Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2009, Article ID 768920, 12 pages
doi:10.1155/2009/768920

Research Article

Existence of Positive Solutions for Multiterm Fractional Differential Equations of Finite Delay with Polynomial Coefficients

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Received 9 February 2009; Revised 7 June 2009; Accepted 1 September 2009

Recommended by Ferhan Atici

Existence of positive solutions has been studied by A. Babakhani and V. Daftardar-Gejji (2003) in case of multiterm nonautonomous fractional differential equations with constant coefficients. In the present paper we discuss existence of positive solutions in case of multiterm fractional differential equations of finite delay with polynomial coefficients.

1. Introduction

In last 30 years, the theory of ordinary differential equations of fractional order has become a new important branch (see, e.g., [1–5] and the references therein). Numerous applications of such equations have been presented [3–10]. Existence of positive solution of fractional ordinary differential equations has been well investigated for fractional functional differential equations [1, 6, 11–14]. Ye et al. [6] have addressed the question of existence of positive solutions for the nonlinear fractional functional differential equation

$$\begin{gathered} D^{\alpha}[x(t)-x(0)] = x(t)f(t,x_t), \quad t\in(0,T], \\ x(t)=\phi(t)\geq 0, \quad t\in[-w,0], \end{gathered} \tag{1.1}$$

by using the sub- and supersolution method, where $0<\alpha<1$, D^{α} is the standard Riemann-Liouville fractional derivative, $\phi\in C$ and $f:[0,T]\times C\to\mathbb{R}^{+}$ is continuous, as usual,

$C = C([-w,0],\mathbb{R}^+)$ is the space of continuous function from $[-w,0]$ to $\mathbb{R}^+$, $w > 0$, equipped with the sup norm:

$$\|\phi\| = \max_{-w\leq\Theta\leq 0}|\phi(t)|, \tag{1.2}$$

and x_t denotes the function in C defined by

$$x_t(\theta) = x(t+\theta), \quad -w \leq \theta \leq 0. \tag{1.3}$$

They require that the nonlinearity $f(t,x_t)$ is increasing in x_t for each $t \in [0,T]$.

As a pursuit of this in the present paper, we deal with the existence of positive solutions in the case of multiterm differential equations with polynomial coefficients of the fractional type:

$$\begin{gathered}\mathcal{L}(D)[x(t)-x(0)] = f(t,x_t), \quad t \in (0,T],\\ x(t) = \phi(t) \geq 0, \quad t \in [-w,0],\end{gathered} \tag{1.4}$$

where

$$\begin{gathered}\mathcal{L}(D) = D^{\alpha_n} - \sum_{j=1}^{n-1} p_j(t) D^{\alpha_{n-j}}, \quad 0 < \alpha_1 < \cdots < \alpha_n < 1,\ p_j(t) = \sum_{k=0}^{N_j} a_{jk}t^k,\\ p_j^{(2m)}(t) \geq 0, \quad p_j^{(2m+1)}(t) \leq 0, \quad m = 0,1,\ldots,\left[\frac{N_j}{2}\right],\ j = 1,2,\ldots,n-1,\end{gathered} \tag{1.5}$$

and D^{α_j} is the standard Riemann-Liouville fractional derivative, $T > 0$, $w > 0$, $\phi \in C = C([-w,0],\mathbb{R}^+)$ and $f : I \times C \to \mathbb{R}^+$ is a given continuous function, $I = [0,T]$.

2. Preliminaties

Let E be a real Banach space with a cone K. K introduces a partial order $\leq$ in E in the following manner [13]:

$$x \leq y \quad \text{if } y - x \in K. \tag{2.1}$$

Definition 2.1 (see [15]). For $x, y \in E$ the order interval $\langle x, y\rangle$ is defined as

$$\langle x, y\rangle = \{z \in E : x \leq z \leq y\}. \tag{2.2}$$

Definition 2.2 (see [15]). A cone K is called normal, if there exists a positive constant r such that $f, g \in K$ and $\vartheta \prec f \prec g$ implies $\|f\| \leq r\|g\|$, where ϑ denotes the zero element of K.

Definition 2.3 (see [16, 17]). Let $f : [a,b] \to \mathbb{R}$, and $f \in L^1[a,b]$. The left-sided Riemann-Liouville fractional integral of f of order α is defined as

$$I_a^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt, \quad \alpha > 0, \; x \in [a,b]. \tag{2.3}$$

Definition 2.4 (see [16, 17]). The left-sided Riemann-Liouville fractional derivative of a function $f : [a,b] \to \mathbb{R}$ is defined as

$$D_a^\alpha f(x) = D^m \left[I_a^{m-\alpha} f(x) \right], \quad x \in [a,b], \tag{2.4}$$

where $m = [\alpha]+1$, $D^m = d^m/dt^m$. We denote D_0^α by D^α and I_0^α by I^α. If the fractional derivative $D_a^\alpha f(x)$ is integrable, then [16, page 71]

$$I_a^\alpha \left(D_a^\beta f(x) \right) = I_a^{\alpha-\beta} f(x) - \left[I_a^{1-\beta} f(x) \right]_{x=a} \frac{x^{\alpha-1}}{\Gamma(\alpha)}, \quad 0 < \beta \leq \alpha < 1. \tag{2.5}$$

If f is continuous on $[a,b]$, then $[I_a^{1-\beta} f(x)]_{x=a} = 0$ and (2.5) reduces to

$$I_a^\alpha \left(D_a^\beta f(x) \right) = I_a^{\alpha-\beta} f(x), \quad 0 < \beta \leq \alpha < 1. \tag{2.6}$$

Proposition 2.5. *Let y be continuous on $[0,T]$, $T > 0$ and let n be a nonnegative integer, then*

$$I^\alpha (t^n x(t)) = \sum_{k=0}^{n} \binom{-\alpha}{k} \left[D^k t^n \right] \left[I^{\alpha+k} x(t) \right] = \sum_{k=0}^{n} \binom{-\alpha}{k} \frac{n! t^{n-k}}{(n-k)!} I^{\alpha+k} x(t), \tag{2.7}$$

where

$$\binom{-\alpha}{k} = (-1)^k \frac{\Gamma(\alpha+1)}{k!\Gamma(\alpha)} = (-1)^k \binom{\alpha}{k} = \frac{\Gamma(1-\alpha)}{\Gamma(k+1)\Gamma(1-\alpha-k)}. \tag{2.8}$$

The proof of the above proposition can be found in [17, page 53].

Corollary 2.6. *Let $x \in C[0,T]$, $T > 0$ and $p_j(t) = \sum_{k=0}^{N_j} a_{jk} t^k$, $N_j \in \mathbb{N} \cup \{0\}$, $j = 1, 2, \ldots, n$, then*

$$I^{\alpha}\left(\sum_{j=1}^{n} p_j(t) x(t)\right) = \sum_{j=1}^{n} \sum_{k=0}^{N_j} \sum_{r=0}^{k} a_{jk} \binom{-\alpha}{r} \frac{k! t^{k-r}}{(k-r)!} [I^{\alpha+r} x(t)]. \tag{2.9}$$

Theorem 2.7 (see [10]). *Let E be a Banach space with $C \subseteq E$ closed and convex. Assume that U is a relatively open subset of C with $0 \in U$ and $F : \overline{U} \to C$ is a continuous and compact map. Then either*

(1) *F has a fixed point in $\overline{U}$, or*

(2) *there exists $u \in \partial U$ and $\lambda \in (0,1)$ with $u = \lambda F(u)$.*

3. Existence of Positive Solution

In this section, we discuss the existence of positive solutions for (1.4). Using (2.5), (2.6), and Corollary 2.6, (1.4) is equivalent to the integral equation

$$x(t) = \begin{cases} x(0) + \mathcal{O}[x(t) - x(0)] + I^{\alpha_n} f(t, x_t), & t \in (0, T], \\ \phi(t), & t \in [-w, 0], \end{cases} \tag{3.1}$$

where

$$\mathcal{O} = \sum_{j=1}^{n-1} \sum_{k=0}^{N_j} \sum_{r=0}^{k} a_{jk} \binom{-\alpha_n}{r} \frac{k! t^{k-r}}{(k-r)!} I^{\alpha_n - \alpha_{n-j} + r}. \tag{3.2}$$

Let $y(\cdot) : [-w, T] \to [0, +\infty)$ be the function defined by

$$y(t) = \begin{cases} \phi(0), & t \in I, \\ \phi(t) \geq 0, & t \in [-w, 0], \end{cases} \tag{3.3}$$

then $y_0 = \phi$, for each $z \in C(I, \mathbb{R})$ with $z(0) = 0$, we denote by $\overline{z}$ the function define by

$$\overline{z}(t) = \begin{cases} z(t), & t \in I, \\ 0, & t \in [-w, 0]. \end{cases} \tag{3.4}$$

We can decompose $x(\cdot)$ as $x(t) = \overline{z}(t) + y(t)$, $t \in [-w, T]$, which implies $x_t = \overline{z}_t + y_t$, for $t \in I$. Therefore, (3.1) is equivalent to the integral equation

$$z(t) = \mathcal{O}z(t) + I^{\alpha_n} f\left(t, \overline{z}_t + y_t\right), \quad t \in I, \tag{3.5}$$

where $\mathcal{O}$ is defined (3.2). Set $A_0 = \{z \in C(I, \mathbb{R}) : z_0 = 0\}$ and let $\|z\|_T$ be the seminorm in A_0 defined by

$$\|z\|_T = \|z_0\| + \|z\| = \|z\| =: \sup\{|z(t)| : t \in I\}, \quad z \in A_0, \tag{3.6}$$

and A_0 is a Banach space with norm $\|\cdot\|_T$. Let K be a cone of A_0, $K = \{z \in A_0;\ z(t) \geq 0,\ t \in I\}$ and

$$K^* = \left\{x \in C([-w, T], \mathbb{R}^+);\ x(t) = \phi(t) \geq 0,\ t \in [-w, 0]\right\}. \tag{3.7}$$

Define the operator $F : K \to K$ by

$$Fz(t) = \mathcal{O}z(t) + I^{\alpha_n} f\left(t, \overline{z}_t + y_t\right), \quad t \in I. \tag{3.8}$$

Theorem 3.1. *Suppose that the following conditions hold:*

(1) *there exist $p, q \in C(I, \mathbb{R}^+)$ such that $f(t, x_t) \leq p(t) + q(t)\|x_t\|$, for $t \in I$, $x_t \in C$, and $\|I^{\alpha_n} p\| = \sup_{t\in[0,T]} I^{\alpha_n} p(t) < \infty$, $\|I^{\alpha_n} q\| = \sup_{t\in[0,T]} I^{\alpha_n} q(t) < \infty$,*

(2) *$1 - \mathcal{O}(T) - \|I^{\alpha_n} q\| > 0$, where*

$$\mathcal{O}(T) = \sum_{j=1}^{n-1} \sum_{k=0}^{N_j} \sum_{r=0}^{k} \left| a_{jk} \binom{-\alpha_n}{r} \right| \frac{k! T^{\alpha_n - \alpha_{n-j} + k}}{(k-r)!\Gamma\left(\alpha_n - \alpha_{n-j} + r\right)}. \tag{3.9}$$

Then (1.4) *has at least a positive solution $x^* \in K^*$, satisfying $\|x^*\| \leq \max\{\|\phi\|, h\}$, where*

$$h = \frac{\lambda\|\phi\|\left\|I^{\alpha_n} q\right\| + \lambda\left\|I^{\alpha_n} p\right\|}{1 - \lambda\mathcal{O}(T) - \lambda\left\|I^{\alpha_n} q\right\|} + 1. \tag{3.10}$$

Proof. We will show that the operator $F : K \to K$ is continuous and completely continuous.

Step 1. The operator $F : K \to K$ is continuous in view of the continuity of f.

Step 2. F maps bounded sets into bounded sets in K.

Let $G \subset K$ be bounded; that is, there exists a positive constant l such that $\|z\|_T \le l$, for all $z \in G$. For each $z \in G$, we have

$$\begin{aligned}|Fz(t)| &\le \sum_{j=1}^{n-1}\sum_{k=0}^{N_j}\sum_{r=0}^{k}\left|a_{jk}\binom{-\alpha_n}{r}\right|\frac{k!t^{k-r}}{(k-r)!}I^{\alpha_n-\alpha_{n-j}+r}|z(t)| + I^{\alpha_n}f\left(t,\overline{z}_t+y_t\right)\\ &\le \sum_{j=1}^{n-1}\sum_{k=0}^{N_j}\sum_{r=0}^{k}\left|a_{jk}\binom{-\alpha_n}{r}\right|\frac{lk!t^{\alpha_n-\alpha_{n-j}+k}}{(k-r)!\Gamma\left(\alpha_n-\alpha_{n-j}+r\right)} + I^{\alpha_n}f\left(t,\overline{z}_t+y_t\right)\\ &\le l\mathcal{O}(T) + I^{\alpha_n}\left\{p(t)+q(t)\left\|\overline{z}_t+y_t\right\|\right\},\end{aligned} \tag{3.11}$$

where $\mathcal{O}(T)$ is defined in (3.9). It follows that

$$\|Fz\|_T \le l\mathcal{O}(T) + \left\|I^{\alpha_n}p\right\| + l\left\|I^{\alpha_n}q\right\| + \left\|\phi(t)\right\|\left\|I^{\alpha_n}q\right\|. \tag{3.12}$$

Hence FG is bounded.

Step 3. F maps bounded sets into equicontinuous sets of K.

We will show that FG is equicontinuous. For each $z \in G$, $t_1, t_2 \in I$ and $t_1 < t_2$, then for given $\epsilon > 0$, choose

$$\delta = \min\left\{\left[\frac{\epsilon C(j,k,r)}{4}\right]^{1/(\alpha_n-\alpha_{n-j}+r)}, \left[\frac{\epsilon\Gamma(\alpha_n+1)}{4\left(\|p\|+\|q\|(l+\|\phi\|)\right)}\right]^{1/\alpha_n}\right\}, \tag{3.13}$$

where $j = 1,2,\ldots,n-1$, $k = 0,1,\ldots,N_j$, $r = 0,1,\ldots,k$,

$$C(j,k,r) = \frac{(k-r)!}{\sum_{i=1}^{n-1}(N_i+1)(N_i+2)} \times \frac{\Gamma\left(\alpha_n-\alpha_{n-j}+r+1\right)}{\left|a_{jk}\binom{-\alpha_n}{r}\right| l\eta k!} \tag{3.14}$$

and $\eta = \max\{1, T^{N_j},\ j = 1,2,\ldots,n-1\}$. If $|t_1 - t_2| < \delta$,

$$
\begin{aligned}
&|Fz(t_1) - Fz(t_2)| \\
&= \Bigg| \sum_{j=1}^{n-1}\sum_{k=0}^{N_j}\sum_{r=0}^{k} \frac{\left|a_{jk}\binom{-\alpha_n}{r}\right| k! {t_1}^{k-r}}{(k-r)!\Gamma(\alpha_n - \alpha_{n-j} + r)} \int_0^{t_1} (t_1 - s)^{\alpha_n - \alpha_{n-j} + r - 1} z(s)\,ds \\
&\quad - \sum_{j=1}^{n-1}\sum_{k=0}^{N_j}\sum_{r=0}^{k} \frac{\left|a_{jk}\binom{-\alpha_n}{r}\right| k! {t_2}^{k-r}}{(k-r)!\Gamma(\alpha_n - \alpha_{n-j} + r)} \int_0^{t_2} (t_2 - s)^{\alpha_n - \alpha_{n-j} + r - 1} z(s)\,ds \Bigg| \\
&\quad + \frac{1}{\Gamma(\alpha_n)} \left| \int_0^{t_1} (t_1 - s)^{\alpha_n - 1} f(s, \overline{z}_s + y_s)\,ds - \int_0^{t_2} (t_2 - s)^{\alpha_n - 1} f(s, \overline{z}_s + y_s)\,ds \right| \\
&\le \Bigg| \sum_{j=1}^{n-1}\sum_{k=0}^{N_j}\sum_{r=0}^{k} \frac{\left|a_{jk}\binom{-\alpha_n}{r}\right| k! {t_2}^{k-r}}{(k-r)!\Gamma(\alpha_n - \alpha_{n-j} + r)} \int_0^{t_1} (t_2 - s)^{\alpha_n - \alpha_{n-j} + r - 1} z(s)\,ds \\
&\quad - \sum_{j=1}^{n-1}\sum_{k=0}^{N_j}\sum_{r=0}^{k} \frac{\left|a_{jk}\binom{-\alpha_n}{r}\right| k! {t_1}^{k-r}}{(k-r)!\Gamma(\alpha_n - \alpha_{n-j} + r)} \int_0^{t_1} (t_1 - s)^{\alpha_n - \alpha_{n-j} + r - 1} z(s)\,ds \Bigg| \\
&\quad + \sum_{j=1}^{n-1}\sum_{k=0}^{N_j}\sum_{r=0}^{k} \frac{\left|a_{jk}\binom{-\alpha_n}{r}\right| k! {t_2}^{k-r}}{(k-r)!\Gamma(\alpha_n - \alpha_{n-j} + r)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha_n - \alpha_{n-j} + r - 1} |z(s)|\,ds \\
&\quad + \frac{\|f\|_\infty}{\Gamma(\alpha_n)} \left\{ \int_0^{t_1} \left[(t_2 - s)^{\alpha_n - 1} - (t_1 - s)^{\alpha_n - 1} \right] ds + \int_{t_1}^{t_2} (t_2 - s)^{\alpha_n - 1} ds \right\} \\
&\le \sum_{j=1}^{n-1}\sum_{k=0}^{N_j}\sum_{r=0}^{k} \frac{\left|a_{jk}\binom{-\alpha_n}{r}\right| l k! T^{k-r}}{(k-r)!\Gamma(\alpha_n - \alpha_{n-j} + r)} \\
&\quad \times \int_0^{t_2} \left\{ (t_2 - s)^{\alpha_n - \alpha_{n-j} + r - 1} - (t_1 - s)^{\alpha_n - \alpha_{n-j} + r - 1} \right\} ds \\
&\quad + \frac{\|p\| + \|q\|(l + \|\phi\|)}{\Gamma(\alpha_n)} \left\{ \int_0^{t_1} \left[(t_2 - s)^{\alpha_n - 1} - (t_1 - s)^{\alpha_n - 1} \right] ds + \int_{t_1}^{t_2} (t_2 - s)^{\alpha_n - 1} ds \right\} \\
&\le \sum_{j=1}^{n-1}\sum_{k=0}^{N_j}\sum_{r=0}^{k} \frac{\left|a_{jk}\binom{-\alpha_n}{r}\right| 2 l k! \eta (t_2 - t_1)^{\alpha_n - \alpha_{n-j} + r}}{(k-r)!\Gamma(\alpha_n - \alpha_{n-j} + r + 1)} + \frac{2(\|p\| + \|q\|(l + \|\phi\|))(t_2 - t_1)^{\alpha_n}}{\Gamma(\alpha_n + 1)} \\
&= \sum_{j=1}^{n-1}\sum_{k=0}^{N_j}\sum_{r=0}^{k} \frac{\left|a_{jk}\binom{-\alpha_n}{r}\right| 2 l k! \eta \delta^{\alpha_n - \alpha_{n-j} + r}}{(k-r)!\Gamma(\alpha_n - \alpha_{n-j} + r + 1)} + \frac{2(\|p\| + \|q\|(l + \|\phi\|))\delta^{\alpha_n}}{\Gamma(\alpha_n + 1)} \\
&\le \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}
\tag{3.15}
$$

Hence FG is equicontiuous. The Arzela-Ascoli theorem implies that $\overline{F(G)}$ is compact and $F : K \to K$ is continuous and completely continuous.

Step 4. We now show that there exists an open set $U \subseteq K$ with $z \neq \lambda F(z)$ for $\lambda \in (0,1)$ and $z \in \partial U$. Let $z \in K$ be any solution of $z = \lambda Fz$, $\lambda \in (0,1)$, where F is given by (3.8); since $F : K \to K$ is continuous and completely continuous, we have

$$\begin{aligned} z(t) &= \lambda Fz(t) \\ &\leq \lambda \sum_{j=1}^{n-1} \sum_{k=0}^{N_j} \sum_{r=0}^{k} \left| a_{jk} \binom{-\alpha_n}{r} \right| \frac{k! t^{k-r}}{(k-r)! \Gamma(\alpha_n - \alpha_{n-j} + r)} \\ &\quad \times \int_0^t (t-s)^{\alpha_n - \alpha_{n-j} + r - 1} z(s) ds + \frac{\lambda}{\Gamma(\alpha_n)} \int_0^t (t-s)^{\alpha_n - 1} f(s, \overline{z}_s + y_s) ds \\ &\leq \lambda \sum_{j=1}^{n-1} \sum_{k=0}^{N_j} \sum_{r=0}^{k} \left| a_{jk} \binom{-\alpha_n}{r} \right| \frac{\|z\| k! t^{k-r}}{(k-r)! \Gamma(\alpha_n - \alpha_{n-j} + r)} \int_0^t (t-s)^{\alpha_n - \alpha_{n-j} + r - 1} ds \\ &\quad + \frac{\lambda}{\Gamma(\alpha_n)} \int_0^t (t-s)^{\alpha_n - 1} \left[p(s) + q(s) \left\| \overline{z}_s + y_s \right\| \right] ds \\ &\leq \lambda \left\{ \|z\| \mathcal{O}(T) + \|z\| I^{\alpha_n} q(t) + \|\phi\| I^{\alpha_n} q(t) + I^{\alpha_n} p(t) \right\}. \end{aligned} \tag{3.16}$$

So

$$\|z\| \left(1 - \lambda \mathcal{O}(T) - \lambda \left\| I^{\alpha_n} p \right\| \right) \leq \lambda \|\phi\| \left\| I^{\alpha_n} q \right\| + \lambda \left\| I^{\alpha_n} p \right\|. \tag{3.17}$$

Now, by (3.10) and (3.17), we know that any solution z of (3.8) satisfies $\|z\| \neq h$; let

$$U = \{ z \in K; \ \|z\| < h \}. \tag{3.18}$$

Therefore, Theorem 2.7 guarantees that (3.1) has at least a positive solution $z \in \overline{U}$. Hence, (1.4) has at least a positive solution $x^* \in K^*$, satisfying $\|x^*\| \leq \max\{\|\phi\|, h\}$ and the proof is complete.

□

Note that we can complete the above mentioned procedure by using only the continuity of f without condition (1), but with our procedure and details of condition (1) in Theorem 3.1 answers all the questions exist in the following remark.

Remark 3.2. When f is continuous on $(0,T] \times C$, $\lim_{t \to 0^+} f(t, \cdot) = +\infty$, (i.e., f is singular at $t = 0$) in (1.4). Suppose $\exists \sigma \in (0, \alpha_n]$, such that $t^\sigma f(t, x_t)$ is a continuous function on $[0,T] \times C$, then $I^{\alpha_n} f(t, x_t) = I^{\alpha_n} t^{-\sigma} t^\sigma f(t, x_t)$ is continuous on $I \times C$ by Lemma 2.1 in [12, page 613]. We also obtain results about the existence to (1.4) by using a nonlinear alternative of Leray-Schauder type. The proof is similar to that of Theorem 3.1 as long as we let

(1) $t^{\sigma} f(t, x_t) \le p(t) + q(t)\|x_t\|$, for $t \in I$, $x_t \in C$, and $\|I^{\alpha_n} t^{-\sigma} p\| < \infty$, $\|I^{\alpha_n} t^{-\sigma} q\| < \infty$,

(2) $1 - \mathcal{O}(T) - \|I^{\alpha_n} t^{-\sigma} q\| > 0$, then (1.4) has at least a positive solution $x^* \in K^*$, satisfying $\|x^*\| \le \max\{\|\phi\|, h\}$, where

$$h = \frac{\lambda \left\| I^{\alpha_n} t^{-\sigma} p \right\| + \lambda \|\phi\| \left\| I^{\alpha_n} t^{-\sigma} q \right\|}{1 - \lambda \mathcal{O}(T) - \lambda \left\| I^{\alpha_n} t^{-\sigma} q \right\|} + 1. \tag{3.19}$$

4. Unique Existence of Solution

In this section, we will give uniqueness of positive solution to (1.4).

Theorem 4.1. *Let $f : I \times C \to \mathbb{R}^+$ be continuous and $\lambda \in L^1([0,T], \mathbb{R}^+)$ with $\|I^{\alpha_n}\lambda\| < \infty$. Further assume*

(i) $|f(t, \overline{u}_t + y_t) - f(t, \overline{v}_t + y_t)| \le \lambda(t)\|\overline{u}_t - \overline{v}_t\|$, *for all* $u, v \in K$, $t \in [0,T]$,

(ii) $\mathcal{O}(T) + \|I^{\alpha_n}\lambda\| < 1$.

Then (1.4) *has unique solution which is positive, where* $\mathcal{O}(T)$ *is given in* (3.9).

Proof. Let $u, v \in K$. Then we obtain

$$\begin{aligned}
|Fu(t) - Fv(t)| &\le \sum_{j=1}^{n-1} \sum_{k=0}^{N_j} \sum_{r=0}^{k} \frac{\left| a_{jk} \binom{-\alpha_n}{r} \right| k! t^{k-r}}{(k-r)!} I^{\alpha_n - \alpha_{n-j} + r} |u(t) - v(t)| \\
&\quad + I^{\alpha_n} \left| f(t, \overline{u}_t + y + t) - f(t, \overline{v}_t + y_t) \right| \\
&\le \|u - v\|_T \left\{ \sum_{j=1}^{n-1} \sum_{k=0}^{N_j} \sum_{r=0}^{k} \frac{\left| a_{jk} \binom{-\alpha_n}{r} \right| k! t^{\alpha_n - \alpha_{n-j} + k}}{(k-r)! \Gamma(\alpha_n - \alpha_{n-j} + r + 1)} + I^{\alpha_n}\lambda(t) \right\} \\
&\le \|u - v\|_T \left\{ \sum_{j=1}^{n-1} \sum_{k=0}^{N_j} \sum_{r=0}^{k} \frac{\left| a_{jk} \binom{-\alpha_n}{r} \right| k! T^{\alpha_n - \alpha_{n-j} + k}}{(k-r)! \Gamma(\alpha_n - \alpha_{n-j} + r + 1)} + I^{\alpha_n}\lambda(t) \right\},
\end{aligned} \tag{4.1}$$

where F is given in (3.8). Hence,

$$\|Fu - Fv\|_T \le (\mathcal{O}(T) + \|I^{\alpha_n}\lambda\|)\|u - v\|_T. \tag{4.2}$$

In view of Banach fixed point theorem F has unique fixed point in K, which is the unique positive solution of (2.7) and (1.4) has a unique positive solution in K^*. □

Remark 4.2. When $\lambda(t) = L > 0$, then condition (i) reduces to the Lipschitz condition.

Example 4.3. Let $\lambda(t) = L > 0$ and $f(t, x_t) = Lx_t + e^t = Lx(t-\omega) + e^t$, $\omega > 0$. Consider the equation

$$\left(D^{1/2} - at^2 D^{1/4} - btD^{1/6} - cD^{1/8}\right)x = Lx(t-w) + e^t, \quad t \in (0,64],$$
$$x(t) = 0, \quad t \in [-w, 0]. \tag{4.3}$$

Then (4.3) is equivalent to the integral equation,

$$x(t) = \sum_{j=1}^{3}\sum_{k=0}^{N_j}\sum_{r=0}^{k} a_{jk}\begin{pmatrix} -\frac{1}{2} \\ r \end{pmatrix}\frac{k!t^{k-r}}{(k-r)!}I^{1/2-\alpha_{3-j}+r}x(t) + I^{1/2}\left(Lx(t-\omega) + e^t\right). \tag{4.4}$$

Here $\alpha_3 = 1/2$, $p_1(t) = \sum_{k=0}^{2} a_{1k}t^k = at^2$, then $N_1 = 2$, $a_{10} = a_{11} = 0$, $a_{12} = a$, $p_2(t) = \sum_{k=0}^{1} a_{2k}t^k = bt$, then $N_2 = 1$, $a_{20} = 0$, $a_{21} = b$ and $p_3(t) = \sum_{k=0}^{0} a_{3k}t^k = c$, then $N_3 = 0$, $a_{30} = c$. Hence

$$\begin{aligned}
x(t) = {} & a_{10}\begin{pmatrix} -\frac{1}{2} \\ 0 \end{pmatrix} I^{1/2-1/4}x + a_{11}\left[\begin{pmatrix} -\frac{1}{2} \\ 0 \end{pmatrix} tI^{1/2-1/4}x + \begin{pmatrix} -\frac{1}{2} \\ 1 \end{pmatrix} I^{1/2-1/4+1}\right] \\
& + a_{12}\left[\begin{pmatrix} -\frac{1}{2} \\ 0 \end{pmatrix} t^2 I^{1/2-1/4}x + 2\begin{pmatrix} -\frac{1}{2} \\ 1 \end{pmatrix} tI^{1/2-1/4+1}x + 2\begin{pmatrix} -\frac{1}{2} \\ 2 \end{pmatrix} I^{1/2-1/4+2}x\right] \\
& + a_{20}\begin{pmatrix} -\frac{1}{2} \\ 0 \end{pmatrix} I^{1/2-1/6}x + a_{21}\left[\begin{pmatrix} -\frac{1}{2} \\ 0 \end{pmatrix} tI^{1/2-1/6}x + \begin{pmatrix} -\frac{1}{2} \\ 1 \end{pmatrix} I^{1/2-1/6+1}x\right] \\
& + a_{30}\begin{pmatrix} -\frac{1}{2} \\ 0 \end{pmatrix} I^{1/2-1/8}x + LI^{1/2}x(t-\omega) + I^{1/2}e^t.
\end{aligned} \tag{4.5}$$

In view of (2.8) and that $\Gamma(1/2) = \sqrt{\pi}$, $\Gamma(-1/2) = -2\sqrt{\pi}$ and $\Gamma(-3/4) = 4\sqrt{\pi}/3$ we obtain

$$\begin{aligned}
x(t) = {} & a\left[t^2 I^{1/4}x(t) - tI^{5/4}x(t) + \frac{3}{4}I^{9/4}x(t)\right] \\
& + b\left[(1+t)I^{1/3}x(t) - \frac{1}{2}I^{4/3}x(t)\right] + cI^{3/8} + LI^{1/2}x(t-\omega) + I^{1/2}e^t.
\end{aligned} \tag{4.6}$$

If $|a| \le 3/5$, $|b| \le 2/5$, $|c| \le 1/5$, $0 < L < 4/5$ in the above equation satisfy the conditions required in Theorem 4.1, the iterated sequence is

$$\begin{aligned}
x_1(t) &= I^{1/2} e^t = t^{1/2} E_{1,3/2}(t), \\
x_2(t) &= \left[a\left(t^2 I^{1/4} - t I^{5/4} + \frac{3}{4} I^{9/4} \right) + b\left((1+t) I^{1/3} - \frac{1}{2} I^{4/3} \right) + c I^{3/8} + L I^{1/2} \right] x_1(t) + x_1(t), \\
x_{n+1}(t) &= \sum_{k=0}^{n} \left[a\left(t^2 I^{1/4} - t I^{5/4} + \frac{3}{4} I^{9/4} \right) + b\left((1+t) I^{1/3} - \frac{1}{2} I^{4/3} \right) + c I^{3/8} + L I^{1/2} \right]^{n-k} x_1(t),
\end{aligned} \tag{4.7}$$

for $n = 1, 2, 3, \ldots$, where $I^{\alpha} x_1 = t^{\alpha+1/2} E_{1,\alpha+3/2}(t)$, $\alpha > 0$, $x(t) = \lim_{n \to \infty} x_n(t)$ is the unique solution, which may not be positive, where $E_{\alpha,\beta}(t) = \sum_{k=0}^{\infty} (t^k / \Gamma(\alpha k + \beta))$ is Mittag-Leffler function.

Acknowledgment

The authors thank the referee's efforts for their remarks and Professor Ferhan Merdivenci Atici, Western Kentucky University, USA, for her regular contact.

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