Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2010, Article ID 123094, 14 pages
doi:10.1155/2010/123094

Research Article

Some Nonunique Fixed Point Theorems of Ćirić Type on Cone Metric Spaces

Erdal Karapınar

Department of Mathematics, Atılım University, 06836 Ankara, Turkey

Correspondence should be addressed to Erdal Karapınar, erdalkarapinar@yahoo.com

Received 18 January 2010; Revised 6 March 2010; Accepted 15 April 2010

Academic Editor: Dumitru Baleanu

Some results of (Ćirić, 1974) on a nonunique fixed point theorem on the class of metric spaces are extended to the class of cone metric spaces. Namely, nonunique fixed point theorem is proved in orbitally T complete cone metric spaces under the assumption that the cone is strongly minihedral. Regarding the scalar weight of cone metric, we are able to remove the assumption of strongly minihedral.

1. Introduction and Preliminaries

In 1980, Rzepecki [1] introduced a generalized metric d_E on a set X in a way that $d_E : X \times X \to S$ where E is a Banach space and S is a normal cone in E with partial order $\preceq$. In that paper, the author generalized the fixed point theorems of Maia type [2].

In 1987, Lin [3] considered the notion of K-metric spaces by replacing real numbers with cone K in the metric function, that is, $d : X \times X \to K$. In that manuscript, some results of Khan and Imdad [4] on fixed point theorems were considered for K-metric spaces. Without mentioning the papers of Lin and Rzepecki, in 2007, Huang and Zhang [5] announced the notion of cone metric spaces (CMSs) by replacing real numbers with an ordering Banach space. In that paper, they also discussed some properties of convergence of sequences and proved the fixed point theorems of contractive mapping for cone metric spaces.

Recently, many results on fixed point theory have been extended to cone metric spaces (see, e.g., [5–13]).

Ćirić type nonunique fixed point theorems were considered by many authors (see, e.g., [14–20]). In this paper, some of the known results (see, e.g., [2, 14, 15]) are extended to cone metric spaces.

Throughout this paper $E := (E, \|\cdot\|)$ stands for a real Banach space. Let $P := P_E$ always be a closed nonempty subset of E. P is called *cone* if $ax + by \in P$ for all $x, y \in P$ and non-negative real numbers a, b where $P \cap (-P) = \{0\}$ and $P \neq \{0\}$.

For a given cone P, one can define a partial ordering (denoted by $\leq$ or $\leq_P$) with respect to P by $x \leq y$ if and only if $y - x \in P$. The notation $x < y$ indicates that $x \leq y$ and $x \neq y$ while $x \ll y$ will show $y - x \in \operatorname{int} P$, where $\operatorname{int} P$ denotes the interior of P. From now on, it is assumed that $\operatorname{int} P \neq \emptyset$.

The cone P is called *normal* if there is a number $K \geq 1$ for which $0 \leq x \leq y \Rightarrow \|x\| \leq K\|y\|$ holds for all $x, y \in E$. The least positive integer K, satisfying this equation, is called the normal constant of P. The cone P is said to be *regular* if every increasing sequence which is bounded from above is convergent, that is, if $\{x_n\}_{n\geq 1}$ is a sequence such that $x_1 \leq x_2 \leq \cdots \leq y$ for some $y \in E$, then there is $x \in E$ such that $\lim_{n\to\infty}\|x_n - x\| = 0$.

Lemma 1.1. (i) *Every regular cone is normal.*

(ii) *For each $k > 1$, there is a normal cone with normal constant $K > k$.*

(iii) *The cone P is regular if every decreasing sequence which is bounded from below is convergent.*

Proof of (i) and (ii) are given in [6] and the last one follows from definition.

Definition 1.2. Let X be a nonempty set. Suppose that the mapping $d : X \times X \to E$ satisfies

($M1$) $0 \leq d(x, y)$ for all $x, y \in X$,

($M2$) $d(x, y) = 0$ if and only if $x = y$,

($M3$) $d(x, y) \leq d(x, z) + d(z, y)$, for all $x, y \in X$,

($M4$) $d(x, y) = d(y, x)$ for all $x, y \in X$,

then d is called cone metric on X, and the pair (X, d) is called a cone metric space (CMS).

Example 1.3. Let $E = \mathbb{R}^3$, $P = \{(x, y, z) \in E : x, y, z \geq 0\}$, and $X = \mathbb{R}$. Define $d : X \times X \to E$ by $d(x, \tilde{x}) = (\alpha|x - \tilde{x}|, \beta|x - \tilde{x}|, \gamma|x - \tilde{x}|)$, where α, β, and γ are positive constants. Then (X, d) is a CMS. Note that the cone P is normal with the normal constant $K = 1$.

Definition 1.4. Let (X, d) be a CMS, $x \in X$, and $\{x_n\}_{n\geq 1}$ a sequence in X. Then

(i) $\{x_n\}_{n\geq 1}$ converges to x whenever for every $c \in E$ with $0 \ll c$ there is a natural number N, such that $d(x_n, x) \ll c$ for all $n \geq N$. It is denoted by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$.

(ii) $\{x_n\}_{n\geq 1}$ is a Cauchy sequence whenever for every $c \in E$ with $0 \ll c$ there is a natural number N, such that $d(x_n, x_m) \ll c$ for all $n, m \geq N$.

(iii) (X, d) is a complete cone metric space if every Cauchy sequence is convergent.

Lemma 1.5 (see [5]). *Let (X, d) be a CMS, P a normal cone with normal constant K, and $\{x_n\}$ a sequence in X. Then,*

(i) *the sequence $\{x_n\}$ converges to x if and only if $d(x_n, x) \to 0$ (or equivalently $\|d(x_n, x)\| \to 0$),*

(ii) *the sequence $\{x_n\}$ is Cauchy if and only if $d(x_n, x_m) \to 0$ (or equivalently $\|d(x_n, x_m)\| \to 0$ as $m, n \to \infty$),*

(iii) *the sequence $\{x_n\}$ converges to x and the sequence $\{y_n\}$ converges to y, then $d(x_n, y_n) \to d(x, y)$.*

Lemma 1.6 (see [8]). *Let (X, d) be a CMS over a cone P in E. Then*

(1) $\operatorname{int}(P) + \operatorname{int}(P) \subseteq \operatorname{int}(P)$ *and* $\lambda \operatorname{int}(P) \subseteq \operatorname{int}(P), \lambda > 0$.

(2) *If $c \gg 0$, then there exists $\delta > 0$ such that $\|b\| < \delta$ implies that $b \ll c$.*

(3) *For any given $c \gg 0$ and $c_0 \gg 0$ there exists $n_0 \in \mathbb{N}$ such that $c_0/n_0 \ll c$.*

(4) *If a_n, b_n are sequences in E such that $a_n \to a$, $b_n \to b$, and $a_n \leq b_n$, for all n, then $a \leq b$.*

Definition 1.7 (see [21]). P is called minihedral cone if $\sup\{x, y\}$ exists for all $x, y \in E$ and strongly minihedral if every subset of E which is bounded from above has a supremum. (equivalently, if every subset of E which is bounded from below has an infimum.)

Lemma 1.8. (i) *Every strongly minihedral normal (not necessarily closed) cone is regular.*
(ii) *Every strongly minihedral (closed) cone is normal.*

The proof of (i) is straightforward, and for (ii) see, for example, [22].

Example 1.9. Let $E = C[0, 1]$ with the supremum norm and $P = \{f \in E : f \geq 0\}$. Then P is a cone with normal constant $M = 1$ which is not regular. This is clear, since the sequence x^n is monotonically decreasing but not uniformly convergent to 0. This cone, by Lemma 1.8, is not strongly minihedral. However, it is easy to see that the cone mentioned in Example 1.3 is strongly minihedral.

2. Non unique Fixed Points on Cone Metric Spaces

Definition 2.1. A mapping T on CMS (X, d) is said to be orbitally continuous if $\lim_{i\to\infty} T^{n_i}(x) = z$ implies that $\lim_{i\to\infty} T(T^{n_i}(x)) = Tz$. A CMS (X, d) is called T orbitally complete if every Cauchy sequence of the form $\{T^{n_i}(x)\}_{i=1}^{\infty}$, $x \in X$, converges in (X, d).

Remark 2.2. It is clear that orbital continuity of T implies orbital continuity of T^m for any $m \in \mathbb{N}$.

Theorem 2.3. *Let $T : X \to X$ be an orbitally continuous mapping on CMS (X, d) over strongly minihedral normal cone P. Suppose that CMS (X, d) is T orbitally complete and that T satisfies the condition*

$$u(x, y) - \inf\{d(x, T(y)), d(T(x), y)\} \leq k d(x, y) \tag{2.1}$$

for all $x, y \in X$ and for some $0 \leq k < 1$, where $u(x, y) \in \{d(x, T(x)), d(T(x), T(y)), d(T(y), y)\}$. Then, for each $x \in X$, the iterated sequence $\{T^n(x)\}$ converges to a fixed point of T.

Proof. Fix $x_0 \in X$. For $n \geq 1$ set $x_1 = T(x_0)$ and recursively $x_{n+1} = T(x_n) = T^{n+1}(x_0)$. It is clear that the sequence x_n is Cauchy when the equation $x_{n+1} = x_n$ holds for some $n \in \mathbb{N}$. Consider the case $x_{n+1} \neq x_n$ for all $n \in \mathbb{N}$. By replacing x and y with x_{n-1} and x_n, respectively, in (2.1), one can get

$$\begin{aligned} &u(x_{n-1}, x_n) - \inf\{d(x_{n-1}, T(x_n)), d(T(x_{n-1}), x_n)\} \\ &\quad = \{d(x_n, x_{n+1}), d(x_{n-1}, x_n)\} \leq k d(x_{n-1}, x_n), \end{aligned} \tag{2.2}$$

where $u(x_{n-1}, x_n) \in \{d(x_{n-1}, T(x_{n-1})), d(T(x_{n-1}), T(x_n)), d(T(x_n), x_n)\}$. Since $k < 1$, the case $d(x_{n-1}, x_n) \leq kd(x_{n-1}, x_n)$ yields contradiction. Thus, $d(x_n, x_{n+1}) \leq kd(x_{n-1}, x_n)$. Recursively, one can observe that

$$d(x_n, x_{n+1}) \leq kd(x_{n-1}, x_n) \leq k^2 d(x_{n-2}, x_{n-1}) \leq \cdots \leq k^n d(x_0, T(x_0)). \tag{2.3}$$

By using the triangle inequality, for any $p \in \mathbb{N}$, one can get

$$\begin{aligned} d(x_n, x_{n+p}) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{n+p-1}, T(x_{n+p})) \\ &\leq \left(k^n + k^{n+1} + \cdots + k^{n+p-1}\right) d(x_0, T(x_0)) \\ &= k^n \left(1 + k + \cdots + k^{p-1}\right) d(x_0, T(x_0)) \leq \frac{k^n}{1-k} d(x_0, T(x_0)). \end{aligned} \tag{2.4}$$

Let $c \in \operatorname{int}(P)$. Choose a natural number M_0 such that $(k^n/(1-k))d(T(x_0), x_0) \ll c$ for all $n > M_0$. Thus, for any $p \in \mathbb{N}$, $d(x_{n+p}, x_n) \ll c$ for all $n > M_0$. So $\{x_n\}$ is a Cauchy sequence in (X, d). Since (X, d) is T orbitally complete, there is some $z \in X$ such that $\lim_{n\to\infty} x_n = \lim_{n\to\infty} T^n(x_0) = z$. Regarding the orbital continuity of T, $T(z) = \lim_{n\to\infty} T(T^n(x_0)) = \lim_{n\to\infty} T^{n+1}(x_0) = z$, that is, z is a fixed point of T. □

A point z is said to be a periodic point of a function T of period m if $T^m(z) = z$, where $T^0(x) = x$ and $T^m(x)$ is defined recursively by $T^m(x) = T(T^{m-1}(x))$.

Theorem 2.4. *Let $T : X \to X$ be an orbitally continuous mapping on T orbitally complete CMS (X, d) over strongly minihedral normal cone P and $c \in \operatorname{int}(P)$. Suppose that there exists a point $x_0 \in X$ such that $d(x_0, T^n(x_0)) \ll c$ for some $n \in \mathbb{N}$ and that T satisfies the condition*

$$0 < d(x, y) \ll c \Longrightarrow u(x, y) \leq kd(x, y) \tag{2.5}$$

for all $x, y \in X$ and for some $k < 1$, where $u(x, y) \in \{d(x, T(x)), d(T(x), T(y)), d(T(y), y)\}$. Then, T has a periodic point.

Proof. Set $M = \{n \in \mathbb{N} : d(x, T^n(x)) \ll c$ for some $x \in X\}$. By assumption of theorem $M \neq \emptyset$. Set $m = \min M$ and let $x \in X$ such that $d(x, T^m(x)) \ll c$ which is equivalent to saying that $c - d(x, T^m(x)) \in \operatorname{int}(P)$.

Suppose that $m = 1$. By replacing $y = T(x)$ in (2.5), one can get

$$u(x, T(x)) \leq kd(x, T(x)), \tag{2.6}$$

where $u(x, T(x)) \in \{d(x, T(x)), d(T(x), T(T(x))), d(T(T(x)), T(x))\}$. There are two cases. Consider the first case, $d(x, T(x)) \leq kd(x, T(x))$, which is a contraction by, regarding $k < 1$. Thus, one has $d(T(x), T(T(x))) = d(T(x), T^2(x)) \leq kd(x, T(x))$.

As in the proof of Theorem 2.3, one can consider the iterative sequence $x_{n+1} = T(x_n), x = x_0$ and observe that $Tz = z$ for some $z \in X$.

Suppose that $m \geq 2$. It is equivalent to saying that for each $y \in X$, the condition

$$c - d(T(y), y) \notin \operatorname{int}(P). \tag{2.7}$$

Taking account of $d(x, T^m(x)) \ll c$ and applying into (2.5), one can get

$$u(x, T^m(x)) \leq kd(x, T^m(x)), \tag{2.8}$$

where $u(x, T^m(x)) \in \{d(x, T(x)), d(T(x), T(T^m(x))), d(T(T^m(x)), T^m(x))\}$.

Recall that $T^m(x) \in X$ and say that $T^m(x) = z$. Then, $d(T(T^m(x)), T^m(x)) = d(T(z), z)$ is observed. Regarding (2.7), $c - d(T(z), z) = c - d(T(T^m(x)), T^m(x)) \notin \operatorname{int}(P)$ and also $c - d(T(x), x) \notin \operatorname{int}(P)$. Thus,

$$\min\{d(x, T(x)), d(T(x), T(T^m(x))), d(T(T^m(x)), T^m(x))\} = d\left(T(x), T^{m+1}(x)\right), \tag{2.9}$$

and hence, (2.8) turns into

$$d\left(T(x), T^{m+1}(x)\right) \leq kd(x, T^m(x)). \tag{2.10}$$

Recursively, one can get

$$d\left(T^2(x), T^{m+2}(x)\right) \leq kd\left(T(x), T^{m+1}(x)\right) \leq k^2 d(x, T^m(x)). \tag{2.11}$$

Continuing in this way, for each $p \in \mathbb{N}$, one can obtain

$$d(T^p(x), T^{m+p}(x)) \leq kd\left(T^{p-1}(x), T^{m+p-1}(x)\right) \leq \cdots \leq k^p d(x, T^m(x)). \tag{2.12}$$

Thus, for the recursive sequence $x_{n+1} = T^m(x_n)$ where $x_0 = x$,

$$d(x_n, x_{n+1}) = d\left(T^{nm}(x_0), T^{(n+1)m}(x_0)\right) = d(T^{nm}(x_0), T^{m+nm}(x_0)) \leq k^{nm} d(x_0, T^m(x_0)). \tag{2.13}$$

By using the triangle inequality, for any $p \in \mathbb{N}$, one can get

$$\begin{aligned} d(x_n, x_{n+p}) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{n+p-1}, x_{n+p}) \\ &= k^{nm}\left(1 + k^m + \cdots + k^{(p-1)m}\right) d(x_0, T(x_0)) \leq \frac{k^{nm}}{1 - k^m} d(x_0, T^m(x_0)). \end{aligned} \tag{2.14}$$

Let $c \in \operatorname{int}(P)$. Choose a natural number M_0 such that $((k^{nm})/(1 - k^m))d(T(x_0), x_0) \ll c$ for all $n > M_0$. Thus, for any $p \in \mathbb{N}$, $d(x_{n+p}, x_n) \ll c$ for all $n > M_0$. So $\{x_n\}$ is a Cauchy sequence in (X,d). Since (X,d) is T orbitally complete, there is some $z \in X$ such that $\lim_{n\to\infty} T^n(x_0) = z$. Regarding Remark 2.2, the orbital continuity of T implies that

$$T^m(z) = \lim_{n\to\infty} T^m(T^{nm}(x_0)) = \lim_{n\to\infty} T^{(n+1)m}(x_0) = z, \tag{2.15}$$

that is, z is a periodic point of T. □

Theorem 2.5. *Let $T : X \to X$ be an orbitally continuous mapping on CMS (X,d) over strongly minihedral normal cone P. Suppose that T satisfies the condition*

$$u(x,y) - \inf\{d(x,T(y)), d(T(x),y)\} < d(x,y) \tag{2.16}$$

for all $x,y \in X, x \neq y$ where $u(x,y) \in \{d(x,T(x)), d(T(x),T(y)), d(T(y),y)\}$. Suppose that the sequence $\{T^n(x_0)\}$ has a cluster point $z \in X$, for some $x_0 \in X$. Then, z is a fixed point of T.

Proof. Suppose that $T^m(x_0) = T^{m-1}(x_0)$ for some $m \in \mathbb{N}$, then $T^n(x_0) = T^m(x_0) = z$ for all $n \geq m$. It is clear that z is a required point.

Suppose that $T^m(x_0) \neq T^{m-1}(x_0)$ for all $m \in \mathbb{N}$. Since $\{T^n(x_0)\}$ has a cluster point $z \in X$, one can write $\lim_{i\to\infty} T^{n_i}(x_0) = z$. By replacing x and y with $T^{n-1}(x_0)$ and $T^n(x_0)$, respectively, in (2.16),

$$\begin{aligned} &u\left(T^{n-1}(x_0), T^n(x_0)\right) - \inf\left\{d\left(T^{n-1}(x_0), T(T^n(x_0))\right), d\left(T\left(T^{n-1}(x_0)\right), T^n(x_0)\right)\right\} \\ &\quad < d\left(T^{n-1}(x_0), T^n(x_0)\right), \end{aligned} \tag{2.17}$$

where $u(T^{n-1}(x_0), T^n(x_0))$ lies in $\{d(T^{n-1}(x_0), T(T^{n-1}(x_0))), d(T(T^{n-1}(x_0)), T(T^n(x_0))), d(T(T^n(x_0)), T^n(x_0))\}$. The case $d(T^{n-1}(x_0), T^n(x_0)) < d(T^{n-1}(x_0), T^n(x_0))$ is impossible. Thus, (2.17) is equivalent to $d(T^n(x_0), T^{n+1}(x_0)) < d(T^{n-1}(x_0), T^n(x_0))$. It shows that

$$\left\{d\left(T^n(x_0), T^{n+1}(x_0)\right)\right\}_1^\infty \tag{2.18}$$

is decreasing. Since the cone P is strongly minihedral, then by Lemma 1.1 (iii) and Lemma 1.8 (i), $\{d(T^n(x_0), T^{n+1}(x_0))\}_1^\infty$ is convergent. Due to Lemma 1.5, and T orbital continuity,

$$\lim_{i\to\infty} d\left(T^{n_i}(x_0), T^{n_i+1}(x_0)\right) = d(z, Tz). \tag{2.19}$$

By $\{d(T^{n_i}(x_0), T^{n_i+1}(x_0))\}_1^\infty \subset \{d(T^n(x_0), T^{n+1}(x_0))\}_1^\infty$ and (2.19),

$$\lim_{n\to\infty} d\left(T^n(x_0), T^{n+1}(x_0)\right) = d(z, Tz). \tag{2.20}$$

Regarding $\lim_{i\to\infty} T^{n_i+1}(x_0) = Tz$, $\lim_{i\to\infty} T^{n_i+2}(x_0) = T^2 z\{d(T^{n_i+1}(x_0), T^{n_i+2}(x_0))\}_1^{\infty} \subset \{d(T^n(x_0), T^{n+1}(x_0))\}_1^{\infty}$, and (2.20),

$$d\left(Tz, T^2 z\right) = d(z, Tz). \tag{2.21}$$

Assume that $Tz \neq z$, that is, $d(z, Tz) > 0$. So, one can replace x and y with z and Tz, respectively, in(2.16)

$$u(z, Tz) - \inf\{d(z, T(T(z))), d(T(z), T(z))\} < d(z, T(z)), \tag{2.22}$$

where $u(z, Tz) \in \{d(z, T(z)), d(T(z), T(T(z))), d(T(T(z)), T(z))\}$.

It yields that $d(Tz, T^2 z) < d(z, Tz)$. But it contradicts (2.21). Thus, $Tz = z$. □

3. Non unique Fixed Points on Scalar Weighted Cone Metric Spaces

Definition 3.1. Let (X, d) be a CMS. The scalar weight of the cone metric d is defined by $d_s(x, y) := \|d(x, y)\|$.

Notice that for normal cone P with the normal constant $K = 1$, the scalar weight of the cone metric d_s behaves as a metric on X. In the following theorems normal constant K has no restriction.

Theorem 3.2. *Let $T : X \to X$ be an orbitally continuous mapping on T orbitally complete CMS (X, d_s) over normal cone P with normal constant K. Suppose that T satisfies the condition*

$$\begin{aligned} &\min\{d_s(x, T(x)), d_s(T(x), T(y)), d_s(T(y), y)\} \\ &\quad - \min\{d_s(x, T(y)), d_s(T(x), y)\} \le k d_s(x, y) \end{aligned} \tag{3.1}$$

for all $x, y \in X$ and for some $k < 1$. Then, for each $x \in X$, the iterated sequence $\{T^n(x)\}$ converges to a fixed point of T.

Proof. Fix $x_0 \in X$. For $n \geq 1$ set $x_1 = T(x_0)$ and recursively $x_{n+1} = T(x_n) = T^{n+1}(x_0)$. It is clear that the sequence x_n is Cauchy when $x_{n+1} = x_n$ hold for some $n \in \mathbb{N}$. Consider the case $x_{n+1} \neq x_n$ for all $n \in \mathbb{N}$. By replacing x and y with x_{n-1} and x_n, respectively, in (3.1), one can get

$$\begin{aligned} &\min\{d_s(x_{n-1}, T(x_{n-1})), d_s(T(x_{n-1}), x_n), d_s(x_n, T(x_n))\} \\ &\quad - \min\{d_s(x_{n-1}, T(x_n)), d_s(T(x_{n-1}), x_n)\} \\ &= \min\{d_s(x_n, x_{n+1}), d_s(x_{n-1}, x_n) \le k d_s(x_{n-1}, x_n). \end{aligned} \tag{3.2}$$

Since $k < 1$, the case $d_s(x_{n-1}, x_n) \leq k d_s(x_{n-1}, x_n)$ yields contradiction. Thus, $d_s(x_n, x_{n+1}) \leq k d_s(x_{n-1}, x_n)$. Recursively, one can observe that

$$d_s(x_n, x_{n+1}) \le k d_s(x_{n-1}, x_n) \le k^2 d_s(x_{n-2}, x_{n-1}) \le \cdots \le k^n d_s(x_0, T(x_0)). \tag{3.3}$$

By using the triangle inequality, for any $p \in \mathbb{N}$, one can get

$$\begin{aligned} d_s(x_n, x_{n+p}) &\leq K\left(d_s(x_n, x_{n+1}) + d_s(x_{n+1}, x_{n+2}) + \cdots + d_s\left(x_{n+p-1}, T\left(x_{n+p}\right)\right)\right) \\ &\leq K\left(\left(k^n + k^{n+1} + \cdots + k^{n+p-1}\right) d_s(x_0, T(x_0))\right) \\ &= Kk^n\left(1 + k + \cdots + k^{p-1}\right) d_s(x_0, T(x_0)) \leq \frac{Kk^n}{1-k} d_s(x_0, T(x_0)). \end{aligned} \tag{3.4}$$

By routine calculation, one can obtain that $\{x_n\}$ is a Cauchy sequence in (X, d). Since (X, d) is T orbitally complete, there is some $z \in X$ such that

$$\lim_{n \to \infty} x_n = \lim_{n \to \infty} T^n(x_0) = z. \tag{3.5}$$

Regarding the orbital continuity of T,

$$T(z) = \lim_{n \to \infty} T(T^n(x_0)) = \lim_{n \to \infty} T^{n+1}(x_0) = z, \tag{3.6}$$

that is, z is a fixed point of T. □

Theorem 3.3. *Let $T : X \to X$ be an orbitally continuous mapping on T orbitally complete CMS (X, d) over normal cone P with normal constant K and $\varepsilon > 0$. Suppose that there exists a point $x_0 \in X$ such that $d_s(x_0, T^n(x_0)) < \varepsilon$ for some $n \in \mathbb{N}$ and that T satisfies the condition*

$$0 < d_s(x, y) < \varepsilon \Longrightarrow \min\{d_s(x, T(x)), d_s(T(x), T(y)), d_s(T(y), y)\} \leq k d_s(x, y) \tag{3.7}$$

for all $x, y \in X$ and for some $k < 1$. Then, T has a periodic point.

Proof. Set $M = \{n \in \mathbb{N} : d_s(x, T^n(x)) < \varepsilon : \text{ for } x \in X\}$. By assumption of the theorem $M \neq \emptyset$. Let $m = \min M$ and $x \in X$ such that $d_s(x, T^m(x)) < \varepsilon$.

Suppose that $m = 1$, that is, $d_s(x, T(x)) < \varepsilon$. By replacing $y = T(x)$ in (3.7), one can get

$$\min\{d_s(x, T(x)), d_s(T(x), T(T(x))), d_s(T(T(x)), T(x))\} \leq k d_s(x, T(x)). \tag{3.8}$$

The case $d_s(x, T(x)) \leq k d_s(x, T(x))$ implies a contraction due to the fact that $k < 1$. Thus, $d_s(T(x), T(T(x))) = d_s(T(x), T^2(x)) \leq k d_s(x, T(x))$.

As in the proof of Theorem 3.2, one can consider the iterative sequence $x_{n+1} = T(x_n)$, $x = x_0$ and observe that $Tz = z$ for some $z \in X$.

Suppose that $m \geq 2$. It is equivalent to saying that the condition

$$d_s(T(y), y) \geq \varepsilon \tag{3.9}$$

holds for each $y \in X$. Then, from $d_s(x, T^m(x)) < \varepsilon$ and (3.7) it follows that

$$\min\{d_s(x, T(x)), d_s(T(x), T(T^m(x))), d_s(T(T^m(x)), T^m(x))\} \leq k d_s(x, T^m(x)). \tag{3.10}$$

Considering $T^m(x) \in X$, say $T^m(x) = z$, one has $d_s(T(T^m(x)), T^m(x)) = d_s(T(z), z)$. Regarding (3.9), $d_s(T(z), z) = d_s(T(T^m(x)), T^m(x)) \geq \varepsilon$ and $d_s(T(x), x) \geq \varepsilon$. Thus,

$$\min\{d_s(x, T(x)), d_s(T(x), T(T^m(x))), d_s(T(T^m(x)), T^m(x))\} = d_s\Big(T(x), T^{m+1}(x)\Big). \tag{3.11}$$

and hence

$$d_s\Big(T(x), T^{m+1}(x)\Big) \leq k d_s(x, T^m(x)). \tag{3.12}$$

Recursively, one can get

$$d_s\Big(T^2(x), T^{m+2}(x)\Big) \leq d_s\Big(T(x), T^{m+1}(x)\Big) \leq k^2 d_s(x, T^m(x)). \tag{3.13}$$

Continuing in this way, for each $p \in \mathbb{N}$, one can obtain

$$d_s(T^p(x), T^{m+p}(x)) \leq d_s\Big(T^{p-1}(x), T^{m+p-1}(x)\Big) \leq \cdots \leq k^p d_s(x, T^m(x)). \tag{3.14}$$

Thus, for the recursive sequence $x_{n+1} = T^m(x_n)$ where $x_0 = x$,

$$d_s(x_n, x_{n+1}) = d_s\Big(T^{nm}(x_0), T^{(n+1)m}(x_0)\Big) = d_s(T^{nm}(x_0), T^{m+nm}(x_0)) \leq k^{nm} d_s(x_0, T^m(x_0)). \tag{3.15}$$

By using the triangle inequality and regarding the normality of the cone, for any $p \in \mathbb{N}$, one can get

$$\begin{aligned} d_s(x_n, x_{n+p}) &\leq K\left[d_s(x_n, x_{n+1}) + d_s(x_{n+1}, x_{n+2}) + \cdots + d_s(x_{n+p-1}, x_{n+p})\right] \\ &= K k^{nm}\Big[1 + k^m + \cdots + k^{(p-1)m}\Big] d_s(x_0, T^m(x_0)) \\ &\leq \frac{K k^{nm}}{1 - k^m} d_s(x_0, T^m(x_0)). \end{aligned} \tag{3.16}$$

Let $\varepsilon > 0$. Choose a natural number M_0 such that $(K k^{nm}/(1-k^m)) d_s(T^m(x_0), x_0) < \varepsilon$ for all $n > M_0$. Thus, for any $p \in \mathbb{N}$, $d_s(x_{n+p}, x_n) < \varepsilon$ for all $n > M_0$. So $\{x_n\}$ is a Cauchy sequence in X. Since X is T orbitally complete, there is some $z \in X$ such that $\lim_{n\to\infty} T^n(x_0) = z$. Regarding Remark 2.2, the orbital continuity of T implies that

$$T^m(z) = T^m\Big(\lim_{n\to\infty} T^{nm}(x_0)\Big) = \lim_{n\to\infty} T^m(T^{nm}(x_0)) = \lim_{n\to\infty} T^{(n+1)m}(x_0) = z, \tag{3.17}$$

that is, z is a periodic point of T. □

Theorem 3.4. *Let $T : X \to X$ be an orbitally continuous mapping on CMS (X, d) over normal cone P with normal constant K. Suppose that T satisfies the condition*

$$\begin{aligned} &\min\{d_s(x, T(x)), d_s(T(x), T(y)), d_s(T(y), y)\} \\ &\quad - \min\{d_s(x, T(y)), d_s(T(x), y)\} < d_s(x, y) \end{aligned} \tag{3.18}$$

for all $x, y \in X, x \neq y$. If the sequence $\{T^n(x_0)\}$ has a cluster point $z \in X$, for some $x_0 \in X$, then z is a fixed point of T.

The proof of Theorem 3.4 is omitted by regarding the analogy with the proof of Theorem 2.5. In the proof of Theorem 2.5, to conclude that the decreasing sequence (2.18) is convergent, we need to use the assumption of strong minihedrality of the cone P. Since we use the scalar weight of cone metric in the proof of Theorem 3.4, we can conclude that the corresponding decreasing sequence of (2.18) is convergent without the assumption of strong minihedrality of the cone P.

Theorem 3.5. *Let $T : X \to X$ be an orbitally continuous mapping on T orbitally complete CMS (X, d) over normal cone P with normal constant K and $\varepsilon > 0$. Suppose that T satisfies the condition*

$$0 < d_s(x, y) < \varepsilon \Longrightarrow \min\{d_s(x, T(x)), d_s(T(x), T(y)), d_s(T(y), y)\} < d_s(x, y) \tag{3.19}$$

for all $x, y \in X$. If for some $x_0 \in X$, the sequence $\{T^n(x_0)\}_{n=1}^{\infty}$ has a cluster point of $z \in X$, then z is a periodic point of T.

Proof. Set $\lim_{i\to\infty} T^{n_i}(x_0) = z$, that is, for any $\varepsilon > 0$ there exists $N_0 \in \mathbb{N}$ such that $d_s(T^{n_i}(x_0), z) < \varepsilon/2K$ for all $i > N_0$. Hence, by triangle inequality and normality of the cone it yields that

$$d_s(T^{n_i}(x_0), T^{n_{i+1}}(x_0)) \leq d_s(T^{n_i}(x_0), z) + d_s(z, T^{n_{i+1}}(x_0)) < \varepsilon. \tag{3.20}$$

Define a set

$$M = \left\{ j \in \mathbb{N} : d_s\left(T^n(x_0), T^{n+j}(x_0)\right) < \varepsilon \text{ for some } n \in \mathbb{N} \right\} \tag{3.21}$$

which is nonempty by assumption of the theorem. Let $m = \min M$. Consider two cases. Suppose $d_s(T^n(x_0), T^{n+m}(x_0)) = 0$ for some $n \in \mathbb{N}$. Then, $z = T^n(x_0) = T^{n+m}(x_0) = T^m(T^n(x_0)) = T^m(z)$ and the assertion of theorem follows.

Suppose that $d_s(T^n(x_0), T^{n+m}(x_0)) > 0$ for all $n \in \mathbb{N}$. Let $r \in \mathbb{N}$ be such that $d_s(T^r(x_0), T^{r+m}(x_0)) < \varepsilon$.

If $m = 1$, then replacing x and y with $T^n(x_0)$ and $T^{n+1}(x_0)$, respectively, in (3.19) one can obtain that

$$\begin{aligned} &\min\left\{d_s(T^n(x_0), T(T^n(x_0))), d_s\left(T(T^n(x_0)), T\left(T^{n+1}(x_0)\right)\right), d_s\left(T\left(T^{n+1}(x_0)\right), T^{n+1}(x_0)\right)\right\} \\ &\quad < d_s\left(T^n(x_0), T^{n+1}(x_0)\right). \end{aligned} \tag{3.22}$$

Since the case $d_s(T^n(x_0), T^{n+1}(x_0)) < d_s(T^n(x_0), T^{n+1}(x_0))$ is impossible, (3.22) turns into $d_s(T^{n+1}(x_0), T^{n+2}(x_0)) < d_s(T^n(x_0), T^{n+1}(x_0))$, that is, the sequence $\{d_s(T^n(x_0), T^{n+1}(x_0))\}$ is decreasing for $n \geq r$. Thus, by routine calculation, one can conclude that $Tz = z$.

Assume that $m \geq 2$, that is, for every $n \in \mathbb{N}$,

$$d_s\left(T^n(x_0), T^{n+1}(x_0)\right) \geq \varepsilon. \tag{3.23}$$

By orbital continuity of T, $\lim_{i\to\infty} T^{n_i+r}(x_0) = T^r(z)$, and by (3.23), one can get

$$d_s\left(T^r(z), T^{r+1}(z)\right) = \lim_{i\to\infty} d_s\left(T^{n_i+r}(x_0), T^{n_i+r+1}(x_0)\right) \geq \varepsilon. \tag{3.24}$$

for every $r \in \mathbb{N}$

Regarding (3.19) under the assumption $0 < d_s(T^j(x_0), T^{j+m}(x_0)) < \varepsilon$ one can obtain

$$\begin{aligned} &\min\left\{d_s\left(T^j(x_0), T^{j+1}(x_0)\right), d_s\left(T^{j+1}(x_0), T^{j+m+1}(x_0)\right), d_s\left(T^{j+m}(x_0), T^{j+m+1}(x_0)\right)\right\} \\ &\quad < d_s\left(T^j(x_0), T^{j+m}(x_0)\right). \end{aligned} \tag{3.25}$$

Thus, due to (3.23), $d_s(T^{j+1}(x_0), T^{j+m+1}(x_0)) < d_s(T^j(x_0), T^{j+m}(x_0)) < \varepsilon$.

By continuing this process, it yields that

$$\cdots < d_s\left(T^{j+2}(x_0), T^{j+m+2}(x_0)\right) < d_s\left(T^{j+1}(x_0), T^{j+m+1}(x_0)\right) < d_s\left(T^j(x_0), T^{j+m}(x_0)\right) < \varepsilon. \tag{3.26}$$

Hence, the sequence $\{d_s(T^n(x_0), T^{n+m}(x_0)) : n \geq j\}$ is decreasing and thus is convergent. Notice that the subsequences $\{d_s(T^{n_i}(x_0), T^{n_i+m}(x_0)) : i \in \mathbb{N}\}$ and $\{d_s(T^{n_i+1}(x_0), T^{n_i+1+m}(x_0)) : i \in \mathbb{N}\}$ are convergent to $d(z, T^m z)$ and $d(Tz, T^{m+1}z)$, respectively. By orbital continuity of T and $\lim_{i\to\infty} T^{n_i}(x_0) = z$, one can get

$$d_s\left(T(z), T^{m+1}(z)\right) = d_s(z, T^m(z)) = \lim_{n\to\infty} d_s(T^n(x_0), T^{n+m}(x_0)). \tag{3.27}$$

One can conclude that $d_s(z, T^m z) < \varepsilon$ from (3.26) and (3.27). If $d_s(z, T^m z) = 0$, then $T^m z = z$. Thus, the desired result is obtained. Suppose that $d_s(z, T^m z) > 0$. Applying (3.19),

$$\min\{d_s(z, T(z)), d_s(T(z), T(T^m(z))), d_s(T(T^m(z)), T^m(z))\} < d_s(z, T^m z) < \varepsilon. \tag{3.28}$$

Taking account of (3.24), (3.28) yields that $d_s(T(z), T^{m+1}(z)) < d_s(z, T^m z)$ which contradicts (3.27). Thus, $d_s(z, T^m z) = 0$, and so $T^m z = z$. □

Theorem 3.6. *Let $T : X \to X$ be an orbitally continuous mapping on T orbitally complete CMS (X, d_s) over normal cone P with normal constant K. Suppose that T satisfies the condition*

$$\min\left\{[d_s(x,T(x))]^2, d_s(x,y)d_s(T(x),T(y)), [d_s(T(y),y)]^2\right\} - \min\{d_s(x,T(x))d_s(T(y),y), d_s(x,T(y))d_s(T(x),y)\} \leq kd_s(x,T(x))d_s(T(y),y) \tag{3.29}$$

for all $x, y \in X$ and for some $k < 1$. Then, for each $x \in X$, the iterated sequence $\{T^n(x)\}$ converges to a fixed point of T.

Proof. As in the proof of Theorem 3.2, fix $x_0 \in X$ and define the sequence $\{x_n\}$ in the following way. For $n \geq 1$ set $x_1 = T(x_0)$ and recursively $x_{n+1} = T(x_n) = T^{n+1}(x_0)$. It is clear that the sequence x_n is Cauchy when $x_{n+1} = x_n$ hold for some $n \in \mathbb{N}$. Consider the case $x_{n+1} \neq x_n$ for all $n \in \mathbb{N}$. By replacing x and y with x_{n-1} and x_n, respectively, in (3.29), one can get

$$\begin{aligned}&\min\left\{[d_s(x_{n-1},T(x_{n-1}))]^2, d_s(x_{n-1},x_n)d_s(T(x_{n-1}),T(x_n)), [d_s(T(x_n),x_n)]^2\right\}\\&\quad - \min\{d_s(x_{n-1},T(x_{n-1}))d_s(T(x_n),x_n), d_s(x_{n-1},T(x_n))d_s(T(x_{n-1}),x_n)\}\\&\leq kd_s(x_{n-1},T(x_{n-1}))d_s(T(x_n),x_n).\end{aligned} \tag{3.30}$$

Since $k < 1$, the case $d_s(x_{n-1},x_n)d_s(x_n,x_{n+1}) \leq kd_s(x_{n-1},x_n)d_s(x_n,x_{n+1})$ yields contradiction. Thus, one gets

$$d_s(x_n,x_{n+1}) \leq kd_s(x_{n-1},x_n). \tag{3.31}$$

Recursively, one can observe that

$$d_s(x_n,x_{n+1}) \leq kd_s(x_{n-1},x_n) \leq k^2 d_s(x_{n-2},x_{n-1}) \leq \cdots \leq k^n d_s(x_0,T(x_0)). \tag{3.32}$$

By routine calculation as in the proof of Theorem 3.2, one can show that T has a fixed point. □

Theorem 3.7. *Let X be a nonempty set endowed in two cone metrics d, ρ, and let T be a mapping of X into itself. Suppose that*

(i) *X is orbitally complete space with respect to d_s,*

(ii) *$d_s(x,y) \leq \rho_s(x,y)$ for all $x, y \in X$,*

(iii) *T is orbitally continuous with respect to d_s,*

(iv) *T satisfies*

$$\min\left\{[\rho_s(T(x),T(y))]^2, \rho_s(x,y)\rho_s(T(x),T(y)), [\rho_s(y,T(y))]^2\right\} - \min\{\rho_s(x,T(x))\rho_s(y,T(y)), \rho_s(x,Ty)\rho_s(y,T(x))\} \leq k\rho_s(x,T(x)),\rho_s(y,Ty) \tag{3.33}$$

for all $x, y \in X$, where $0 \leq k < 1$.

Then T has a fixed point in X.

Proof. As in the proof of Theorem 3.2, fix $x_0 \in X$ and define the sequence $\{x_n\}$ in the following way. For $n \geq 1$ set $x_1 = T(x_0)$ and recursively $x_{n+1} = T(x_n) = T^{n+1}(x_0)$. Replacing x, y with x_{n-1}, x_n, respectively, in (3.33), one can get

$$\begin{aligned}\min&\left\{ \left[\rho_s(T(x_{n-1}),T(x_n))\right]^2, \rho_s(x_{n-1},x_n)\rho_s(T(x_{n-1}),T(x_n)), \left[\rho_s(x_n,T(x_n))\right]^2 \right\}\\ &- \min\{\rho_s(x_{n-1},T(x_{n-1}))\rho_s(x_n,T(x_n)), \rho_s(x_{n-1},T(x_n))\rho_s(x_n,T(x_{n-1}))\}\\ &\leq k\rho_s(x_{n-1},T(x_{n-1})), \rho_s(x_n,T(x_n)).\end{aligned} \tag{3.34}$$

Since the case $k\rho_s(x_{n-1},T(x_{n-1})), \rho_s(x_n,T(x_n)) \leq k\rho_s(x_{n-1},T(x_{n-1})), \rho_s(x_n,T(x_n))$, (3.34) is equivalent to $\rho_s(x_n,x_{n+1}) \leq k\rho_s(x_{n-1},x_n)$. Recursively one can obtain

$$\rho_s(x_n,x_{n+1}) \leq k\rho_s(x_{n-1},x_n) \leq \cdots \leq k^n\rho_s(x_0,x_1). \tag{3.35}$$

Regarding the triangle inequality and the normality of the cone, (3.35) implies that

$$\rho_s(x_n,x_{n+p}) \leq \frac{Kk^n}{1-k}\rho_s(x_0,x_1), \tag{3.36}$$

for any $p \in \mathbb{N}$. Taking account of assumption (ii) of the theorem, one can get

$$d_s(x_n,x_{n+p}) \leq \frac{Kk^n}{1-k}\rho_s(x_0,x_1). \tag{3.37}$$

Thus, $\{x_n\}$ is a Cauchy sequence with respect to d_s. Since X is T orbitally complete, there exists $z \in X$ such that $\lim_{n\to\infty}T^n(x) = z$. From orbital continuity of T, one can get the desired result, that is, $Tz = \lim_{n\to\infty}T(T^n(x)) = z$.

Remark 3.8. Theorem 3.6 can be restated by replacing (3.29) with

$$\begin{aligned}\min&\left\{ \left[d_s(T(x),T(y))\right]^2, d_s(x,y)d_s(T(x),T(y)), \left[d_s(T(y),y)\right]^2 \right\}\\ &- \min\{d_s(x,T(y)), d_s(y,T(x))\} \leq kd_s(x,T(x))d_s(T(y),y).\end{aligned} \tag{3.38}$$

Note also that, Theorem 3.7 remains valid by replacing (3.33) with

$$\begin{aligned}\min&\left\{ \left[\rho_s(T(x),T(y))\right]^2, \rho_s(x,y)\rho_s(T(x),T(y)), \left[\rho_s(T(y),y)\right]^2 \right\}\\ &- \min\{v_s(x,T(y)), \rho_s(y,T(x))\} \leq k\rho_s(x,T(x))\rho_s(T(y),y).\end{aligned} \tag{3.39}$$

□

References

[1] B. Rzepecki, "On fixed point theorems of Maia type," *Institut Mathématique*, vol. 28, pp. 179–186, 1980.

[2] M. G. Maia, "Un'osservazione sulle contrazioni metriche," *Rendiconti del Seminario Matematico della Università di Padova*, vol. 40, pp. 139–143, 1968.

[3] S. D. Lin, "A common fixed point theorem in abstract spaces," *Indian Journal of Pure and Applied Mathematics*, vol. 18, no. 8, pp. 685–690, 1987.

[4] M. S. Khan and M. Imdad, "A common fixed point theorem for a class of mappings," *Indian Journal of Pure and Applied Mathematics*, vol. 14, no. 10, pp. 1220–1227, 1983.

[5] L.-G. Huang and X. Zhang, "Cone metric spaces and fixed point theorems of contractive mappings," *Journal of Mathematical Analysis and Applications*, vol. 332, no. 2, pp. 1468–1476, 2007.

[6] Sh. Rezapour and R. Hamlbarani, "Some notes on the paper: "Cone metric spaces and fixed point theorems of contractive mappings"," *Journal of Mathematical Analysis and Applications*, vol. 345, no. 2, pp. 719–724, 2008.

[7] T. Abdeljawad, "Completion of cone metric spaces," *Hacettepe Journal of Mathematics and Statistics*, vol. 39, no. 1, pp. 67–74, 2010.

[8] D. Turkoglu and M. Abuloha, "Cone metric spaces and fixed point theorems in diametrically contractive mappings," *Acta Mathematica Sinica (English Series)*, vol. 26, no. 3, pp. 489–496, 2010.

[9] D. Turkoglu, M. Abuloha, and T. Abdeljawad, "KKM mappings in cone metric spaces and some fixed point theorems," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 72, no. 1, pp. 348–353, 2010.

[10] T. Abdeljawad, D. Turkoglu, and M. Abuloha, "Some theorems and examples of cone metric spaces," *Journal of Computational Analysis and Applications*, vol. 12, no. 4, pp. 739–753, 2010.

[11] E. Karapınar, "Fixed point theorems in cone Banach spaces," *Fixed Point Theory and Applications*, vol. 2009, Article ID 609281, 9 pages, 2009.

[12] E. Karapınar, "Couple fixed point theorems for nonlinear contractions in cone metric spaces," *Computers & Mathematics with Applications*, vol. 59, no. 12, pp. 3656–3668, 2010.

[13] T. Abdeljawad and E. Karapınar, "Quasi-cone metric spaces and generalizations of Caristi Kirk's theorem," *Fixed Point Theory and Applications*, vol. 2009, Article ID 574387, 9 pages, 2009.

[14] L. B. Ćirić, "On some maps with a nonunique fixed point," *Institut Mathématique*, vol. 17, pp. 52–58, 1974.

[15] B. G. Pachpatte, "On Ćirić type maps with a nonunique fixed point," *Indian Journal of Pure and Applied Mathematics*, vol. 10, no. 8, pp. 1039–1043, 1979.

[16] J. Achari, "On Ćirić's non-unique fixed points," *Matematički Vesnik*, vol. 13(28), no. 3, pp. 255–257, 1976.

[17] S. Gupta and B. Ram, "Non-unique fixed point theorems of Ćirić type," *Vijnana Parishad Anusandhan Patrika*, vol. 41, no. 4, pp. 217–231, 1998.

[18] F. Zhang, S. M. Kang, and L. Xie, "On Ćirić type mappings with a nonunique coincidence points," *Fixed Point Theory and Applications*, vol. 6, pp. 187–190, 2007.

[19] Z. Liu, Z. Guo, S. M. Kang, and S. K. Lee, "On Ćirić type mappings with nonunique fixed and periodic points," *International Journal of Pure and Applied Mathematics*, vol. 26, no. 3, pp. 399–408, 2006.

[20] Z. Q. Liu, "On Ćirić type mappings with a nonunique coincidence points," *Filiale de Cluj-Napoca. Mathematica*, vol. 35(58), no. 2, pp. 221–225, 1993.

[21] K. Deimling, *Nonlinear Functional Analysis*, Springer, Berlin, Germany, 1985.

[22] C. D. Aliprantis and R. Tourky, *Cones and Duality*, vol. 84 of *Graduate Studies in Mathematics*, American Mathematical Society, Providence, RI, USA, 2007.