widetilde{G}(s,u_2(s))\right),\overline{u}(s)\right)\right)dW(s)\\
&- \mu\int_0^t\int_0^t \rho(s)\|u_2(s)\|^2_{D(A)}\|\overline{u}(s)\|^2 ds,\end{aligned} \tag{4.79}$$

for all $t \in [0,T]$.

By young's inequality, we have

$$2\widetilde{c}\|u_2(s)\|_{D(A)}\|\overline{u}(s)\|_{D(A)}\|\overline{u}(s)\| \le \frac{\widetilde{\alpha}}{2}\|\overline{u}(s)\|^2_{D(A)} + \frac{2\widetilde{c}^2}{\widetilde{\alpha}}\|u_2(s)\|^2_{D(A)}\|\overline{u}(s)\|^2$$
$$2L_{\widetilde{F}}\|\overline{u}(s)\|_{D(A)}\|\overline{u}(s)\| \le \frac{\widetilde{\alpha}}{2}\|\overline{u}(s)\|^2_{D(A)} + \frac{2L^2_{\widetilde{F}}}{\widetilde{\alpha}}\|\overline{u}(s)\|^2. \tag{4.80}$$

If we take $\mu = 2(\tilde{c}^2/\tilde{\alpha})$, we obtain from (4.79) that

$$\rho(t)\|\overline{u}(t)\|^2 \le \left(L_{\tilde{G}}^2 + \frac{2L_{\tilde{F}}^2}{\tilde{\alpha}}\right)\int_0^t \rho(s)\|\overline{u}(s)\|^2 ds + 2\int_0^t \left(\left(\rho(s)\left(\tilde{G}(s,u_1(s)) - \tilde{G}(s,u_2(s)), \overline{u}(s)\right)\right)\right) dW. \tag{4.81}$$

As $0 < \rho(t) \le 1$, the expectation of the stochastic integral in (4.81) vanishes and

$$E\rho(t)\|\overline{u}(t)\|^2 \le \left(L_{\tilde{G}}^2 + \frac{2L_{\tilde{F}}^2}{\tilde{\alpha}}\right) E\int_0^t \rho(s)\|\overline{u}(s)\|^2 ds. \tag{4.82}$$

The Gronwall lemma implies that $\overline{u}(t) = 0$, P-a.s. for all $t \in [0,T]$. Also, the corollary is proved. □

Remark 4.7. Using the famous Yamada-Watanabe theorem [11], Corollary 3.3 implies the existence of a unique strong solution of (1.1).

Acknowledgment

The research of the authors is supported by the University of Pretoria and the National Research Foundation South Africa.

References

[1] S. Chen, C. Foias, D. D. Holm, E. Olson, E. S. Titi, and S. Wynne, "A connection between the Camassa-Holm equations and turbulent flows in pipes and channels," *Physics of Fluids*, vol. 11, no. 8, pp. 2343–2353, 1999.

[2] S. Chen, C. Foias, D. D. Holm, E. Olson, E. S. Titi, and S. Wynne, "The Camassa-Holm equations and turbulence," *Physica D*, vol. 133, no. 1–4, pp. 49–65, 1999.

[3] S. Chen, C. Foias, D. D. Holm, E. Olson, E. S. Titi, and S. Wynne, "Camassa-Holm equations as a closure model for turbulent channel and pipe flow," *Physical Review Letters*, vol. 81, no. 24, pp. 5338–5341, 1989.

[4] S. Chen, D. D. Holm, L. G. Margolin, and R. Zhang, "Direct numerical simulations of the Navier-Stokes alpha model," *Physica D*, vol. 133, no. 1–4, pp. 66–83, 1999.

[5] C. Foias, D. D. Holm, and E. S. Titi, "The Navier-Stokes-alpha model of fluid turbulence," *Physica D*, vol. 152-153, pp. 505–519, 2001.

[6] D. Coutand, J. Peirce, and S. Shkoller, "Global well-posedness of weak solutions for the Lagrangian averaged Navier-Stokes equations on bounded domains," *Communications on Pure and Applied Analysis*, vol. 1, no. 1, pp. 35–50, 2002.

[7] J. E. Marsden and S. Shkoller, "Global well-posedness for the LANS—α equations on bounded domains," *Philosophical Transactions of The Royal Society A*, vol. 359, no. 1784, pp. 1449–1468, 2001.

[8] C. Foias, D. D. Holm, and E. S. Titi, "The three dimensional viscous Camassa-Holm equations, and their relation to the Navier-Stokes equations and turbulence theory," *Journal of Dynamics and Differential Equations*, vol. 14, no. 1, pp. 1–35, 2002.

[9] T. Caraballo, J. Real, and T. Taniguchi, "The existence and uniqueness of solutions to stochastic three-dimensional Lagrangian averaged Navier-Stokes equations," *Proceedings of The Royal Society of London A*, vol. 462, no. 2066, pp. 459–479, 2006.

[10] T. Caraballo, A. M. Marquez-Duran, and J. Real, "On the stochastic 3D-Lagrangian averaged Navier-Stokes α-model with finite delay," *Stochastics and Dynamics*, vol. 5, no. 2, pp. 189–200, 2005.

[11] N. Ikeda and S. Watanabe, *Stochastic Differential Equations and Diffusions Processes*, North-Holland, Amsterdam, The Netherlands, 1981.

[12] Y. V. Prohorov, "Convergence of random processes and limit theorems in probability theory," *Teorija Verojatnostii Primenenija*, vol. 1, pp. 177–238, 1956 (Russian).

[13] A. V. Skorohod, "Limit theorems for stochastic processes," *Teorija Verojatnosteii Primenenija*, vol. 1, pp. 289–319, 1956.

[14] R. Temam, *Navier Stokes Equations, Theory and Numerical Analysis*, vol. 2 of *Studies in Mathematics and Its Applications*, North-Holland, Amsterdam, The Netherlands, 3rd edition, 1997.

[15] I. Karatzas and S. E. Shreve, *Brownian Motion and Stochastic Calculus*, vol. 113 of *Graduate Texts in Mathematics*, Springer, New York, NY, USA, 1987.

[16] A. Bensoussan, "Stochastic Navier-Stokes equations," *Acta Applicandae Mathematicae*, vol. 38, no. 3, pp. 267–304, 1995.

[17] J. L. Lions, *Quelques Méthodes de Resolution des Problèmes aux Limites Non Linéaires*, Dunod, Gauthier-Villars, Paris, France, 1969.

[18] N. V. Krylov and B. L. Rozovskii, "Stochastic evolution equations," in *Current Problems in Mathematics, Vol. 14 (Russian)*, pp. 71–147, Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Informatsii, Moscow, Russia, 1979.

[19] E. Pardoux, *Equations aux dérivées partielles stochastiques non linéaires monotones*, Ph.D. thesis, Université Paris 15, Paris, France, 1975.

[20] J. A. Langa, J. Real, and J. Simon, "Existence and regularity of the pressure for the stochastic Navier-Stokes equations," *Applied Mathematics and Optimization*, vol. 48, no. 3, pp. 195–210, 2003.