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Research Article

Long-Term Behavior of Solutions of the Difference Equation $x_{n+1} = x_{n-1}x_{n-2} - 1$

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We investigate the long-term behavior of solutions of the following difference equation: $x_{n+1} = x_{n-1}x_{n-2} - 1$, $n \in \mathbb{N}_0$, where the initial values x_{-2}, x_{-1}, and x_0 are real numbers. Numerous fascinating properties of the solutions of the equation are presented.

1. Introduction and Preliminaries

Recently there has been great interest in studying nonlinear difference equations which do not stem from differential equations (see, e.g., [1–28] and the references therein). Standard properties which have been studied are boundedness [5, 9, 23–25], periodicity [2, 5, 9, 10, 27], asymptotic periodicity [3, 4, 8, 11–14, 16, 17, 19, 20, 23], and local and global stability [5, 9–11, 23–26], as well as existence of specific solutions such as monotone or nontrivial solutions [1, 6, 7, 13, 15, 18–22].

In this paper, we investigate the long-term behavior of solutions of the third-order difference equation

$$x_{n+1} = x_{n-1}x_{n-2} - 1, \quad n \in \mathbb{N}_0, \tag{1.1}$$

where the initial values x_{-2}, x_{-1}, x_0 are real numbers.

The difference equation (1.1) belongs to the class of equations of the form

$$x_{n+1} = x_{n-k}x_{n-l} - 1, \quad n \in \mathbb{N}_0, \tag{1.2}$$

where $k, l \in \mathbb{N}_0$, $k < l$, and $\gcd(k;\ l) = 1$. The case $k = 0, l = 1$ has been recently investigated in [8].

2. The Equilibria and Periodic Solutions of (1.1)

This section is devoted to the study of the equilibria and periodic solutions of (1.1).

2.1. Equilibria of (1.1)

If $\overline{x}$ is an equilibrium of (1.1), then it satisfies the equation

$$\overline{x}^2 - \overline{x} - 1 = 0. \tag{2.1}$$

Hence, (1.1) has exactly two equilibria, one positive and one negative, which we denote by $\overline{x}_1$ and $\overline{x}_2$, respectively:

$$\overline{x}_1 =: \frac{1+\sqrt{5}}{2}, \qquad \overline{x}_2 =: \frac{1-\sqrt{5}}{2} \tag{2.2}$$

(the golden number and its conjugate).

2.2. Periodic Solutions of (1.1)

Here, we study the existence of periodic solutions of (1.1). For related results, see, for example, [2, 5, 9, 10, 27] and the references therein. The first two results are simple, but we will prove them for the completeness, the benefit of the reader, and since we use them in the sequel.

Theorem 2.1. *There are no eventually constant solutions of difference equation* (1.1).

Proof. If $\{x_n\}_{n=-2}^{\infty}$ is an eventually constant solution of (1.1), then $x_N = x_{N+1} = x_{N+2} = \overline{x}$, for some $N \in \mathbb{N}_0$, where $\overline{x}$ is an equilibrium point. In this case, (1.1) gives $x_{N+2} = x_N x_{N-1} - 1$, which implies

$$x_{N-1} = \frac{x_{N+2}+1}{x_N} = \frac{\overline{x}+1}{\overline{x}} = \overline{x}. \tag{2.3}$$

Repeating this procedure, we obtain $x_n = \overline{x}$ for $-2 \le n \le N+2$. Hence, there are no eventually constant solutions. □

Theorem 2.2. *Difference equation* (1.1) *has no nontrivial period two solutions nor eventually period two solutions.*

Proof. Assume that $x_N = x_{N+2k}$ and $x_{N+1} = x_{N+2k+1}$, for every $k \in \mathbb{N}_0$, and some $N \geq -2$, with $x_N \neq x_{N+1}$. Then, we have

$$x_{N+4} = x_{N+2}x_{N+1} - 1 = x_N x_{N+1} - 1 = x_{N+3} = x_{N+1}. \tag{2.4}$$

From this and since $x_{N+4} = x_N$, we obtain a contradiction, finishing the proof of the result. □

Theorem 2.3. *There are no periodic or eventually periodic solutions of* (1.1) *with prime period three.*

Proof. If

$$x_N = x_{N+3k}, \qquad x_{N+1} = x_{N+3k+1}, \qquad x_{N+2} = x_{N+3k+2}, \quad k \geq 0, \tag{2.5}$$

for some $N \geq -2$, we have

$$\begin{aligned} x_{N+3} &= x_{N+1}x_N - 1 = x_N, \\ x_{N+4} &= x_{N+2}x_{N+1} - 1 = x_{N+1}, \\ x_{N+5} &= x_{N+3}x_{N+2} - 1 = x_N x_{N+2} - 1 = x_{N+2}. \end{aligned} \tag{2.6}$$

If $x_N = 0$, $x_{N+1} = 0$, or $x_{N+2} = 0$, then from (2.6) we easily obtain contradictions in all these cases. Hence, we may assume that $x_N \neq 0$, $x_{N+1} \neq 0$, and $x_{N+2} \neq 0$. Equalities in (2.6) also imply that

$$x_{N+1} = \frac{x_N + 1}{x_N}, \qquad x_{N+2} = \frac{x_{N+1} + 1}{x_{N+1}}, \qquad x_N = \frac{x_{N+2} + 1}{x_{N+2}}. \tag{2.7}$$

From (2.7), we get

$$x_N = \frac{x_{N+2} + 1}{x_{N+2}} = \frac{2x_{N+1} + 1}{x_{N+1} + 1} = \frac{3x_N + 2}{2x_N + 1}, \tag{2.8}$$

which implies that $x_N^2 - x_N - 1 = 0$, that is, $x_N = (1 \pm \sqrt{5})/2$. From this and (2.7) we obtain $x_{N+1} = x_{N+2} = (1 \pm \sqrt{5})/2$, implying $x_n = (1 \pm \sqrt{5})/2, n \geq N$, from which the result follows. □

Theorem 2.4. *There are no periodic or eventually periodic solutions of* (1.1) *with prime period four.*

Proof. Assume

$$x_N = x_{N+4k}, \qquad x_{N+1} = x_{N+4k+1}, \qquad x_{N+2} = x_{N+4k+2}, \qquad x_{N+3} = x_{N+4k+3}, \quad k \geq 0, \tag{2.9}$$

for some $N \geq -2$. Then we have

$$\begin{aligned} x_{N+4} &= x_{N+2}x_{N+1} - 1 = x_N, \\ x_{N+5} &= x_{N+3}x_{N+2} - 1 = x_{N+1}, \\ x_{N+6} &= x_{N+4}x_{N+3} - 1 = x_N x_{N+3} - 1 = x_{N+2}, \\ x_{N+7} &= x_{N+5}x_{N+4} - 1 = x_{N+1}x_N - 1 = x_{N+3}. \end{aligned} \tag{2.10}$$

If $x_{N+i} = 0$ for some $i \in \{0,1,2,3\}$, then from (2.10) we easily obtain a contradiction. For example, if $x_N = 0$, then from (2.10) we get $x_{N+2} = -1 = x_{N+3}$. This implies $x_{N+1} = x_{N+3}x_{N+2} - 1 = 0$. From this and since $x_{N+2}x_{N+1} - 1 = x_N$ we would get $x_N = -1$, a contradiction. The other cases are proved analogously.

Hence we may assume that $x_{N+i} \neq 0$, $i \in \{0,1,2,3\}$. From (2.10) we have

$$x_N = \frac{x_{N+3}+1}{x_{N+1}} = \frac{(x_{N+2}+1)/x_N + 1}{(x_N+1)/x_{N+2}} = \frac{x_{N+2}(x_{N+2}+x_N+1)}{x_N(x_N+1)} = \frac{(x_N+1)(x_{N+1}+1)}{x_N x_{N+1}^2}, \tag{2.11}$$

that is,

$$(x_N x_{N+1})^2 = (x_N+1)(x_{N+1}+1). \tag{2.12}$$

From (2.10) and (2.12), we obtain

$$(x_{N+3}+1)^2 = (x_N+1)(x_{N+1}+1). \tag{2.13}$$

Since the relations in (2.12) are cyclic, we also obtain that

$$\begin{aligned} (x_N+1)^2 &= (x_{N+1}+1)(x_{N+2}+1), \\ (x_{N+1}+1)^2 &= (x_{N+2}+1)(x_{N+3}+1), \\ (x_{N+2}+1)^2 &= (x_{N+3}+1)(x_N+1). \end{aligned} \tag{2.14}$$

Equalities (2.13) and (2.14) imply that the expressions $x_{N+i}+1, i \in \{0,1,2,3\}$ have the same sign. Assume that they are all positive (the case when they are all negative is considered similarly so it is omitted). We have

$$\begin{aligned} (x_{N+3}+1)^2 &= (x_N+1)(x_{N+1}+1) = (x_N+1)\sqrt{(x_{N+3}+1)(x_{N+2}+1)} \\ &= (x_N+1)\sqrt{(x_{N+3}+1)\sqrt{(x_{N+3}+1)(x_N+1)}}, \end{aligned} \tag{2.15}$$

from which easily follows that $x_{N+3} = x_N$. From this, (2.13), and (2.14), it follows that $x_{N+i} = x_N$, $i \in \{1,2,3\}$, which implies the result. □

The following result shows that there exist periodic solutions of (1.1) with prime period five.

Theorem 2.5. *A solution of* (1.1) *is of period five if and only if*

(i) $x_{-2} = a$, $x_{-1} = b$, $ab \neq 1$, *and* $x_0 = (1 + a)/(ab - 1)$, *or*

(ii) $x_{-2} = x_{-1} = -1$, *or*

(iii) $x_{-2} = a$, $x_{-1} = 0$, *and* $x_0 = -a - 1$.

Proof. If the initial conditions are as given, then by some calculations it is easy to see that these solutions are of period five. Now we assume that a solution $\{x_n\}_{n=-2}^{\infty}$ is of period five. Then, we can write terms of the solution of (1.1) as

$$
\begin{aligned}
x_{-2} &= a, \\
x_{-1} &= b, \\
x_0 &= c, \\
x_1 &= d = ab - 1, \\
x_2 &= e = bc - 1, \\
x_3 &= f = cd - 1 = c(ab - 1) - 1 = a, \\
x_4 &= g = (ab - 1)(bc - 1) - 1 = b, \\
x_5 &= h = (bc - 1)f - 1 = (bc - 1)a - 1 = c.
\end{aligned}
\tag{2.16}
$$

Note that the expressions for f and h both yield the same condition, namely, $abc = a + c + 1$. The condition for g gives $ab^2c - ab - bc = b$, so that $b(abc - a - c - 1) = 0$. Hence, either $b = 0$ or $abc = a + c + 1$, which was a restriction coming from f and h. If $b = 0$, then $f = a = -c - 1$. The second restriction can be rewritten as $c = (1 + a)/(ab - 1)$, provided $ab \neq 1$. If $ab = 1$, then $f = a = -1 = b$. □

Remark 2.6. If $x_k = 0$, where $k \geq -1$, then x_k starts a 5-cycle (in particular, in view of the above theorem, we have a 5 cycle if the initial conditions are $x_{-2} = a, x_{-1} = 0, x_0 = c$, where a and c are any real numbers). If $k = -1$, then we have

$$
\begin{aligned}
x_{-1} &= 0, \\
x_0 &= x_0, \\
x_1 &= x_{-1}x_{-2} - 1 = -1, \\
x_2 &= x_0 x_{-1} - 1 = -1, \\
x_3 &= x_1 x_0 - 1 = -x_0 - 1, \\
x_4 &= x_2 x_1 - 1 = 0, \\
x_5 &= x_3 x_2 - 1 = -(-x_0 - 1) - 1 = x_0, \\
x_6 &= x_4 x_3 - 1 = -1,
\end{aligned}
\tag{2.17}
$$

from which the statement follows in this case.

If $k \geq 0$, then by direct calculation we have

$$\begin{gathered}
x_k = 0, \\
x_{k+1} = x_{k-1}x_{k-2} - 1, \\
x_{k+2} = x_k x_{k-1} - 1 = -1, \\
x_{k+3} = x_{k+1}x_k - 1 = -1, \\
x_{k+4} = x_{k+2}x_{k+1} - 1 = -(x_{k-1}x_{k-2} - 1) - 1 = -x_{k-1}x_{k-2}, \\
x_{k+5} = x_{k+3}x_{k+2} - 1 = 0, \\
x_{k+6} = x_{k+4}x_{k+3} - 1 = -(-x_{k-1}x_{k-2}) - 1 = x_{k-1}x_{k-2} - 1, \\
x_{k+7} = x_{k+5}x_{k+4} - 1 = -1,
\end{gathered} \tag{2.18}$$

from which the statement follows in this case.

Remark 2.7. There are period-five solutions of (1.1) that do not have a zero term. It is enough to use $a \neq -1$ and $b \in \mathbb{R} \setminus \{-1, 0\}$ such that $ab \neq 1$. For example, we can choose $x_{-2} = 2$, $x_{-1} = 2, x_0 = 1$.

Remark 2.8. There exist solutions that are eventually of period five; for example, choose $x_{-2} = 4$, $x_{-1} = 0.5$, $x_0 = 1$, or $x_{-2} = 0$, $x_{-1} = 4$, $x_0 = -0.5$.

3. Solutions in the Interval $(-1, 0)$

Here, we study the solutions of (1.1) with initial values in the interval $(-1, 0)$ or for which there are three subsequent terms that are eventually in the interval.

The next result shows that the interval $(-1, 0)$ is an invariant interval for (1.1).

Theorem 3.1. *If* $-1 < x_{-2}, x_{-1}, x_0 < 0$*, then* $-1 < x_n < 0$ *for all* $n \geq -2$.

Proof. If $-1 < x_{-2}, x_{-1}, x_0 < 0$, then $-1 < x_1 = x_{-1}x_{-2} - 1 < 0$. From (1.1) and by induction, we then have that $-1 < x_n < 0$ for all $n \geq -2$. □

Remark 3.2. There are solutions that eventually enter the interval $(-1, 0)$. One example of such a solution is one with the initial conditions $x_{-2} = 1.3$, $x_{-1} = 1.5$, and $x_0 = 1.6$.

3.1. Convergence to Period-Five Solutions

The next theorem is devoted to the convergence of solutions of (1.1) with initial conditions in the interval $(-1, 0)$ to period-five solutions. For related results on the asymptotic periodicity of difference equations, see [3, 4, 8, 10–14, 16, 17, 19, 20, 23].

Note that if $a, b \in (-1, 0)$, then $ab \neq 1$, and we have that $(1 + a)/(ab - 1) \in (-1, 0)$. Indeed, since $a, b \in (-1, 0)$, then clearly $(1 + a)/(ab - 1) < 0$, and $ab < -a$ so that $1 + a < 1 - ab$, from which it follows that $(1 + a)/(ab - 1) > -1$. Hence, by Theorems 2.5 and 3.1 it follows that these solutions are periodic of period five belonging to the interval $(-1, 0)$.

Before we formulate and prove the main result in this section, we need an auxiliary result. Relations of this type were first discovered and used by Stević in [12] and then subsequently used in several papers (e.g., in [16]).

Lemma 3.3. *Any solution of* (1.1) *satisfies the following equality*

$$x_{n+5} - x_n = x_n(x_{n+4} - x_{n-1}), \quad \text{for } n \geq -1. \tag{3.1}$$

Proof. If $n \geq -1$, we have

$$\begin{aligned} x_{n+2} &= x_n x_{n-1} - 1, \\ x_{n+3} &= x_{n+1} x_n - 1, \\ x_{n+4} &= x_{n+2} x_{n+1} - 1 = (x_n x_{n-1} - 1) x_{n+1} - 1 \\ &= x_{n+1} x_n x_{n-1} - x_{n+1} - 1, \end{aligned} \tag{3.2}$$

$$\begin{aligned} x_{n+5} &= x_{n+3} x_{n+2} - 1 = (x_{n+1} x_n - 1)(x_n x_{n-1} - 1) - 1 \\ &= x_{n+1} (x_n)^2 x_{n-1} - x_{n+1} x_n - x_n x_{n-1} \\ &= x_n (x_{n+1} x_n x_{n-1} - x_{n+1} - x_{n-1}). \end{aligned} \tag{3.3}$$

Using (3.3) and then (3.2), we get

$$\begin{aligned} x_{n+5} - x_n &= x_n (x_{n+1} x_n x_{n-1} - x_{n+1} - x_{n-1}) - x_n \\ &= x_n (x_{n+1} x_n x_{n-1} - x_{n+1} - x_{n-1} - 1) \\ &= x_n [(x_{n+1} x_n x_{n-1} - x_{n+1} - 1) - x_{n-1}] \\ &= x_n (x_{n+4} - x_{n-1}), \end{aligned} \tag{3.4}$$

which is equality (3.1). □

Theorem 3.4. *Consider the difference equation* (1.1) *with initial conditions* $x_{-2}, x_{-1}, x_0 \in (-1, 0)$. *Then, this solution converges to a period-five solution.*

Proof. Consider five subsequences $\{x_{5n}\}_{n=0}^{\infty}$, $\{x_{5n+1}\}_{n=0}^{\infty}$, $\{x_{5n+2}\}_{n=0}^{\infty}$, $\{x_{5n+3}\}_{n=0}^{\infty}$, and $\{x_{5n+4}\}_{n=0}^{\infty}$. By Lemma 3.3, we know that our solution satisfies (3.1) for all $n \geq -1$. Thus, we have

$$\begin{aligned} x_{n+10} - x_{n+5} &= x_{n+5}(x_{n+9} - x_{n+4}) \\ &= x_{n+5} x_{n+4} (x_{n+8} - x_{n+3}) \\ &= x_{n+5} x_{n+4} x_{n+3} (x_{n+7} - x_{n+2}) \\ &= x_{n+5} x_{n+4} x_{n+3} x_{n+2} (x_{n+6} - x_{n+1}) \\ &= x_{n+5} x_{n+4} x_{n+3} x_{n+2} x_{n+1} (x_{n+5} - x_n). \end{aligned} \tag{3.5}$$

Therefore,

$$|x_{n+10} - x_{n+5}| = |x_{n+5}||x_{n+4}||x_{n+3}||x_{n+2}||x_{n+1}||x_{n+5} - x_n|, \tag{3.6}$$

and, since the values of the sequence are in the interval $(-1, 0)$, we have,

$$|x_{n+10} - x_{n+5}| \le |x_{n+5} - x_n|. \tag{3.7}$$

Without loss of generality, we will consider the subsequence $\{x_{5n}\}_{n=0}^{\infty}$. Let

$$a_n = x_{5(n+1)} - x_{5n}, \quad n \in \mathbb{N}_0. \tag{3.8}$$

By inequality (3.7), we have

$$|a_0| \ge |a_1| \ge |a_2| \ge \cdots . \tag{3.9}$$

Since the sequence $\{|a_n|\}_{n=0}^{\infty}$ is positive, nonincreasing, and bounded below by 0, it must converge to, say, A. We will prove that $A = 0$. Assume to the contrary that $A \neq 0$. Observe that from (3.6) we have

$$\left|x_{5(n+1)} - x_{5n}\right| = \left(\prod_{k=1}^{5n} |x_k|\right) |x_5 - x_0|. \tag{3.10}$$

Since the sequence $\{|a_n|\}_{n=0}^{\infty}$ converges to a nonzero value, it follows that $\prod_{k=1}^{\infty} |x_k|$ converges to a nonzero value. Hence, $\lim_{k \to \infty} |x_k| = 1$. Therefore, x_k is close to -1 for k large. Suppose k is large. Then, $x_{k+3} = x_{k+1}x_k - 1$ is close to 0, which contradicts the fact that $|x_{k+3}| \to 1$ as $k \to \infty$. Hence, $A = 0$, as desired. □

Remark 3.5. As it has been already mentioned, there are solutions that have initial values outside the interval $(-1, 0)$ and enter the interval. Such solutions also converge to a period five solution by the previous theorem.

Corollary 3.6. *Assume that a solution of* (1.1) *has initial conditions* $x_{-2} = 0, x_{-1}, x_0 \in (-1, 0)$. *Then,* $x_1 = -1$, *all future terms are in the interval* $(-1, 0)$, *and thus this solution converges to a periodic solution with period five.*

Proof. Clearly $x_1 = x_{-1}x_{-2} - 1 = -1$. Since $0 < -x_0 < 1$ and $0 < -x_{-1} < 1$, we have $0 < x_0 x_{-1} < 1$, and so $-1 < x_2 < 0$. Furthermore, we have $x_3 = x_1 x_0 - 1 = -x_0 - 1 \in (-1, 0)$. Similarly, $x_4 = x_2 x_1 - 1 = -x_2 - 1 \in (-1, 0)$. The rest of the proof is the same as in Theorem 3.1, and so is omitted. □

4. Stability and Convergence of Solutions of (1.1)

In this section, we determine the stability nature of the two equilibria of (1.1) and leave open for the reader the possibility of convergence of solutions to the negative equilibrium.

Lemma 4.1. *The positive equilibrium of* (1.1), $\overline{x}_1$, *is unstable.*

Proof. The characteristic equation of the equilibrium $\overline{x}_1$ is the following:

$$\lambda^3 - \overline{x}_1\lambda - \overline{x}_1 = 0. \tag{4.1}$$

Let $P_1(x) = x^3 - \overline{x}_1 x - \overline{x}_1$. Since $P_1(1) = 1 - 2\overline{x}_1 = -\sqrt{5} < 0$ and $\lim_{x\to+\infty} P_1(x) = +\infty$, it follows that there is a $\lambda_0 > 1$ such that $P_1(\lambda_0) = 0$, from which the result follows for the equilibrium $\overline{x}_1$. □

Remark 4.2. Consider the characteristic equation of the negative equilibrium of (1.1), $\overline{x}_2$,

$$\lambda^3 - \overline{x}_2\lambda - \overline{x}_2 = 0 \tag{4.2}$$

and the function $P_2(x) = x^3 - \overline{x}_2 x - \overline{x}_2$. Observe the following:

(i) $P_2(\overline{x}_2) = 0$.

(ii) $P_2'(x) = 3x^2 - \overline{x}_2 > 0$ for all $x \in \mathbb{R}$.

Hence, we have one negative eigenvalue, $\lambda_1 = \overline{x}_2 \in (-1, 0)$ and a complex conjugate pair of eigenvalues, λ_2 and $\overline{\lambda}_2$, such that

$$\lambda_1 |\lambda_2|^2 = \overline{x}_2. \tag{4.3}$$

It follows that $|\lambda_2| = |\overline{\lambda}_2| = 1$, and so $\overline{x}_2$ is a nonhyperbolic equilibrium where the roots of the characteristic equation have absolute values less than or equal to one.

Open problem 4.3. Determine the stability nature of the negative equilibrium of (1.1), $\overline{x}_2$.

5. More on Invariant Intervals

In this section, we discuss invariant intervals regarding the subsequences $\{x_{5n}\}_{n=0}^{\infty}$, $\{x_{5n+1}\}_{n=0}^{\infty}$, $\{x_{5n+2}\}_{n=0}^{\infty}$, $\{x_{5n+3}\}_{n=0}^{\infty}$, and $\{x_{5n+4}\}_{n=0}^{\infty}$ of a solution $\{x_n\}_{n=-2}^{\infty}$ of (1.1).

Theorem 5.1. *Assume that a solution of* (1.1) *has initial conditions* $x_{-2} = 0, x_{-1} \in (0, 1)$ *and* $x_0 \in (-1, 0)$. *Then,* $x_{5n+2}, x_{5n+6} \in (-2, -1)$, $x_{5n+3}, x_{5n+5} \in (-1, 0)$, *and* $x_{5n+4} \in (0, 1)$ *for* $n \in \mathbb{N}_0$ *and also* $x_1 = -1$.

Proof. Since $x_{-2} = 0$, we have $x_1 = x_{-1}x_{-2} - 1 = -1$. Now, we prove the above statement for $n = 0$. From $0 < x_{-1} < 1$ and $-1 < x_0 < 0$, we have $-1 < x_0 x_{-1} < 0$. From this and since $x_2 = x_0 x_{-1} - 1$, we get $x_2 \in (-2, -1)$.

Furthermore, we have that

$$\begin{aligned} x_3 &= x_1 x_0 - 1 = -x_0 - 1 \in (-1, 0), \\ x_4 &= x_2 x_1 - 1 = -x_2 - 1 \in (0, 1). \end{aligned} \tag{5.1}$$

Since $0 < -x_3 < 1$ and $1 < -x_2 < 2$, it follows that $x_5 = x_3x_2 - 1 \in (-1, 1)$. Now, we verify that x_5 must be in a more restrictive interval $(-1, 0)$. We prove this by contradiction. Assume that $x_5 \in [0, 1)$. Since $0 \le x_5 = x_3x_2 - 1 < 1$ with $x_3 = x_1x_0 - 1 = -x_0 - 1$ and $x_2 = x_0x_{-1} - 1$, we must have that

$$0 \le (-x_0 - 1)(x_0x_{-1} - 1) - 1 < 1 \Longleftrightarrow 0 \le x_0(1 - x_{-1} - x_0x_{-1}) < 1, \tag{5.2}$$

which is equivalent to

$$0 \le x_0(-x_2 - x_{-1}) < 1. \tag{5.3}$$

On the other hand, since $1 < -x_2 < 2$ and $-1 < -x_{-1} < 0$, we have $0 < -x_2 - x_{-1} < 2$. However, since $x_0 < 0$, it follows that $x_0(-x_2 - x_{-1}) < 0$, which is a contradiction. Hence, $x_5 \in (-1, 0)$, as claimed.

Next, since $0 < x_4 < 1$ and $0 < -x_3 < 1$, we have $-1 < x_4x_3 < 0$, and so, $-2 < x_6 < -1$, finishing the proof for the case $n = 0$.

Now assume that $x_{5n+2}, x_{5n+6} \in (-2, -1)$, $x_{5n+3}, x_{5n+5} \in (-1, 0)$, and $x_{5n+4} \in (0, 1)$ for $0 \le n \le k$.

Since $x_{5k+4} \in (0, 1)$ and $x_{5k+5} \in (-1, 0)$, we have

$$x_{5(k+1)+2} = x_{5k+5}x_{5k+4} - 1 \in (-2, -1). \tag{5.4}$$

Since $x_{5k+6} \in (-2, -1)$ and $x_{5k+5} \in (-1, 0)$, we have

$$x_{5(k+1)+3} = x_{5k+6}x_{5k+5} - 1 \in (-1, 1). \tag{5.5}$$

Assume that $x_{5(k+1)+3} \in [0, 1)$. Then, we have

$$0 \le x_{5(k+1)+3} = x_{5k+6}x_{5k+5} - 1 = (x_{5k+4}x_{5k+3} - 1)(x_{5k+3}x_{5k+2} - 1) - 1 < 1 \tag{5.6}$$

which is equivalent to

$$0 \le x_{5k+3}(x_{5k+4}x_{5k+3}x_{5k+2} - x_{5k+4} - x_{5k+2}) < 1. \tag{5.7}$$

Now note that $x_{5k+4}x_{5k+3}x_{5k+2} > 0$ and $-x_{5k+4} - x_{5k+2} > 0$ since $-1 < -x_{5k+4} < 0$ and $1 < -x_{5k+2} < 2$. Hence, $x_{5k+4}x_{5k+3}x_{5k+2} - x_{5k+4} - x_{5k+2} > 0$ which along with $x_{5k+3} < 0$ implies that $x_{5k+3}(x_{5k+4}x_{5k+3}x_{5k+2} - x_{5k+4} - x_{5k+2}) < 0$, which is a contradiction. Hence,

$$x_{5(k+1)+3} \in (-1, 0). \tag{5.8}$$

Since $-2 < x_{5(k+1)+2}, x_{5k+6} < -1$, we have

$$x_{5(k+1)+4} = x_{5(k+1)+2}x_{5k+6} - 1 \in (0, 3). \tag{5.9}$$

Assume that $x_{5(k+1)+4} \in [1, 3)$.

Since by Lemma 3.3

$$x_{5(k+1)+4} - x_{5k+4} = x_{5k+4}\left(x_{5(k+1)+3} - x_{5k+3}\right) \tag{5.10}$$

and $0 < x_{5k+4} < 1 \leq x_{5(k+1)+4}$, we have $x_{5(k+1)+4} - x_{5k+4} > 0$. Therefore, $x_{5k+4}(x_{5(k+1)+3} - x_{5k+3}) > 0$. From this and since $x_{5k+4} > 0$, we have that $x_{5(k+1)+3} - x_{5k+3} > 0$. We then have

$$x_{5(k+1)+3} - x_{5k+3} = x_{5k+3}\left(x_{5(k+1)+2} - x_{5k+2}\right) > 0. \tag{5.11}$$

Since $x_{5k+3} < 0$, we have $x_{5(k+1)+2} - x_{5k+2} < 0$. Furthermore we have

$$x_{5(k+1)+2} - x_{5k+2} = x_{5k+2}\left(x_{5(k+1)+1} - x_{5k+1}\right) < 0. \tag{5.12}$$

Since $x_{5k+2} < 0$, we have $x_{5(k+1)+1} - x_{5k+1} > 0$.

By Lemma 3.3, for each $1 \leq n \leq k$, we have

$$x_{5(n+1)+1} - x_{5n+1} = x_{5n+1}x_{5n}x_{5n-1}x_{5n-2}x_{5n-3}\left(x_{5n+1} - x_{5(n-1)+1}\right). \tag{5.13}$$

By the inductive hypothesis, we have that $x_{5n+1}x_{5n}x_{5n-1}x_{5n-2}x_{5n-3} > 0$ so that the sign of the difference is the same as the sign of $x_{5(k+1)+1} - x_{5k+1}$, that is,

$$x_{5(n+1)+1} - x_{5n+1} > 0, \quad 0 \leq n \leq k. \tag{5.14}$$

On the other hand, $x_6 - x_1 = x_6 + 1 < 0$, which is a contradiction. Hence,

$$x_{5(k+1)+4} \in (0,1). \tag{5.15}$$

Since $-2 < x_{5(k+1)+2} < -1$ and $x_{5(k+1)+3} \in (-1,0)$, we have

$$x_{5(k+1)+5} = x_{5(k+1)+3}x_{5(k+1)+2} - 1 \in (-1,1). \tag{5.16}$$

Assume that $x_{5(k+1)+5} \in [0,1)$. Then, we have

$$0 \leq (x_{5k+6}x_{5k+5} - 1)(x_{5k+5}x_{5k+4} - 1) - 1 < 1 \tag{5.17}$$

which is equivalent to

$$0 \leq x_{5k+5}(x_{5k+6}x_{5k+5}x_{5k+4} - x_{5k+6} - x_{5k+4}) < 1. \tag{5.18}$$

On the other hand, since $x_{5k+6}x_{5k+5}x_{5k+4} > 0$ and since from $1 < -x_{5k+6} < 2$ and $-1 < -x_{5k+4} < 0$ we obtain $-x_{5k+6} - x_{5k+4} > 0$, it follows that

$$x_{5k+6}x_{5k+5}x_{5k+4} - x_{5k+6} - x_{5k+4} > 0. \tag{5.19}$$

This fact along with $x_{5k+5} < 0$ implies

$$x_{5k+5}(x_{5k+6}x_{5k+5}x_{5k+4} - x_{5k+6} - x_{5k+4}) < 0 \tag{5.20}$$

which is a contradiction. Hence,

$$x_{5(k+1)+5} \in (-1, 0). \tag{5.21}$$

Finally, since $0 < x_{5(k+1)+4} < 1$ and $-1 < x_{5(k+1)+3} < 0$, we obtain

$$x_{5(k+1)+6} = x_{5(k+1)+4}x_{5(k+1)+3} - 1 \in (-2, -1). \tag{5.22}$$

From (5.4)–(5.22) and by the method of induction, the proof follows. □

Theorem 5.2. *Any solution of* (1.1) *with initial values satisfying the following conditions* $x_{-2} = 0, x_{-1} \in (0, 1)$, *and* $x_0 \in (-1, 0)$ *converges to a period-five solution.*

Proof. By Theorem 5.1, we have that $x_{-2} = 0$, $x_{-1} \in (0, 1)$, $x_0 \in (-1, 0)$, $x_1 = -1$, $x_{5n+2}, x_{5n+6} \in (-2, -1)$, $x_{5n+3}, x_{5n+5} \in (-1, 0)$, and $x_{5n+4} \in (0, 1)$ for $n \in \mathbb{N}_0$.

We prove by induction that all the subsequences $\{x_{5n+3}\}_{n=-1}^{\infty}$, $\{x_{5n+4}\}_{n=-1}^{\infty}$, $\{x_{5n+5}\}_{n=-1}^{\infty}$, $\{x_{5n+6}\}_{n=-1}^{\infty}$, and $\{x_{5n+7}\}_{n=-1}^{\infty}$ are monotone.

Assume $n = -1$. Then, by Lemma 3.3 and above comments, we have

$$\begin{aligned}
x_3 - x_{-2} &= x_3 < 0,\\
x_4 - x_{-1} &= x_{-1}(x_3 - x_{-2}) = x_{-1}x_3 < 0,\\
x_5 - x_0 &= x_0(x_4 - x_{-1}) > 0,\\
x_6 - x_1 &= x_1(x_5 - x_0) < 0,\\
x_7 - x_2 &= x_2(x_6 - x_1) > 0.
\end{aligned} \tag{5.23}$$

Assume that we have proved

$$\begin{aligned}
-1 &< x_{5n+3} < x_{5(n-1)+3} < 0 = x_{-2},\\
0 &< x_{5n+4} < x_{5(n-1)+4} < 1,\\
0 &> x_{5n+5} > x_{5(n-1)+5} > -1,\\
-2 &< x_{5n+6} < x_{5(n-1)+6} < -1,\\
-1 &> x_{5n+7} > x_{5(n-1)+7} > -2,
\end{aligned} \tag{5.24}$$

for $0 \le n \le k$.

Since

$$x_{5(k+1)+j} - x_{5k+j} = x_{5k+j}x_{5k+j-1}x_{5k+j-2}x_{5k+j-3}x_{5k+j-4}\left(x_{5k+j} - x_{5(k-1)+j}\right), \tag{5.25}$$

for each $k \in \mathbb{N}_0$ and $j = 3,4,5,6,7$, and since by Theorem 5.1, we obtain

$$x_{5k+j}x_{5k+j-1}x_{5k+j-2}x_{5k+j-3}x_{5k+j-4} > 0, \tag{5.26}$$

$k \in \mathbb{N}_0$, $j = 3,4,5,6,7$, we have that the differences $x_{5(k+1)+j} - x_{5k+j}$, $k \in \mathbb{N}_0$ have the same sign for each $j \in \{3,4,5,6,7\}$. From this and by Theorem 5.1, the claim follows.

Since all the subsequences $\{x_{5n+3}\}_{n=-1}^{\infty}$, $\{x_{5n+4}\}_{n=-1}^{\infty}$, $\{x_{5n+5}\}_{n=-1}^{\infty}$, $\{x_{5n+6}\}_{n=-1}^{\infty}$ and $\{x_{5n+7}\}_{n=-1}^{\infty}$ are monotone and bounded, they are convergent, and consequently the solution $\{x_n\}_{n=-2}^{\infty}$ converges to a period-five solution, as claimed. □

6. Unbounded Solutions of (1.1)

In this section, we find sets of initial conditions of (1.1) for which unbounded solutions exist.

First, observe that when the initial values $x_{-2}, x_{-1}, x_0 > \overline{x}_1$ or $x_{-2}, x_{-1}, x_0 < -1$, then existence of unbounded solutions appears. Specifically, the following two theorems will show existence of unbounded solutions relative to the set of these initial conditions.

Theorem 6.1. *If $x_{-2}, x_{-1}, x_0 > \overline{x}_1 = (1+\sqrt{5})/2$, then the following statements hold true:*

(a) $x_{-1} < x_1 < x_3 < \cdots$ *and* $x_0 < x_2 < x_4 < \cdots$;

(b) *the solution tends to* $+\infty$.

Proof. (a) Since $x_{-1} > (1+\sqrt{5})/2$, we have $1/x_{-1} < 2/(1+\sqrt{5}) = (\sqrt{5}-1)/2$. Thus,

$$1 + \frac{1}{x_{-1}} < 1 + \frac{\sqrt{5}-1}{2} = \frac{1+\sqrt{5}}{2} < x_{-2}. \tag{6.1}$$

Therefore, $x_{-2} > 1 + 1/x_{-1}$. Thus, $x_{-1}x_{-2} > x_{-1} + 1$. Rewriting, $x_{-1}x_{-2} - 1 > x_{-1}$. Hence, $x_1 > x_{-1}$. One can follow the same steps to prove that $x_2 > x_0$, and the rest of the proof goes by a simple inductive argument.

(b) Suppose, on the contrary, that one of these subsequences given in part (a) is bounded. Then, by the relationship

$$x_{n-2} = \frac{1 + x_{n+1}}{x_{n-1}}, \quad n \in \mathbb{N}_0, \tag{6.2}$$

it would follow that both subsequences $\{x_{2n}\}_{n=0}^{\infty}$ and $\{x_{2n-1}\}_{n=0}^{\infty}$ converge. Hence the whole solution either converges to a period-two solution or to an equilibrium. However, (1.1) does not have any nontrivial period-two solution. Thus, it must converge to an equilibrium. But, this is not possible because the largest equilibrium point is smaller than x_{-1} and x_0. This is a contradiction. Hence, the proof is complete. □

Theorem 6.2. *Assume that $a, b > 0$ and $ab > 1$. Then, each solution with the initial conditions $x_{-2} = a$, $x_{-1} = b$, and $x_0 > (1+a)/(ab-1)$ tends to plus infinity.*

Proof. We have $x_1 = ab - 1 > 0$, $x_2 = x_0 x_{-1} - 1 > (b+1)/(ab-1) > 0$, and

$$x_3 = x_1 x_0 - 1 > (ab-1)\frac{1+a}{ab-1} - 1 = a = x_{-2} > 0. \tag{6.3}$$

By Lemma 3.3, we have

$$x_n - x_{n-5} = (x_3 - x_{-2})\prod_{j=-1}^{n-5} x_j, \quad n \geq 4. \tag{6.4}$$

Using (6.4) and the fact $x_3 - x_{-2} > 0$, by induction it easily follows that

$$x_{5k+j} > x_{5(k-1)+j} > 0, \quad k \in \mathbb{N}_0, \; j = 3,4,5,6,7, \tag{6.5}$$

that is, the subsequences $\{x_{5k+3}\}_{k=-1}^{\infty}$, $\{x_{5k+4}\}_{k=-1}^{\infty}$, $\{x_{5k+5}\}_{k=-1}^{\infty}$, $\{x_{5k+6}\}_{k=-1}^{\infty}$ and $\{x_{5k+7}\}_{k=-1}^{\infty}$ are increasing.

We also have

$$\begin{aligned} x_{5k+3} - x_{5(k-1)+3} &= (x_3 - x_{-2})\prod_{j=1}^{k}\left(x_{5(j-1)+3}x_{5(j-1)+2}x_{5(j-1)+1}x_{5(j-1)}x_{5(j-1)-1}\right) \\ &\geq (x_3 - x_{-2})(x_3 x_2 x_1 x_0 x_{-1})^k \\ &\geq (x_3 - x_{-2})\left(\frac{ab(a+1)(b+1)}{ab-1}\right)^k. \end{aligned} \tag{6.6}$$

Note that

$$q := \frac{ab(a+1)(b+1)}{ab-1} > 1. \tag{6.7}$$

Using this fact and (6.6) we obtain

$$x_{5k+3} > x_{5(k-1)+3} + (x_3 - x_{-2})q^k \Longrightarrow x_{5k+3} > x_{-2} + (x_3 - x_{-2})\sum_{j=0}^{k} q^j, \tag{6.8}$$

from which it follows that $x_{5k+3} \to \infty$ as $k \to \infty$.

From (6.4), the monotonicity of those five subsequences, and (6.8), we get

$$\begin{aligned} x_{5k+3+i} - x_{5(k-1)+3+i} &= x_{5(k-1)+3+i} \cdots x_{5(k-1)+3+1}\left(x_{5k+3} - x_{5(k-1)+3}\right) \\ &> x_{i-2} \cdots x_{-1}\left(x_{5k+3} - x_{5(k-1)+3}\right) \\ &> x_{i-2} \cdots x_{-1}(x_3 - x_{-2})q^k, \end{aligned} \tag{6.9}$$

for $i = 1,2,3,4$, from which it follows that $x_{5k+3+i} \to \infty$ as $k \to \infty$ for each $i \in \{1,2,3,4\}$, finishing the proof of the theorem. □

Remark 6.3. Note that the last theorem shows that if initial values are moved to the right with respect to the initial values of a positive solution of period five then such solutions go to plus infinity.

7. Case $x_{-2}, x_{-1}, x_0 \in (1, \overline{x}_1)$

Here, we consider the case $x_{-2}, x_{-1}, x_0 \in (1, \overline{x}_1)$. The next theorem shows that there is a large class of eventually nondecreasing solutions of (1.1) converging to $\overline{x}_1$. For some results of this type, see, for example, [1, 6, 7, 10, 13, 15, 18–22] and the related references therein.

Theorem 7.1. *Assume $x_{-2}, x_{-1}, x_0 \in (1, \overline{x}_1)$ and $x_{-2}, x_{-1} \leq x_0 \leq x_{-1}x_{-2} - 1$. Then, every solution with such initial values is eventually nondecreasing and converges to $\overline{x}_1$.*

Proof. Multiplying the assumption $x_{-2} \leq x_0$, by x_{-1} we obtain $x_{-1}x_{-2} \leq x_{-1}x_0$. From this and since $1 < x_{-2}, x_{-1}, x_0 < \overline{x}_1$, we obtain $0 < x_{-1}x_{-2} - 1 \leq x_{-1}x_0 - 1 < \overline{x}_1^2 - 1 = \overline{x}_1$, that is, $0 < x_1 \leq x_2 < \overline{x}_1$. Hence, $1 < x_{-2}, x_{-1} \leq x_0 \leq x_1 \leq x_2 < \overline{x}_1$. Now assume

$$1 < x_{-2}, x_{-1} \leq x_0 \leq x_1 \leq x_2 \leq \cdots \leq x_n < \overline{x}_1, \tag{7.1}$$

for some $n \geq 2$. Multiplying the inequality $x_{n-3} \leq x_{n-1}$ by x_{n-2} and using (7.1), it follows that

$$x_{n-3}x_{n-2} - 1 \leq x_{n-1}x_{n-2} - 1 < \overline{x}_1^2 - 1 = \overline{x}_1, \tag{7.2}$$

that is, $x_n \leq x_{n+1} < \overline{x}_1$. Hence, by induction we have proved that the sequence $\{x_n\}_{n=-1}^{\infty}$ is nondecreasing and bounded above by $\overline{x}_1$, from which the result easily follows. □

Theorem 7.2. *Assume $x_{-2}, x_{-1}, x_0 \in (1, \overline{x}_1)$, $x_0 \leq \max\{x_{-1}, x_{-2}\}$, $x_{-1}x_{-2} - 1 \leq \min\{x_{-2}, x_{-1}, x_0\}$, and $x_0 x_{-1} - 1 \leq x_0$. Then, for every solution with such initial values there is an $N \geq 2$, such that*

$$x_2 \geq x_3 \geq \cdots \geq x_N \geq 0 > x_{N+1}. \tag{7.3}$$

Proof. According to the assumptions we have

$$0 = 1 \cdot 1 - 1 < x_1 = x_{-1}x_{-2} - 1 \leq x_{-1} < \overline{x}_1, \tag{7.4}$$

$$0 = 1 \cdot 1 - 1 < x_2 = x_0 x_{-1} - 1 \leq x_0 < \overline{x}_1. \tag{7.5}$$

If $x_0 \leq x_{-2}$, then by multiplying by x_{-1}, we obtain

$$x_2 = x_0 x_{-1} - 1 \leq x_{-1}x_{-2} - 1 = x_1. \tag{7.6}$$

If $x_0 \le x_{-1}$, then since $x_1 = x_{-1}x_{-2} - 1 \le x_{-2}$, we obtain

$$x_3 = x_1 x_0 - 1 \le x_{-1}x_{-2} - 1 = x_1. \tag{7.7}$$

Multiplying the inequality $x_1 \le x_{-1}$ (see (7.4)) by x_0, we obtain

$$x_3 \le x_2, \tag{7.8}$$

which, together with inequality (7.6), implies again $x_3 \le x_1$.

Note that now we cannot guarantee the positivity of x_3. Similarly, from the inequality $x_2 \le x_0$ (see (7.5)), we obtain

$$x_4 \le x_3, \tag{7.9}$$

and from the inequality $x_3 \le x_1$, we obtain

$$x_5 \le x_4. \tag{7.10}$$

Now assume that

$$0 < x_n \le x_{n-1} \le \cdots x_3 \le x_2 < \overline{x}_1 \tag{7.11}$$

and $x_{n+1} > 0$. Then, by multiplying the inequality $x_{n-1} \le x_{n-3}$ by x_{n-2} and subtracting 1, we obtain $x_{n+1} \le x_n$. Hence, we proved by induction that (7.11) holds as far as x_n is positive.

If $x_n > 0$ for all $n \ge 2$, the sequence $\{x_n\}_{n=-2}^{\infty}$ is convergent and its limit is nonnegative. However, this is not possible since the only nonnegative equilibrium of (1.1) is $\overline{x}_1$. From this, the result follows. □

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