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Research Article

On the Differential Operators with Periodic Matrix Coefficients

O. A. Veliev

Department of Mathematics, Dogus University, Acıbadem, Kadiköy, Istanbul, Turkey

Correspondence should be addressed to O. A. Veliev, oveliev@dogus.edu.tr

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We obtain asymptotic formulas for eigenvalues and eigenfunctions of the operator generated by a system of ordinary differential equations with summable coefficients and quasiperiodic boundary conditions. Then by using these asymptotic formulas, we find conditions on the coefficients for which the number of gaps in the spectrum of the self-adjoint differential operator with the periodic matrix coefficients is finite.

Let $L(P_2, P_3, \ldots, P_n)$ be the differential operator generated in the space $L_2^m(-\infty, \infty)$ of vector-valued functions by the differential expression

$$(-i)^n y^{(n)}(x) + (-i)^{n-2} P_2(x) y^{(n-2)}(x) + \sum_{v=3}^{n} P_v(x) y^{(n-v)}(x), \tag{1}$$

where n is an integer greater than 1 and $P_k(x)$, for $k = 2, 3, \ldots, n$, is the $m \times m$ matrix with the complex-valued summable entries $p_{k,i,j}(x)$ satisfying $p_{k,i,j}(x+1) = p_{k,i,j}(x)$ for all $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, m$. It is well known that (see [1–4]) the spectrum of the operator $L(P_2, P_3, \ldots, P_n)$ is the union of the spectra of the operators $L_t(P_2, P_3, \ldots, P_n)$ for $t \in [0, 2\pi)$ generated in $L_2^m(0,1)$ by expression (1) and the quasiperiodic conditions

$$U_\nu(y) \equiv y^{(\nu)}(1) - e^{it} y^{(\nu)}(0) = 0, \quad \nu = 0, 1, \ldots, (n-1). \tag{2}$$

Note that $L_2^m(a,b)$ is the set of vector-valued functions $f = (f_1, f_2, \ldots, f_m)$ with $f_k \in L_2(a,b)$ for $k = 1, 2, \ldots, m$. The norm $\|\cdot\|$ and inner product $(\cdot,\cdot)$ in $L_2^m(a,b)$ are defined by

$$\|f\|^2 = \int_a^b |f(x)|^2 dx, \qquad (f,g) = \int_a^b \langle f(x), g(x) \rangle dx, \tag{3}$$

where $|\cdot|$ and $\langle \cdot, \cdot \rangle$ are the norm and inner product in $\mathbb{C}^m$.

The first works concerned with the differential operator $L_t(P_2, P_3, \ldots, P_n)$ were by Birkhoff [5], Tamarkin [6] in the beginning of 20th century. There exist enormously many papers concerning with the operators $L_t(P_2, P_3, \ldots, P_n)$ and $L(P_2, P_3, \ldots, P_n)$. For the list of these papers one can look to the monographs [1, 7–10]. Here we only note that in these classical investigations in order to obtain the asymptotic formulas of high accuracy, by using the classical asymptotic expansions for solutions of the matrix equation

$$(-i)^n Y^{(n)} + (-i)^{n-2} P_2 Y^{(n-2)} + \sum_{v=3}^{n} P_v Y^{(n-v)} = \lambda Y, \tag{4}$$

it is required that the coefficients must be differentiable. Thus, these classical methods never permit us to obtain the asymptotic formulas of high accuracy for the operator $L_t(P)$ with nondifferentiable coefficients. However, the method suggested in this paper is independent of smootness of the coefficients. Using this method we obtain an asymptotic formulas of high accuracy for eigenvalues and eigenfunctions of the operator $L_t(P_2, P_3, \ldots, P_n)$ generated by a system of ordinary differential equations with only summable coefficients and then by using these formulas we consider the spectrum of the operator $L(P_2, P_3, \ldots, P_n)$.

Let us introduce some preliminary results and describe the results of this paper. Clearly,

$$\varphi_{k,1,t} = \left(e^{i(2\pi k+t)x}, 0, \ldots, 0\right), \varphi_{k,2,t} = \left(0, e^{i(2\pi k+t)x}, 0, \ldots, 0\right), \ldots, \varphi_{k,m,t} = \left(0, 0, \ldots, 0, e^{i(2\pi k+t)x}\right) \tag{5}$$

are the eigenfunctions of the operator $L_t(0)$ corresponding to the eigenvalue $(2\pi k + t)^n$, where $k \in \mathbb{Z}$, and the operator $L_t(P_2, \ldots, P_n)$ is denoted by $L_t(0)$ when $P_2(x) = 0, \ldots, P_n(x) = 0$. Furthermore, for brevity of notation, the operators $L_t(P_2, \ldots, P_n)$ and $L(P_2, \ldots, P_n)$ are denoted by $L_t(P)$ and $L(P)$, respectively. It easily follows from the classical investigations [7, Chapter 3, Theorem 2] that the large eigenvalues of the operator $L_t(P)$ consist of m sequences

$$\{\lambda_{k,1}(t) : |k| \geq N\}, \{\lambda_{k,2}(t) : |k| \geq N\}, \ldots, \{\lambda_{k,m}(t) : |k| \geq N\}, \tag{6}$$

satisfying the following, uniform with respect to t in $[0, 2\pi)$, asymptotic formulas:

$$\lambda_{k,j}(t) = (2\pi k + t)^n + O\left(k^{n-1-1/2m}\right) \tag{7}$$

for $j = 1, 2, \ldots, m$, where N is a sufficiently large positive number, that is, $N \gg 1$. We say that the formula $f(k,t) = O(h(k))$ is uniform with respect to t in a set S if there exists a positive constant c_1, independent of t, such that $|f(k,t))| < c_1 |h(k)|$ for all $t \in S$ and $k \in \mathbb{Z}$. Thus formula (7) means that there exist positive numbers N and c_1, independent of t, such that

$$\left|\lambda_{k,j}(t) - (2\pi k + t)^n\right| < c_1 |k|^{n-1-1/2m}, \quad \forall |k| \geq N, \ \forall t \in [0, 2\pi). \tag{8}$$

In this paper, by the suggested method, we obtain the uniform asymptotic formulas of high accuracy for the eigenvalues $\lambda_{k,j}(t)$ and for the corresponding normalized eigenfunctions $\Psi_{k,j,t}(x)$ of $L_t(P)$ when the entries $p_{2,i,j}(x), p_{3,i,j}(x), \ldots, p_{n,i,j}(x)$ of $P_2(x), P_3(x), \ldots, P_n(x)$ belong to $L_1[0,1]$, that is, when there is not any condition about smoothness of the coefficients. Then using these formulas, we find the conditions on the coefficient $P_2(x)$ for which the number of the gaps in the spectrum of the self-adjoint differential operator $L(P)$ is finite.

Now let us describe the scheme of the paper. Inequality (8) shows that the eigenvalue $\lambda_{k,j}(t)$ of $L_t(P)$ is close to the eigenvalue $(2k\pi + t)^n$ of $L_t(0)$. To analyze the distance of the eigenvalue $\lambda_{k,j}(t)$ of $L_t(P)$ from the other eigenvalues $(2p\pi + t)^n$ of $L_t(0)$, which is important in perturbation theory, we take into account the following situations. If the order n of the differential expression (1) is odd number, $n = 2r - 1$, and $|k| \gg 1$, then the eigenvalue $(2\pi k + t)^n$ of $L_t(0)$ lies far from the other eigenvalues $(2p\pi + t)^n$ of $L_t(0)$ for all values of $t \in [0, 2\pi)$. We have the same situation if $n = 2r$ and t does not lie in the small neighborhoods of 0 and π. However, if n is even number and t lies in the neighborhoods of 0 and π, then the eigenvalue $(2\pi k + t)^n$ is close to the eigenvalues $(2\pi(-k) + t)^n$ and $(2\pi(-k-1) + t)^n$, respectively. For this reason instead of $[0, 2\pi)$ we consider $t \in [-\pi/2, 3\pi/2)$ and use the following notation.

Notation 1.

Case 1. (a) $n = 2r - 1$ and $t \in [-\pi/2, 3\pi/2)$, (b) $n = 2r$ and $t \in T(k)$, where

$$T(k) = \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right) \setminus \left(\left(-\frac{1}{\ln|k|}, \frac{1}{\ln|k|}\right) \cup \left(\pi - \frac{1}{\ln|k|}, \pi + \frac{1}{\ln|k|}\right)\right). \tag{9}$$

Case 2. $n = 2r$ and $t \in (-(\ln|k|)^{-1}, (\ln|k|)^{-1})$.

Case 3. $n = 2r$ and $t \in (\pi - (\ln|k|)^{-1}, \pi + (\ln|k|)^{-1})$.

Denote by $A(k,n,t)$ the sets $\{k\}$, $\{k,-k\}$, $\{k,-k-1\}$ for Cases 1, 2, and 3, respectively. By (8) there exists a positive constant c_2, independent of t, such that the inequalities

$$\begin{gathered} \left|(2k\pi + t)^n - (2\pi p + t)^n\right| > c_2(\ln|k|)^{-1}\left(\left||k| - |p|\right| + 1\right)\left(|k| + |p|\right)^{n-1}, \\ \left|\lambda_{k,j}(t) - (2\pi p + t)^n\right| > c_2(\ln|k|)^{-1}\left(\left||k| - |p|\right| + 1\right)\left(|k| + |p|\right)^{n-1}, \end{gathered} \tag{10}$$

where $|k| > N$, hold in Cases 1, 2, and 3 for $p \neq k$, for $p \neq k, -k$, and for $p \neq k, -(k+1)$, respectively. To avoid the listing of these cases, using Notation 1, we see that the inequalities in (10) hold for $p \notin A(k,n,t)$. To obtain the asymptotic formulas we essentially use the following lemma that easily follows from (8) and (10).

Lemma 1. *The equalities*

$$\sum_{p:|p|>d} \frac{p^{n-2}}{\left|\lambda_{k,j}(t) - (2\pi p + t)^n\right|} = O\left(\frac{1}{d}\right), \tag{11}$$

$$\sum_{p:p \notin A(k,n,t)} \frac{p^{n-2}}{\left|\lambda_{k,j}(t) - (2\pi p + t)^n\right|} = O\left(\frac{\ln|k|}{k}\right), \tag{12}$$

$$\sum_{p:p \notin A(k,n,t)} \frac{k^{2n-4}}{\left|\lambda_{k,j}(t) - (2\pi p + t)^n\right|^2} = O\left(\frac{(\ln|k|)^2}{k^2}\right), \tag{13}$$

$$\sum_{p:p \notin A(k,n,t)} \frac{p^{2n-4}}{\left|\lambda_{p,j}(t) - (2\pi k + t)^n\right|^2} = O\left(\frac{(\ln|k|)^2}{k^2}\right) \tag{14}$$

hold uniformly with respect to t in $[-\pi/2, 3\pi/2)$, *where* $d \geq 2|k|$, $|k| \geq N \gg 1$.

Proof. The proof of (11). It follows from (8) that if $|p| > d \geq 2|k|$, then

$$\left|\lambda_{k,j}(t) - (2\pi p + t)^n\right| > |p|^n, \quad \forall t \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right). \tag{15}$$

Therefore the left-hand side of (11) is less than

$$\sum_{p:|p|>d} \frac{1}{p^2}, \quad \forall t \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right) \tag{16}$$

which is $O(1/d)$. Thus (11) holds uniformly with respect to $t \in [-\pi/2, 3\pi/2)$.

The proof of (12). The summation in the left-hand side of (12) is taking over all $p \in \mathbb{Z} \setminus A(k,n,t)$. Since

$$\mathbb{Z} \setminus A(k,n,t) = S(1) \cup S(2) \cup S(3), \tag{17}$$

where $S(1) = \{p : |p| > 2|k|\}$, $S(2) = \{p : |p| \leq 2|k|,\ p \notin A(k)\}$, $S(3) = \{p : p \in A(k) \setminus A(k,n,t)\}$, and $A(k) = \{\pm k, \pm(k+1), \pm(k+2)\}$, the left-hand side of (12) can be written as $S(9,1) + S(9,2) + S(9,3)$, where

$$\begin{aligned} S(9,1) &= \sum_{p:p \in S(1)} \frac{p^{n-2}}{\left|\lambda_{k,j}(t) - (2\pi p + t)^n\right|}, \\ S(9,2) &= \sum_{p:p \in S(2)} \frac{p^{n-2}}{\left|\lambda_{k,j}(t) - (2\pi p + t)^n\right|}, \\ S(9,3) &= \sum_{p:p \in S(3)} \frac{p^{n-2}}{\left|\lambda_{k,j}(t) - (2\pi p + t)^n\right|}. \end{aligned} \tag{18}$$

Taking $d = 2|k|$, from (11) we obtain that $S(9,1) = O(k^{-1})$. If $p \notin A(k)$, then using (8) one can readily see that

$$\begin{gathered}\left|\lambda_{k,j}(t) - (2\pi p + t)^n\right| > \left(\left||k| - |p|\right|\right)\left(|k| + |p|\right)^{n-1}, \\ \frac{\left|p^{n-2}\right|}{\left|\lambda_{k,j}(t) - (2\pi p + t)^n\right|} < \frac{1}{\left(\left||k| - |p|\right|\right)|k|}\end{gathered} \tag{19}$$

for all $t \in [-\pi/2, 3\pi/2)$. Let $s = ||k| - |p||$. Clearly, if $|p| \leq 2|k|$, $p \notin A(k)$, then $2 < s \leq |k|$ and the number s attains the same value at most 4 times. Therefore

$$|S(9,2)| \leq 4\frac{1}{|k|}\left(\sum_{s=3}^{k}\frac{1}{s}\right) = O\left(\frac{\ln|k|}{k}\right). \tag{20}$$

Since the set $S(3)$ has at most 6 elements, and for $p \notin A(k,n,t)$ the inequalities in (10) hold, we have

$$S(9,3) = O\left(\frac{\ln|k|}{k}\right). \tag{21}$$

Now, estimations for $S(9,1)$, $S(9,2)$, $S(9,3)$ imply (12).

The proofs of (13) *and* (14). The proofs of (13) and (14) are similar to the proof of (12). Namely, again we consider the left-hand sides of (13) and (14) as $S(10,1)+S(10,2)+S(10,3)$ and $S(11,1)+S(11,2)+S(11,3)$, respectively, where the summations in $S(10,i)$ and in $S(11,i)$ are taking over $p \in S(i)$ $(i = 1,2,3)$. Then, repeating the arguments by which we estimated $S(9,1)$, $S(9,2)$, and $S(9,3)$, we get

$$\begin{aligned}S(10,1) &= O\left(k^{-3}\right), & S(11,1) &= O\left(k^{-3}\right), \\ S(10,2) &= O\left(k^{-2}\right), & S(11,1) &= O\left(k^{-2}\right), \\ S(10,3) &= O\left(\frac{(\ln|k|)^2}{k^2}\right), & S(11,3) &= O\left(\frac{(\ln|k|)^2}{k^2}\right).\end{aligned} \tag{22}$$

These equalities imply the proof of (13) and (14). □

To obtain the asymptotic formulas we use (11)–(14) and consider the operator $L_t(P)$ as perturbation of $L_t(C)$ by $L_t(P) - L_t(C)$, where $C = \int_0^1 P_2(x)dx$, $L_t(C)$ is the operator generated by (2) and by the expression

$$(-i)^n y^{(n)}(x) + (-i)^{n-2} C y^{(n-2)}(x). \tag{23}$$

Therefore, first of all, we analyze the eigenvalues and eigenfunction of the operator $L_t(C)$. We assume that C is the Hermitian matrix. Then the expression in (23) is the self-adjoint expression. Since the boundary conditions (2) are self-adjoint, the operator $L_t(C)$ is also self-adjoint. The eigenvalues of C, counted with multiplicity, and the corresponding orthonormal eigenvectors are denoted by $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_m$ and $v_1, v_2, \ldots, v_m$. Thus

$$Cv_j = \mu_j v_j, \quad \langle v_i, v_j \rangle = \delta_{i,j}. \tag{24}$$

One can easily verify that the eigenvalues and eigenfunctions of $L_t(C)$ are $\mu_{k,j}(t) = (2\pi k + t)^n + \mu_j (2\pi k + t)^{n-2}$, $\Phi_{k,j,t}(x) = v_j e^{i(2\pi k+t)x}$, that is,

$$\left(L(C) - \mu_{k,j}(t)\right)\Phi_{k,j,t}(x) = 0. \tag{25}$$

To prove the asymptotic formulas for the eigenvalues $\lambda_{k,j}(t)$ and for the corresponding normalized eigenfunctions $\Psi_{k,j,t}$ of $L_t(P)$ we use the formula

$$\left(\lambda_{k,j}(t) - \mu_{p,s}(t)\right)\left(\Psi_{k,j,t}, \Phi_{p,s,t}\right) = (-i)^{n-2}\left((P_2 - C)\Psi_{k,j,t}^{(n-2)}, \Phi_{p,s,t}\right) + \sum_{\nu=3}^{n}\left(P_\nu \Psi_{k,j,t}^{n-\nu}, \Phi_{p,s,t}\right) \tag{26}$$

which can be obtained from

$$L(P)\Psi_{k,j,t}(x) = \lambda_{k,j}(t)\Psi_{k,j,t}(x) \tag{27}$$

by multiplying both sides by $\Phi_{p,s,t}(x)$ and using (25). Then we estimate the right-hand side of (26) (see Lemma 3) by using Lemma 2. At last, estimating $(\Psi_{k,j,t}, \Phi_{p,s,t})$ (see Lemma 4) and using these estimations in (26), we find the asymptotic formulas for the eigenvalues and eigenfunctions of $L_t(P)$ (see Theorems 5 and 6). Then using these formulas, we find the conditions on the eigenvalues of the matrix C for which the number of the gaps in the spectrum of the operator $L(P)$ is finite (see Theorem 7). Some of these results for differentiable $P_2(x)$ are obtained in [3, 11] by using the classical asymptotic expansions for the solutions of (4). The case $n = 2$ is investigated in [12]. In this case, an interesting spectral estimates were done in the paper [13], whose main goal was to reformulate some spectral problems for the differential operator with periodic matrix coefficients as problems of conformal mapping theory. In this paper we consider the more complicated case $n > 2$.

To estimate the right-hand side of (26) we use (11), (12), the following lemma, and the formula

$$\left(\lambda_{k,j}(t) - (2\pi p + t)^n\right)\left(\Psi_{k,j,t}, \varphi_{p,s,t}\right) = (-i)^{n-2}\left(P_2 \Psi_{k,j,t}^{n-2}, \varphi_{p,s,t}\right) + \sum_{\nu=3}^{n}\left(P_\nu \Psi_{k,j,t}^{n-\nu}, \varphi_{p,s,t}\right) \tag{28}$$

which can be obtained from (27) by multiplying both sides by $\varphi_{p,s,t}$ and using $L_t(0)\varphi_{p,s,t} = (2\pi p + t)^n \varphi_{p,s,t}$.

Lemma 2. *Let $\Psi_{k,j,t}(x)$ be normalized eigenfunction of $L_t(P)$. Then*

$$\sup_{x\in 0,1]}\left|\Psi^{(\nu)}_{k,j,t}(x)\right| = O(k^{\nu}) \tag{29}$$

for $\nu = 0,1,\dots,n-2$. Equality (29) *is uniform with respect to t in $[-\pi/2, 3\pi/2)$.*

Proof. To prove (29) we use the arguments of the proof of the asymptotic formulas (6) and take into consideration the uniformity with respect to t. The eigenfunction $\Psi_{k,j,t}$ corresponding to the eigenvalue $\lambda_{k,j}(t)$ has the form

$$\Psi_{k,j,t}(x) = Y_1(x,\rho_{k,j})a_1 + Y_2(x,\rho_{k,j})a_2 + \cdots + Y_n(x,\rho_{k,j})a_n, \tag{30}$$

where $a_k \in \mathbb{C}^m$, $\rho_{k,j}(t) = i(\lambda_{k,j}(t))^{1/n}$, $Y_s(x,\rho_{k,j}(t))$ for $s = 1,2,\dots,n$ are linearly independent $m \times m$ matrix solutions of (4) for $\lambda = \lambda_{k,j}(t)$ satisfying

$$\frac{d^{\nu}Y_s(x,\rho_{k,j}(t))}{dx^{\nu}} = (\rho_{k,j}(t))^{\nu} e^{\rho_{k,j}(t)\omega_s x}\left[\omega_s^{\nu} I + O\left(\frac{1}{k}\right)\right] \tag{31}$$

for $\nu = 0,1,\dots,(n-1)$. Here I is unit matrix, $\omega_1,\omega_2,\dots,\omega_n$ are the nth root of 1, and $O(1/k)$ is an $m \times m$ matrix satisfying the following conditions:

$$O\left(\frac{1}{k}\right) = \frac{A(x,t,k)}{k}, \quad |A(x,t,k)| < \frac{c_3}{|k|}, \quad \forall x \in [0,1],\ \forall t \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right), \tag{32}$$

where $k > N$ and c_3 is a positive constant, independent of t. To consider the uniformity, with respect to t, of (29) we use (32).

The proof of (29) *in the case $n = 2r-1$, $r > 1$.* Denote by $(\lambda_{k,j}(t))^{1/n}$ the root of $\lambda_{k,j}(t)$ lying in $O(k^{-1/2m})$ neighborhood of $(2k\pi + t)$ and put $\rho_{k,j}(t) = i(\lambda_{k,j}(t))^{1/n}$. Then we have

$$\rho_{k,j}(t) = (2k\pi + t)i + O\left(k^{-1/2m}\right). \tag{33}$$

Suppose $\omega_1,\omega_2,\dots,\omega_n$ are ordered in such a way that

$$\omega_r = 1, \quad \mathcal{R}(\rho_{k,j}(t)\omega_s) < 0, \quad \forall s < r, \quad \mathcal{R}(\rho_{k,j}(t)\omega_s) > 0, \quad \forall s > r, \tag{34}$$

where $\mathcal{R}(z)$ is the real part of z. Using (31), (34), (2), and (33), we get

$$Y_s^{(\nu-1)}(1,\rho_{k,j}(t)) = (\rho_{k,j}(t))^{\nu-1}e^{\rho_{k,j}(t)\omega_s}\left[\omega_s^{\nu-1}I\right], \qquad Y_s^{(\nu-1)}(0,\rho_{k,j}(t)) = (\rho_{k,j}(t))^{\nu-1}\left[\omega_s^{\nu-1}I\right], \tag{35}$$

$$\begin{aligned} U_\nu(Y_s(x,\rho_{k,j}(t))) &= -(\rho_{k,j}(t))^{\nu-1}e^{it}\left[\omega_s^{\nu-1}I\right], \quad \forall s<r, \\ U_\nu(Y_s(x,\rho_{k,j}(t))) &= (\rho_{k,j}(t))^{\nu-1}e^{\rho_{k,j}(t)\omega_s}\left[\omega_s^{\nu-1}I\right], \quad \forall s>r, \end{aligned} \tag{36}$$

$$U_\nu(Y_r(x,\rho_{k,j}(t))) = (\rho_{k,j}(t))^{\nu-1}O\left(k^{-1/2m}\right), \tag{37}$$

where $[\omega_s^{\nu-1}I] = \omega_s^{\nu-1}I + O(1/k)$ and $O(1/k)$ satisfies the relation (32). Now using these relations and the notations of (30), we prove that

$$Y_s(x,\rho_{k,j}(t))a_s = O\left((|a_r|)k^{-1/2m}\right), \quad \forall s \neq r. \tag{38}$$

Since $\Psi_{k,j,t}(x)$ satisfies (2) and (30), we have the system of equations

$$\sum_{s\neq r} U_\nu(Y_s(x,\rho_{k,j}(t)))a_s = -U_\nu(Y_r(x,\rho_{k,j}(t)))a_r, \quad \nu = 0,1,\ldots,(n-2) \tag{39}$$

with respect to $a_{s,q}$ for $s \neq r$ and $q = 1,2,\ldots,m$, where $a_{s,q}$ are coordinates of the vector a_s. Using (36) and (37) in (39) and then dividing both parts of $(\nu+1)$th equation of (39), for $\nu = 0,1,\ldots,(n-2)$, by $(\rho_{k,j}(t))^\nu$, we get the system of equations whose coefficient matrix A is

$$\begin{pmatrix} -e^{it}[I] & \cdots & -e^{it}[I] & e^{\rho_{k,j}\omega_{r+1}}[I] & \cdots & e^{\rho_{k,j}\omega_n}[I] \\ -e^{it}[\omega_1 I] & \cdots & -e^{it}[\omega_{r-1}I] & e^{\rho_{k,j}\omega_{r+1}}[\omega_{r+1}I] & \cdots & e^{\rho_{k,j}\omega_n}[\omega_n I] \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -e^{it}\left[\omega_1^{n-2}I\right] & \cdots & -e^{it}\left[\omega_{r-1}^{n-2}I\right] & e^{\rho_{k,j}\omega_{r+1}}\left[\omega_{r+1}^{n-2}I\right] & \cdots & e^{\rho_{k,j}\omega n}\left[\omega_n^{n-2}I\right] \end{pmatrix}, \tag{40}$$

and the right-hand side is $O((|a_r|)k^{-1/2m})$. To estimate $\det A$ let us denote by $\widetilde{A}(m)$ the matrix obtained from A by replacing $[\omega_s^j I]$ with $\omega_s^j I$ and by dividing the sth column (note that the entries of the sth column are the $m \times m$ matrices) for $s<r$ and for $s>r$ by $-e^{it}$ and by $e^{\rho_{k,j}\omega_s}$, respectively. Clearly,

$$\widetilde{A}(1) = \begin{pmatrix} 1 & \cdots & 1 & 1 & \cdots & 1 \\ \omega_1 & \cdots & \omega_{r-1} & \omega_{r+1} & \cdots & \omega_n \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \omega_1^{n-2} & \cdots & \omega_{r-1}^{n-2} & \omega_{r+1}^{n-2} & \cdots & \omega_n^{n-2} \end{pmatrix} \tag{41}$$

and $\det \widetilde{A}(1) \neq 0$. Besides, interchanging the rows and then interchanging the columns of $\widetilde{A}(m)$, we obtain $\det \widetilde{A}(m) = (\widetilde{A}(1))^m$. Using this and solving (39) by Cramer's rule, we get

$$a_{s,q} = \frac{\det A_{s,q}}{\det A} = O\Big((|a_r|)e^{-\rho_{k,j}\omega_s}k^{-1/2m}\Big), \quad \forall s > r, \tag{42}$$

since $A_{s,q}$ is obtained from A by replacing the $((s-1)m+q)$th column of A, which is the qth column of

$$\begin{pmatrix} e^{\rho_{k,j}\omega_s}[I] \\ e^{\rho_{k,j}\omega_s}[\omega_k I] \\ \vdots \\ e^{\rho_{k,j}\omega_s}\left[\omega_k^{n-2}\right] \end{pmatrix}, \quad \text{with} \quad \begin{pmatrix} O\left(|a_r|k^{-1/2m}\right) \\ O\left(|a_r|k^{-1/2m}\right) \\ \vdots \\ O\left(|a_s|k^{-1/2m}\right) \end{pmatrix}. \tag{43}$$

In the same way, we obtain

$$a_{s,q} = O\Big((|a_r|)k^{-1/2m}\Big), \quad \forall s < r. \tag{44}$$

Now (38) follows from (44), (42), and (34). Therefore, the normalization condition $\|\Psi_{k,j,t}\| = 1$, and (38), (30), (31), (33), and (34) imply that

$$\Psi_{k,j,t}(x) = \left(Y_r\left(x, \rho_{k,j}(t)\right)\right)a_r + O\Big(k^{-1/2m}\Big) = e^{i(2k\pi+t)x}a_r + O\Big(k^{-1/2m}\Big), \tag{45}$$

where $|a_r|^2 = 1 + O(k^{-1/2m})$, from which we get the proof of (29) for $\nu = 0$. Differentiating both sides of (30) and using (42) and (44), we get the proof of (29) for arbitrary ν in the case $n = 2r - 1$.

The proof of (29) *in the case* $n = 2r$. In this case the nth roots $\omega_1, \omega_2, \ldots, \omega_n$ of 1 are ordered in such a way that

$$\omega_r = 1, \quad \omega_{r+1} = -1, \quad \mathcal{R}\left(\rho_{k,j}\omega_s\right) < 0, \quad \forall s < r; \quad \mathcal{R}\left(\rho_{k,j}\omega_s\right) > 0, \quad \forall s > r+1. \tag{46}$$

Hence we have

$$\begin{gathered} U_\nu\left(Y_s\left(x, \rho_{k,j}(t)\right)\right) = -\left(\rho_{k,j}(t)\right)^{\nu-1}e^{it}\Big[\omega_s^{\nu-1}I\Big], \quad \forall s < r, \\ U_\nu\left(Y_s\left(x, \rho_{k,j}(t)\right)\right) = \left(\rho_{k,j}(t)\right)^{\nu-1}e^{\rho_{k,j}(t)\omega_s}\Big[\omega_s^{\nu-1}I\Big], \quad \forall s > r+1. \end{gathered} \tag{47}$$

Now using these equalities, we prove that

$$Y_s\left(x, \rho_{k,j}(t)\right)a_s = O\Big((|a_r| + |a_{r+1}|)k^{-1/2m}\Big), \quad \forall s \neq r, r+1. \tag{48}$$

Using (47) and arguing as in the case $n = 2r - 1$, we get the system of equations

$$\sum_{s \neq r,r+1} U_\nu(Y_s(x,\rho_{k,j}(t)))a_s = -\sum_{s=r,r+1} U_\nu(Y_s(x,\rho_{k,j}(t)))a_s \tag{49}$$

for $\nu = 0,1,2,\ldots,(n-3)$. Arguing as in the proof of (42)–(45) and using (46), we get

$$\begin{gathered} a_{s,q} = O\Big((|a_r| + |a_{r+1}|)e^{-\rho_{k,j}\omega_s}k^{-1/2m}\Big), \quad \forall s > r+1, \\ a_{s,q} = O\Big((|a_r| + |a_{r+1}|)k^{-1/2m}\Big), \quad \forall s < r, \\ \Psi_{k,j,t}(x) = e^{i(2k\pi+t)x}a_r + e^{-i(2k\pi+t)x}a_{r+1} + O\Big(k^{-1/2m}\Big), \end{gathered} \tag{50}$$

where $|a_r|^2 + |a_{r+1}|^2 = 1 + O(k^{-1/2m})$, which implies the proof of (29) in the case $n = 2r$. □

It follows from this lemma that the equalities

$$\Big(P_\nu \Psi^{n-\nu}_{k,j,t}, \varphi_{p,s,t}\Big) = O(k^{n-\nu}), \qquad \Big(P_\nu \Psi^{n-\nu}_{k,j,t}, \Phi_{p,s,t}\Big) = O(k^{n-\nu}) \tag{51}$$

for $\nu = 2,3,\ldots,n$ and for $j = 1,2,\ldots,m$ hold uniformly with respect to t in $[-\pi/2, 3\pi/2)$. Now (51) together with (28) implies that

$$\big|(\Psi_{k,j,t}, \varphi_{p,s,t})\big| \le \frac{c_4|k|^{n-2}}{\big|\lambda_{k,j}(t) - (2\pi p + t)^n\big|} \tag{52}$$

for $p \notin A(k,n,t)$, $|k| \ge N$, and $s, j = 1,2,\ldots,m$, where c_4 is a positive constant, independent of t. Using this we prove the following lemma.

Lemma 3. *Let $b_{s,q}(x)$ be the entries of $P_2(x)$ and $b_{s,q,p} = \int_0^1 b_{s,q}(x)e^{-2\pi i p x}dx$. Then*

$$\Big(\Psi^{(n-2)}_{k,j,t}, P_2\varphi_{p,s,t}\Big) = \sum_{q=1,2,\ldots m; l\in\mathbb{Z}} b_{s,q,p-l}\Big(\Psi^{(n-2)}_{k,j,t}, \varphi_{l,q,t}\Big), \tag{53}$$

$$\Big(\Psi^{(n-2)}_{k,j,t}, (P_2 - C)\Phi_{p,s,t}\Big) = O\Big(k^{n-3}\ln|k|\Big) + O\Big(k^{n-2}b_k\Big) \tag{54}$$

for $p \in A(k,n,t)$ and $s = 1,2,\ldots,m$, where

$$b_k = \max\big\{\big|b_{i,j,p}\big| : i,j = 1,2,\ldots,m;\ p = 2k, -2k, 2k+1, -2k-1\big\}, \tag{55}$$

and C is the Hermitian matrix defined in (23). Formula (54) is uniform with respect to t in $[-\pi/2, 3\pi/2)$. Moreover, in Case 1 of Notation 1 the formula

$$\Big(\Psi^{(n-2)}_{k,j,t}, (P_2 - C)\Phi_{k,s,t}\Big) = O\Big(k^{n-3}\ln|k|\Big) \tag{56}$$

holds. If $n = 2r - 1$, then (56) is uniform with respect to t in $[-\pi/2, 3\pi/2)$.

Proof. Note that if the entries $b_{s,q}$ of P_2 belong to $L_2[0,1]$, then (53) is obvious, since $\{\varphi_{l,q,t} : l \in \mathbb{Z},\ q = 1,2,\ldots,m\}$ is an orthonormal basis in $L_2^m[0,1]$. Now we prove (53) in case $b_{s,q} \in L_1[0,1]$. Using (2), (52), and the integration by parts, we see that there exists a constant c_5, independent of t, such that

$$\left|\left(\Psi_{k,j,t}^{(n-2)}, \varphi_{l,q,t}\right)\right| = \left|(2\pi l + t)^{n-2}\left(\Psi_{k,j,t}, \varphi_{l,q,t}\right)\right| \leq \frac{c_5|k|^{n-2}|l|^{n-2}}{\left|\lambda_{k,j}(t) - (2\pi l + t)^n\right|}, \tag{57}$$

for $l \notin A(k,n,t)$, $|k| \geq N$. This and (11) imply that there exists a constant c_6, independent of t, such that

$$\sum_{l:|l|>d}\left|\left(\Psi_{k,j,t}^{(n-2)}, \varphi_{l,q,t}\right)\right| < \frac{c_6|k|^{n-2}}{d}, \tag{58}$$

where $d \geq 2|k|$, $t \in [-\pi/2, 3\pi/2)$. Hence the decomposition of $\Psi_{k,j,t}^{(n-2)}$ by the basis $\{\varphi_{l,q,t} : l \in \mathbb{Z},\ q = 1,2,\ldots,m\}$ has the form

$$\begin{gathered}\Psi_{k,j,t}^{(n-2)}(x) = \sum_{|l|\leq d; q=1,2,\ldots,m}\left(\Psi_{k,j,t}^{(n-2)}, \varphi_{l,q,t}\right)\varphi_{l,q,t}(x) + g_d(x), \\ \text{where } \sup_{x\in 0,1]}\left|g_d(x)\right| < \frac{c_6|k|^{n-2}}{d}.\end{gathered} \tag{59}$$

Using (59) in $(\Psi_{k,j,t}^{(n-2)}, P_2\varphi_{p,s,t})$ and letting d tend to ∞, we obtain (53).

Since $\Phi_{p,s,t}(x) \equiv v_s e^{i(2\pi p+t)x}$, to prove (54), it is enough to show that

$$\left(\Psi_{k,j,t}^{(n-2)}, (P_2 - C)\varphi_{p,s,t}\right) = O\left(k^{n-3}\ln|k|\right) + O\left(k^{n-2}b_k\right) \tag{60}$$

for $s = 1,2,\ldots,m$ and $p \in A(k,n,t)$. Using the obvious relation

$$\left(\Psi_{k,j,t}^{(n-2)}, C\varphi_{p,s,t}\right) = \sum_{q=1,2,\ldots,m} b_{s,q,0}\left(\Psi_{k,j,t}^{(n-2)}, \varphi_{p,q,t}\right) \tag{61}$$

and (53), we see that

$$\begin{aligned}\left(\Psi_{k,j,t}^{(n-2)}, (P_2 - C)\varphi_{p,s,t}\right) = &\sum_{l:l\in A(k,n,t)\setminus p; q=1,2,\ldots,m} b_{s,q,p-l}\left(\Psi_{k,j,t}^{(n-2)}, \varphi_{l,q,t}\right) \\ &+ \sum_{l:l\notin A(k,n,t); q=1,2,\ldots,m} b_{s,q,p-l}\left(\Psi_{k,j,t}^{(n-2)}, \varphi_{l,q,t}\right).\end{aligned} \tag{62}$$

Since

$$\left|b_{j,i,s}\right| \leq \max_{p,q=1,2,\ldots,m} \int_0^1 \left|b_{p,q}(x)\right| dx = O(1) \tag{63}$$

for all j, i, s, using (57) and (12), we see that the second summation of the right-hand side of (62) is $O((k^{n-3} \ln|k|)$. Besides, it follows from (29) and (55) that the first summation of the right-hand side of (62) is $O(k^{n-2} b_k)$, since for $p \in A(k,n,t)$ and $l \in A(k,n,t) \setminus p$, we have $p - l \in \{2k, -2k, 2k+1, -2k-1\}$. Hence (54) is proved. In Case 1 of Notation 1 the first summation of the right-hand side of (62) is absent, since in this case $A(k,n,t) = \{k\}$ and $A(k,n,t) \setminus p = \emptyset$ for $p \in A(k,n,t)$. Thus (56) is proved. The uniformity of formulas (54) and (56) follows from the uniformity of (29), (11), and (12). □

Lemma 4. *There exists a positive number N_0, independent of t, such that for $|k| > N_0$ and for $p \in A(k,n,t)$ the following assertions hold.*

(a) *If C is Hermitian matrix, then for each eigenfunction $\Psi_{k,j,t}$ of $L_t(P)$ there exists an eigenfunction $\Phi_{p,s,t}$ of $L_t(C)$ satisfying*

$$\left|\left(\Psi_{k,j,t}, \Phi_{p,s,t}\right)\right| > \frac{1}{3m}. \tag{64}$$

(b) *If $L_t(P)$ is self-adjoint operator, then for each eigenfunction $\Phi_{k,j,t}$ of $L_t(C)$ there exists an eigenfunction $\Psi_{p,s,t}$ of $L_t(P)$ satisfying*

$$\left|\left(\Phi_{k,j,t}, \Psi_{p,s,t}\right)\right| > \frac{1}{3m}. \tag{65}$$

Proof. It follows from (52) and (13) that

$$\sum_{s=1,2,\ldots,m} \left(\sum_{p:p \notin A(k,n,t)} \left|\left(\Psi_{k,j,t}, \varphi_{p,s,t}\right)\right|^2 \right) = O\left(\frac{(\ln|k|)^2}{k^2} \right). \tag{66}$$

Hence using the equality $\Phi_{p,s,t}(x) = v_s e^{i(2\pi p + t)x}$, where v_s is the normalized eigenvectors of C, and the Parseval equality, we get

$$\sum_{s=1,2,\ldots,m} \left(\sum_{p:p \notin A(k,n,t)} \left|\left(\Psi_{k,j,t}, \Phi_{p,s,t}\right)\right|^2 \right) = O\left(\frac{(\ln|k|)^2}{k^2} \right), \tag{67}$$

$$\sum_{s=1,2,\ldots,m;p\in A(k,n,t)} \left|\left(\Psi_{k,j,t}, \Phi_{p,s,t}\right)\right|^2 = 1 + O\left(\frac{(\ln|k|)^2}{k^2} \right). \tag{68}$$

Since the number of the eigenfunctions $\Phi_{p,s,t}(x)$ for $p \in A(k,n,t)$, $s = 1,2,\ldots,m$ is less than $2m$ (see Notation 1), (64) follows from (68).

Using (52) and (14), we get

$$\sum_{s=1,2,\ldots,m}\left(\sum_{p:p\notin A(k,n,t)}\left|\left(\varphi_{k,j,t},\Psi_{p,s,t}\right)\right|^{2}\right)=O\left(\frac{(\ln|k|)^{2}}{k^{2}}\right). \tag{69}$$

Therefore, arguing as in the proof of (64) and taking into account that the eigenfunctions of the self-adjoint operator $L_t(P)$ form an orthonormal basis in $L_2^m(0,1)$, we get the proof of (65). □

Theorem 5. *Let $L_t(P)$ be a self-adjoint operator, and let C be a Hermitian matrix. If $n=2r-1$, then for arbitrary t, if $n=2r$, then for $t\neq 0,\pi$ the large eigenvalues of $L_t(P)$ consist of m sequences* (6) *satisfying*

$$\lambda_{k,j}(t)=(2\pi k+t)^{n}+\mu_{j}(2\pi k+t)^{n-2}+O\left(k^{n-3}\ln|k|\right), \tag{70}$$

and the normalized eigenfunction $\Psi_{k,j,t}$ corresponding to $\lambda_{k,j}(t)$ satisfies

$$\left\|\Psi_{k,j,t}-E\Psi_{k,j,t}\right\|=O\left(\frac{(\ln|k|)}{k}\right) \tag{71}$$

for $j=1,2,\ldots,m$, where $\mu_1\leq\mu_2\leq\cdots\leq\mu_m$ are the eigenvalues of C and E is the orthogonal projection onto the eigenspace of $L_t(C)$ corresponding to $\mu_{k,j}(t)$. If μ_j is a simple eigenvalue of C, then the eigenvalue $\lambda_{k,j}(t)$ satisfying (70) *is a simple eigenvalue, and the corresponding eigenfunction satisfies*

$$\Psi_{k,j,t}(x)=v_{j}e^{i(2\pi k+t)x}+O\left(\frac{(\ln|k|)}{k}\right), \tag{72}$$

where v_j is the eigenvector of C corresponding to the eigenvalue μ_j. In the case $n=2r-1$ the formulas (70)–(72) *are uniform with respect to t in $[-\pi/2,3\pi/2)$.*

Proof. By (51) and (56) the right-hand side of (26) is $O(k^{n-3}\ln|k|)$. On the other hand by Notation 1 if $t\neq 0,\pi$, then there exists N such that $t\in T(k)$, and hence $A(k,n,t)=\{k\}$, for $|k|\geq N$. Thus dividing (26) by $(\Psi_{k,j,t},\Phi_{p,s,t})$, where $p\in A(k,n,t)$, and hence $p=k$, and using (64), we get

$$\{\lambda_{k,1}(t),\lambda_{k,2}(t),\ldots,\lambda_{k,m}(t)\}\subset\bigcup_{j=1}^{m}\left(U\left(\mu_{k,j}(t),\delta_{k}\right)\right), \tag{73}$$

where $U(\mu,\delta)=\{z\in\mathbb{R}:|\mu-z|<\delta\}$, $|k|\geq\max\{N,N_0\}$, $\delta_k=O(|k|^{n-3}\ln|k|)$. Instead of (64) using (65), in the same way, we obtain

$$U\left(\mu_{k,s}(t),\delta_{k}\right)\cap\{\lambda_{k,1}(t),\lambda_{k,2}(t),\ldots,\lambda_{k,m}(t)\}\neq\emptyset \tag{74}$$

for $|k| \geq \max\{N, N_0\}$ and $s = 1, 2, \ldots, m$. Hence to prove (70) we need to show that if the multiplicity of the eigenvalue μ_j is q then there exist precisely q eigenvalues of $L_t(P)$ lying in $U(\mu_{k,j}(t), \delta_k)$ for $|k| \geq \max\{N, N_0\}$. The eigenvalues of $L_t(P)$ and $L_t(C)$ can be numbered in the following way: $\lambda_{k,1}(t) \leq \lambda_{k,2}(t) \leq \cdots \leq \lambda_{k,m}(t)$ and $\mu_{k,1}(t) \leq \mu_{k,2}(t) \leq \cdots \leq \mu_{k,m}(t)$. If C has ν different eigenvalues $\mu_{j_1}, \mu_{j_2}, \ldots, \mu_{j_\nu}$ with multiplicities $j_1, j_2 - j_1, \ldots, j_\nu - j_{\nu-1}$, then we have

$$\begin{gathered} j_1 < j_2 < \cdots < j_\nu = m, \qquad \mu_{j_1} < \mu_{j_2} < \cdots < \mu_{j_\nu}, \qquad \mu_1 = \mu_2 = \cdots = \mu_{j_1}, \\ \mu_{j_1+1} = \mu_{j_1+2} = \cdots = \mu_{j_2}, \ldots, \qquad \mu_{j_{\nu-1}+1} = \mu_{j_{\nu-2}+2} = \cdots = \mu_{j_\nu}. \end{gathered} \tag{75}$$

Suppose there exist precisely $s_1, s_2, \ldots, s_\nu$ eigenvalues of $L_t(P)$ lying in the intervals

$$U(\mu_{k,j_1}(t), \delta_k), U(\mu_{k,j_2}(t), \delta_k), \ldots, U(\mu_{k,j_\nu}(t), \delta_k), \tag{76}$$

respectively. Since

$$\delta_k \ll \left(\min_{p=1,2,\ldots,\nu-1} \left| \left(\mu_{j_p+1} - \mu_{j_p} \right) (2\pi k + t)^{n-2} \right| \right) \quad \text{for } |k| \gg 1, \tag{77}$$

these intervals are pairwise disjoints. Therefore using (6) and (7), we get

$$s_1 + s_2 + \cdots + s_\nu = m. \tag{78}$$

Now let us prove that $s_1 = j_1, s_2 = j_2 - j_1, \ldots, s_\nu = j_\nu - j_{\nu-1}$. Due to the notations the eigenvalues $\lambda_{k,1}(t), \lambda_{k,2}(t), \ldots, \lambda_{k,s_1}(t)$ of the operator $L_t(P)$ lie in $U(\mu_{k,1}(t), \delta_k)$ and by the definition of δ_k we have

$$\left| \lambda_{k,j}(t) - \mu_{k,s}(t) \right| > \frac{1}{2} \left(\min_{p:p>j_1} \left| (\mu_1 - \mu_p)(2\pi k + t)^{n-2} \right| \right) \tag{79}$$

for $j \leq s_1$ and $s > j_1$. Hence using (26) for $p = k$ and (56), (51), we get

$$\sum_{s:s>j_1} \left| (\Psi_{k,j,t}, \Phi_{k,s,t}) \right|^2 = O\left(\frac{(\ln|k|)^2}{k^2} \right), \quad \forall j \leq s_1. \tag{80}$$

Using this, (67), and taking into account that $A(k, n, t) = \{k\}$ for $|k| \geq N$, we conclude that there exists normalized eigenfunction, denoted by $\Phi_{k,j,t}(x)$, of $L_t(P)$ corresponding to $\mu_{k,1}(t) = \mu_{k,2}(t) = \cdots = \mu_{k,j_1}(t)$ such that

$$\Psi_{k,j,t}(x) = \Phi_{k,j,t}(x) + O\left(k^{-1} \ln|k| \right) \tag{81}$$

for $j \leq s_1$. Since $\Psi_{k,1,t}, \Psi_{k,2,t}, \ldots, \Psi_{k,s_1,t}$ are orthonormal system we have

$$\left(\Phi_{k,j,t}, \Phi_{k,s,t} \right) = \delta_{s,j} + O\left(k^{-1} \ln|k| \right), \quad \forall s, j = 1, 2, \ldots, s_1. \tag{82}$$

This formula implies that the dimension j_1 of the eigenspace of $L_t(C)$ corresponding to the eigenvalue $\mu_{k,1}(t)$ is not less than s_1. Thus $s_1 \leq j_1$. In the same way we prove that $s_2 \leq j_2 - j_1, \ldots, s_\nu \leq j_\nu - j_{\nu-1}$. Now (78) and the equality $j_\nu = m$ (see (75)) imply that $s_1 = j_1, s_2 = j_2 - j_1, \ldots, s_\nu = j_\nu - j_{\nu-1}$. Therefore, taking into account that, the eigenvalues of $L_t(P)$ consist of m sequences satisfying (7), we get (70). The proof of (71) follows from (81).

Now suppose that μ_j is a simple eigenvalue of C. Then $\mu_{k,j}(t)$ is a simple eigenvalues of $L_t(C)$ and, as it was proved above, there exists unique eigenvalues $\lambda_{k,j}(t)$ of $L_t(P)$ lying in $U(\mu_{k,s}(t), \delta_k)$, where $|k| \geq \max\{N, N_0\}$, and the eigenvalues $\lambda_{k,j}(t)$ for $|k| \geq \max\{N, N_0\}$ are the simple eigenvalues. Hence (72) is the consequence of (71), since there exists unique eigenfunction $\Phi_{k,j,t}(x) = v_j e^{i(2\pi k+t)x}$ corresponding to the eigenvalue $\mu_{k,j}(t)$. The uniformity of the formulas (70)–(72) follows from the uniformity of (56), (51), (64), and (65). □

Theorem 6. *Let $L_t(P)$ be a self-adjoint operator, let C be a Hermitian matrix, let $n = 2r$, μ_j be a simple eigenvalue of C, let α_j be a positive constant satisfying $\alpha_j < \min_{q:q\neq j}|\mu_j - \mu_q|$, and let $B(\alpha_j, k, \mu_j)$ be a set defined by $B(\alpha_j, k, \mu_j) = B(0, \alpha_j, k, \mu_j) \cup B(\pi, \alpha_j, k, \mu_j)$, where*

$$\begin{aligned} B(0, \alpha_j, k, \mu_j) &= \bigcup_{s=1,2\ldots,m} \left(\frac{\mu_s - \mu_j - \alpha_j}{4n\pi k}, \frac{\mu_s - \mu_j + \alpha_j}{4n\pi k} \right), \\ B(\pi, \alpha_j, k, \mu_j) &= \bigcup_{s=1,2\ldots,m} \left(\pi + \frac{\mu_s - \mu_j - \alpha_j}{2n\pi(2k + n - 1)}, \pi + \frac{\mu_s - \mu_j + \alpha_j}{2n\pi(2k + n - 1)} \right). \end{aligned} \tag{83}$$

There exist a positive number N_1 such that if $|k| \geq N_1$ and $t \notin B(\alpha_j, k, \mu_j)$, then there exists a unique eigenvalue, denoted by $\lambda_{k,j}(t)$, of $L_t(P)$ lying in $U(\mu_{k,j}, \varepsilon_k)$, where $\varepsilon_k = c_7(|k|^{n-3} \ln |k|) + |k|^{n-2} b_k$, b_k is defined by (55)*, and c_7 is a positive constant, independent of t. The eigenvalue $\lambda_{k,j}(t)$ is a simple eigenvalue of $L_t(P)$ and the corresponding normalized eigenfunction $\Psi_{k,j,t}(x)$ satisfies*

$$\Psi_{k,j,t}(x) = v_j e^{i(2\pi k+t)x} + O\left(k^{-1} \ln|k|\right) + O(b_k). \tag{84}$$

Proof. To consider the simplicity of $\mu_{k,j}(t)$ and $\lambda_{k,j}(t)$ we introduce the set

$$S(k, j, p, s) = \left\{ t \in \left[-\frac{\pi}{2}, \frac{3\pi}{2} \right) : \left| \mu_{k,j}(t) - \mu_{p,s}(t) \right| < \alpha_j |k|^{n-2} \right\} \tag{85}$$

for $(p, s) \neq (k, j)$. It follows from (10) that $S(k, j, p, s) = \emptyset$ for $p \neq k, -k, -k - 1$. Moreover, if μ_j is a simple eigenvalue, then $S(k, j, k, s) = \emptyset$ for $s \neq j$, since

$$\left| \mu_{k,j}(t) - \mu_{k,s}(t) \right| = \left| (\mu_j - \mu_s)(2\pi k + t)^{n-2} \right| > \alpha_j |k|^{n-2}. \tag{86}$$

It remains to consider the sets $S(k, j, -k, s)$, $S(k, j, -k - 1, s)$. Using the equality $\mu_{k,j}(t) - \mu_{-k,s}(t) = (2\pi k)^{n-2}(4nk\pi t + \mu_j - \mu_s) + O(k^{n-3})$, we see that

$$S(k, j, -k, s) \subset \left(\frac{\mu_s - \mu_j - \alpha_j}{4n\pi k}, \frac{\mu_s - \mu_j + \alpha_j}{4n\pi k} \right). \tag{87}$$

Similarly, by using the obvious equality

$$\begin{aligned}
&\mu_{k,j}(t) - \mu_{-k-1,s}(t) \\
&\quad = (2\pi k + t)^n + \mu_j(2\pi k + t)^{n-2} - (2\pi k + 2\pi - t)^n - \mu_j(2\pi k + 2\pi - t)^{n-2} \\
&\quad = (2\pi k)^{n-2}\Big(n2\pi k t - n2\pi k(2\pi - t) + \frac{1}{2}n(n-1)\Big(t^2 - (2\pi - t)^2\Big) + \mu_j - \mu_s\Big) + O\Big(k^{n-3}\Big) \\
&\quad = (2\pi k)^{n-2}\left[(t-\pi)(2k + (n-1))2\pi n + \mu_j - \mu_s\right] + O\Big(k^{n-3}\Big),
\end{aligned} \tag{88}$$

we get

$$S(k,j,-k-1,s) \subset \left(\pi + \frac{\mu_s - \mu_j - \alpha_j}{2n\pi(2k+n-1)}, \pi + \frac{\mu_s - \mu_j + \alpha_j}{2n\pi(2k+n-1)}\right). \tag{89}$$

Using these relations and the definition of $B(\alpha_j, k, \mu_j)$, we obtain

$$\bigcup_{\substack{p\in\mathbb{Z}, s=1,2,\ldots,m, \\ (p,s)\neq(k,j)}} S(k,j,p,s) = \bigcup_{\substack{p=-k,-k-1, \\ s=1,2,\ldots,m}} S(k,j,p,s) \subset B(\alpha_j, k, \mu_j). \tag{90}$$

Therefore it follows from (85) that if $t \notin B(\alpha_j, k, \mu_j)$, then

$$\left|\mu_{k,j}(t) - \mu_{p,s}(t)\right| \geq \alpha_j |k|^{n-2} \tag{91}$$

for all $(p,s) \neq (k,j)$. Hence $\mu_{k,j}(t)$ is a simple eigenvalue of $L_t(C)$ for $t \notin B(\alpha_j, k, \mu_j)$. Instead of (56) using (54) and arguing as in the proof of (74), we obtain that there exists N_1 such that if $|k| \geq N_1$, then there exists an eigenvalue, denoted by $\lambda_{k,j}(t)$, of $L_t(P)$ lying in $U(\mu_{k,j}(t), \varepsilon_k)$. Now using the definition of ε_k and then (91), we see that

$$\left|\lambda_{k,j}(t) - \mu_{k,j}(t)\right| < \varepsilon_k = o\Big(k^{n-2}\Big), \qquad \left|\lambda_{k,j}(t) - \mu_{p,s}(t)\right| > \frac{1}{2}\alpha_j |k|^{n-2} \tag{92}$$

for $|k| \geq N_1$, $s = 1,2,\ldots,m$, $(p,s) \neq (k,j)$ and for any eigenvalue $\lambda_{k,j}(t)$ lying in $U(\mu_{k,j}(t), \varepsilon_k)$. Let $\Psi_{k,j,t}(x)$ be any normalized eigenfunction corresponding to $\lambda_{k,j}(t)$. Dividing both sides of (26) by $\lambda_{k,j}(t) - \mu_{p,s}(t)$ and using (54), (51), and (92), we get

$$\left(\Psi_{k,j,t}, \Phi_{p,s,t}\right) = O\Big(k^{-1}\ln|k|\Big) + O(b_k) \tag{93}$$

for $(p,s) \neq (k,j)$ and $p \in A(k,n,t)$. This, (67) and (68) imply that $\Psi_{k,j,t}(x)$ satisfies (84). Thus we have proved that (84) holds for any normalized eigenfunction of $L_t(P)$ corresponding to any eigenvalue lying in $U(\mu_{k,j}(t), \varepsilon_k)$. If there exist two different eigenvalues of $L_t(P)$ lying in $U(\mu_{k,j}(t), \varepsilon_k)$ or if there exists a multiple eigenvalue of $L_t(P)$ lying in $U(\mu_{k,j}(t), \varepsilon_k)$, then we obtain that there exist two orthonormal eigenfunctions satisfying (84) which is impossible.

Therefore there exists unique eigenvalue $\lambda_{k,j}(t)$ of $L_t(P)$ lying in $U(\mu_{k,j}(t),\varepsilon_k)$ and $\lambda_{k,j}(t)$ is a simple eigenvalue of $L_t(P)$. □

Theorem 7. *Let $L(P)$ be self-adjoint operator generated in $L_2^m(-\infty,\infty)$ by the differential expression* (1)*, and let C be Hermitian matrix.*

(a) *If n and m are odd numbers then the spectrum $\sigma(L(P))$ of $L(P)$ coincides with $(-\infty,\infty)$.*

(b) *If n is odd number, $n>1$, and the matrix C has at least one simple eigenvalue, then the number of the gaps in $\sigma(L(P))$ is finite.*

(c) *Suppose that n is even number, and the matrix C has at least three simple eigenvalues $\mu_{j_1}<\mu_{j_2}<\mu_{j_3}$ such that* $\operatorname{diam}(\{\mu_{j_1}+\mu_{i_1},\mu_{j_2}+\mu_{i_2},\mu_{j_3}+\mu_{i_3}\})\neq 0$ *for each triple (i_1,i_2,i_3), where $i_p=1,2,\ldots,m$ for $p=1,2,3$ and* $\operatorname{diam}(A)$ *is the diameter $\sup_{x,y\in A}|x-y|$ of the set A. Then the number of the gaps in the spectrum of $L(P)$ is finite.*

Proof. (a) In case $m=1$ the assertion (a) is proved in [4]. Our proof is carried out analogous fashion. Since $L(P)$ is self-adjoint, $\sigma(L(P))$ is a subset of $(-\infty,\infty)$. Therefore we need to prove that $(-\infty,\infty)\subset\sigma(L(P))$. Suppose to the contrary that there exists a real number λ such that $\lambda\notin\sigma(L(P))$. It is not hard to see that the characteristic determinant $\Delta(\lambda,t)=\det(U_\nu(Y_s(x,\lambda)))$ of $L_t(P)$ has the form

$$\Delta(\lambda,t)=e^{inmt}+a_1(\lambda)e^{i(nm-1)t}+a_2(\lambda)e^{i(nm-2)t}+\cdots+a_{nm}(\lambda), \tag{94}$$

that is, $\Delta(\lambda,t)$ is a polynomial $S_\lambda(u)$ of $u=e^{it}$ of order nm with entire coefficients $a_1(\lambda),a_2(\lambda),\ldots$. It is well known that if $\lambda\notin\sigma(L(P))$, then the absolute values of all roots $u_1=e^{it_1},u_2=e^{it_2},\ldots,u_{nm}=e^{it_{nm}}$ of $S_\lambda(u)=0$ differ from 1, that is, $t_k\neq\overline{t_k}$ and λ is the eigenvalue of $L_{t_k}(P)$ for $k=1,2,\ldots,nm$. It is not hard to see that $L_{t_k}^*=L_{\overline{t_k}}$, $\lambda=\overline{\lambda}\in\sigma(L_{\overline{t_k}})$. Moreover, if λ is the eigenvalue of $L_{t_k}(P)$ of multiplicity m_k then $\overline{\lambda}$ is the eigenvalue of $L_{\overline{t_k}}(P)$ of the same multiplicity m_k. Now taking into account that $u_k=e^{it_k}$ is the root of $S_\lambda(u)=0$ of multiplicity m_k if and only if λ is the eigenvalue of $L_{t_k}(P)$ of multiplicity m_k, we obtain that $e^{i\overline{t_k}}$ is also root of $S_\lambda(u)=0$ of the same multiplicity m_k. Since $e^{i\overline{t_k}}\neq e^{it_k}$, we see that the number nm of the roots of $S_\lambda(u)=0$ (see (94)) is an even number which contradicts the assumption that n and m are odd numbers.

(b) It follows from the uniform asymptotic formula (70) that there exists a positive numbers N_2, c_8, independent of t, such that if $|k|\geq N_2$ and μ_j is a simple eigenvalue of the matrix C then there exists unique simple eigenvalue $\lambda_{k,j}(t)$ of $L_t(P)$ lying in $U(\mu_{k,j}(t),\delta_k)$, where $\delta_k=c_8|k|^{n-3}\ln|k|$ and $t\in[-\pi/2,3\pi/2)$. Therefore $\lambda_{k,j}(t_0)$ for $t_0\in(-\pi/2,3\pi/2)$, $|k|\geq N_2$ is a simple zero of the characteristic determinant $\Delta(\lambda,t_0)$. By implicit function theorem there exists a neighborhood $U(t_0)\subset(-\pi/2,3\pi/2)$ of t_0 and a continuous in $U(t_0)$ function $\Lambda(t)$ such that $\Lambda(t_0)=\lambda_{k,j}(t_0)$, $\Lambda(t)$ is an eigenvalue of $L_t(P)$ for $t\in U(t_0)$ and $|\Lambda(t)-\mu_{k,j}(t)|<\delta_k$, for all $t\in U(t_0)$, since $|\Lambda(t_0)-\mu_{k,j}(t_0)|=|\lambda_{k,j}(t_0)-\mu_{k,j}(t_0)|<\delta_k$ and the functions $\Lambda(t)$, $\mu_{k,j}(t)$ are continuous. Now taking into account that there exists unique eigenvalue of $L_t(P)$ lying in $U(\mu_{k,j}(t),\delta_k)$, we obtain that $\Lambda(t)=\lambda_{k,j}(t)$ for $t\in U(t_0)$, and hence $\lambda_{k,j}(t)$ is continuous at $t_0\in(-\pi/2,3\pi/2)$. Therefore the sets $\Gamma_{k,j}=\{\lambda_{k,j}(t):t\in(-\pi/2,3\pi/2)\}$ for $|k|\geq N_2$ are intervals and $\Gamma_{k,j}\subset\sigma(L(P))$. Similarly there exists a neighborhood

$$U\left(-\frac{\pi}{2}\right)=\left(-\frac{\pi}{2}-\beta,-\frac{\pi}{2}+\beta\right) \tag{95}$$

of $-\pi/2$ and a continuous function $A(t)$ such that $A(-\pi/2) = \lambda_{k+1,j}(-\pi/2)$, where $k \geq N_2$, $A(t)$ is an eigenvalue of $L_{t+2\pi}(P)$ $(L_t(P) = L_{t+2\pi}(P))$ for $t \in (-\pi/2 - \beta, -\pi/2]$ and $A(t)$ is an eigenvalue of $L_t(P)$ for $t \in [-\pi/2, -\pi/2 + \beta)$ and

$$\begin{aligned} \left|A(t) - \mu_{k+1,j}(t)\right| < \delta_k \quad \forall t \in \left[-\frac{\pi}{2}, -\frac{\pi}{2} + \beta\right), \\ \left|A(t) - \mu_{k,j}(t + 2\pi)\right| < \delta_k \quad \forall t \in \left(-\frac{\pi}{2} - \beta, -\frac{\pi}{2}\right), \end{aligned} \tag{96}$$

since $|A(-\pi/2) - \mu_{k+1,j}(-\pi/2)| = |\lambda_{k+1,j}(-\pi/2) - \mu_{k+1,j}(-\pi/2)| < \delta_k$, $\mu_{k,j}(t+2\pi) = \mu_{k+1,j}(t)$ and the functions $\Lambda(t)$, $\mu_{k,j}(t)$ are continuous. Again taking into account that there exists unique eigenvalue of $L_t(P)$ lying in $U(\mu_{k+1,j}(t), \delta_k)$ for $t \in [-\pi/2, -\pi/2+\beta)$ and lying in $U(\mu_{k,j}(t), \delta_k)$ for $t \in (3\pi/2 - \beta, 3\pi/2)$, we obtain that

$$A(t) = \lambda_{k+1,j}(t), \quad \forall t \in \left[-\frac{\pi}{2}, -\frac{\pi}{2} + \beta\right), \qquad A(t) = \lambda_{k,j}(t + 2\pi), \quad \forall t \in \left(-\frac{\pi}{2} - \beta, -\frac{\pi}{2}\right). \tag{97}$$

Thus one part of the interval $\{A(t) : t \in (-\pi/2 - \beta, -\pi/2 + \beta)\}$ lies in $\Gamma_{k,j}$ and the other part lies in $\Gamma_{k+1,j}$, that is, the interval $\Gamma_{k,j}$ and $\Gamma_{k+1,j}$ are connected for $k \geq N_2$. Similarly the interval $\Gamma_{k,j}$ and $\Gamma_{k-1,j}$ are connected for $k \leq -N_2$. Therefore the number of the gaps in the spectrum of $L(P)$ is finite.

(c) In Theorem 6 we proved that if $|k| \geq N_1$ and $t \notin B(\alpha_{j_p}, k, \mu_{j_p})$, where $p = 1, 2, 3$, then there exists a unique eigenvalue, denoted by $\lambda_{k,j_p}(t)$, of $L_t(P)$ lying in $U(\mu_{k,j_p}(t), \varepsilon_k)$ and it is a simple eigenvalue. Let us prove that $\lambda_{k,j_p}(t)$ is continuous at

$$t_0 \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right) \setminus B\left(\alpha_{j_p}, k, \mu_{j_p}\right). \tag{98}$$

Since $\lambda_{k,j_p}(t_0)$ is a simple eigenvalue it is a simple zero of the characteristic determinant $\Delta(\lambda, t)$ of the operator $L_t(P)$. Therefore repeating the argument of the proof of the continuity of $\lambda_{k,j}(t)$ in the proof of (b), we obtain that $\lambda_{k,j_p}(t)$ is continuous at t_0 for $|k| \geq N_1$. Now we prove that there exists H such that

$$(H, \infty) \subset \left\{\lambda_{k,j_p}(t) : t \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right) \setminus B\left(\alpha_{j_p}, k, \mu_{j_p}\right),\ k = N_1, N_1 + 1, \ldots\right\}. \tag{99}$$

It is clear that

$$(h, \infty) \subset \left\{\mu_{k,j_p}(t) : t \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right),\ k = N_1, N_1 + 1, \ldots\right\}, \quad \forall p = 1, 2, 3, \tag{100}$$

where $h = \mu_{N_1,j_3}(-\pi/2)$. Since $\mu_{k,j_p}(t)$ is increasing function for $k \geq N_1$, it follows from the obvious equality

$$\begin{aligned}\mu_{k,j_p}\left(\frac{\mu_s - \mu_{j_p} \mp \alpha_{j_p}}{4n\pi k}\right) &= (2\pi k)^n + n(2\pi k)^{n-1}\frac{\mu_s - \mu_{j_p} \mp \alpha_{j_p}}{4n\pi k} + \mu_{j_p}(2\pi k)^{n-2} + O\left(k^{n-4}\right) \\ &= (2\pi k)^n + (2\pi k)^{n-2}\frac{\mu_s + \mu_{j_p} \mp \alpha_{j_p}}{2} + O\left(k^{n-4}\right)\end{aligned} \tag{101}$$

and from the definition of $B(0, \alpha_{j_p}, k, \mu_{j_p})$ that

$$\left\{\mu_{k,j_p}(t) : t \in B\left(0, \alpha_{j_p}, k, \mu_{j_p}\right)\right\} \subset \bigcup_{s=1,2,\ldots,m} C\left(0, k, j_p, s, \alpha_{j_p}\right), \tag{102}$$

where $C(0, k, j_p, s, \alpha_{j_p}) = \{x \in \mathbb{R} : |x - (2\pi k)^n + (2\pi k)^{n-2}((\mu_s + \mu_{j_p})/2)| < \alpha_{j_p}(2\pi k)^{n-2}\}$. This inclusion with (100) implies that the set

$$(h, \infty) \setminus \bigcup_{k:k\geq N_1;s=1,2,\ldots,m} C\left(0, k, j_p, s, \alpha_{j_p}\right) \tag{103}$$

is a subset of the set $\{\mu_{k,j_p}(t) : t \in [-\pi/2, 3\pi/2) \setminus B(0, \alpha_{j_p}, k, \mu_{j_p}),\ k \geq N_1\}$. Similarly, using

$$\mu_{k,j_p}\left(\pi + \frac{\mu_s - \mu_{j_p} \mp \alpha_{j_p}}{4n\pi k}\right) = (2\pi k + \pi)^n + (2\pi k)^{n-2}\frac{\mu_s + \mu_{j_p} \mp \alpha_{j_p}}{2} + O\left(k^{n-3}\right), \tag{104}$$

which can be proved by direct calculations, we obtain that the set

$$(h, \infty) \setminus \bigcup_{k:k\geq N_1;s=1,2,\ldots,m} C\left(\pi, k, j_p, s, \alpha_{j_p}\right), \tag{105}$$

where $C(\pi, k, j_p, s, \alpha_{j_p}) = \{x \in \mathbb{R} : |x - (2\pi k + \pi)^n + (2\pi k)^{n-2}((\mu_s + \mu_{j_p})/2)| < \alpha_{j_p}(2\pi k)^{n-2}\}$, is a subset of

$$\left\{\mu_{k,j_p}(t) : t \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right) \setminus B\left(\pi, \alpha_{j_p}, k, \mu_{j_p}\right),\ k \geq N_1\right\}. \tag{106}$$

Now using (92) and the continuity of $\lambda_{k,j_p}(t)$ on $[-\pi/2, 3\pi/2) \setminus B(\alpha_{j_p}, k, \mu_{j_p})$, we see that the set

$$(H,\infty) \setminus \left(\bigcup_{k:k\geq N_1;s=1,2,\dots,m} C\left(k, j_p, s, 2\alpha_{j_p}\right) \right), \tag{107}$$

where $H = h+1$, $C(k, j_p, s, 2\alpha_{j_p}) = C(0, k, j_p, s, 2\alpha_{j_p}) \cup C(\pi, k, j_p, s, 2\alpha_{j_p})$, is a subset of the set $\{\lambda_{k,j_p}(t) : t \in [-\pi/2, 3\pi/2) \setminus B(\alpha_{j_p}, k, \mu_{j_p}),\ k \geq N_1\}$. Thus we have

$$\bigcup_{p=1,2,3} \left((H,\infty) \setminus \left(\bigcup_{k\geq N_1;s=1,2,\dots,m} C\left(k, j_p, s, 2\alpha_{j_p}\right) \right) \right) \subset \sigma(L(P)). \tag{108}$$

To prove the inclusion $(H,\infty) \subset \sigma(L(P))$ it is enough to show that the set

$$\bigcap_{p=1,2,3} \left(\bigcup_{k\geq N_1;s=1,2,\dots,m} C\left(k, j_p, s, 2\alpha_{j_p}\right) \right) \tag{109}$$

is empty. If this set contains an element x, then

$$x \in \bigcup_{k\geq N_1;s=1,2,\dots,m} C\left(k, j_p, s, \alpha_{j_p}\right) \tag{110}$$

for all $p = 1, 2, 3$. Using this and the definition of $C(k, j_p, s, \alpha_{j_p})$, we obtain that there exist $k \geq N_1$; $\nu = 0, 1$ and $s = i_p$ such that

$$\left| x - (\pi(2k+\nu))^n - \frac{\mu_{j_p} + \mu_{i_p}}{2}(2\pi k)^{n-2} \right| < 2\alpha_{j_p}(2\pi k)^{n-2} \tag{111}$$

for all $p = 1, 2, 3$ and hence

$$\left| \frac{\mu_{j_q} + \mu_{i_q}}{2} - \frac{\mu_{j_p} + \mu_{i_p}}{2} \right| < 4\alpha_{j_p} \tag{112}$$

for all $p, q = 1, 2, 3$. Clearly, the constant α_{j_p} can be chosen so that

$$8\alpha_{j_p} < \min_{i_1,i_2,i_3}\left(\operatorname{diam}\left(\{\mu_{j_1} + \mu_{i_1}, \mu_{j_2} + \mu_{i_2}, \mu_{j_3} + \mu_{i_3}\}\right)\right), \tag{113}$$

since, by assumption of the theorem, the right-hand side of (113) is a positive constant. If (113) holds then (112) and hence (110) do not hold which implies that $(H,\infty) \subset \sigma(L(P))$. Hence the number of the gaps in the spectrum of $L(P)$ is finite. □

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