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Research Article

Positive Solutions for Second-Order Three-Point Eigenvalue Problems

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With the help of the fixed point index theorem in cones, we get an existence theorem concerning the existence of positive solution for a second-order three-point eigenvalue problem $x''(t)+\lambda f(t,x(t)) = 0$, $0 \le t \le 1$, $x(0) = 0$, $x(1) = x(\eta)$, where λ is a parameter. An illustrative example is given to demonstrate the effectiveness of the obtained result.

1. Introduction

Motivated by the work of Bitsadze and Samarskii [1] and Ilyin and Moiseev [2], much attention has been paid to the study of certain nonlocal boundary value problems (BVPs) in recent years.

In the last twenty years, many mathematician, have considered the existence of positive solutions of nonlinear three-point boundary value problems; see, for example, Graef et al. [3] Webb [4], Gupta and Trofimchuk [5], Infante [6], Ehrke [7], Ma [8], Feng [9], He and Ge [10], Bai and Fang [11], and Guo [12]. Recently, by applying the Avery-Henderson [13] double fixed point theorem, Henderson [14] studied the existence of two positive solutions of the three-point boundary value problem for the second-order differential equation

$$\begin{gathered} y'' + f(y) = 0, \quad 0 \le t \le 1, \\ y(0) = 0, \qquad y(p) - y(1) = 0, \end{gathered} \tag{1.1}$$

where $0 < p < 1$ and $f : R \to [0,+\infty)$ is continuous.

In this paper, motivated and inspired by the above work and Wong [15], we apply a fixed point index theorem in cones to investigate the existence of positive solutions for nonlinear three-point eigenvalue problems

$$\begin{aligned} x''(t) + \lambda f(t, x(t)) = 0, \quad & 0 \le t \le 1, \\ x(0) = 0, \quad & x(1) = x(\eta), \end{aligned} \tag{1.2}$$

where $0 < \eta < 1$ and $f \in C([0,1] \times [0,+\infty), [0,\infty))$.

We need the following well-known lemma. See [16] for a proof and further discussion of the fixed point index $i(A, K_r, K)$.

Lemma 1.1. *Assume that E is a Banach space, and $K \subset E$ is a cone in E. Let $K_r = \{x \in K : \|x\| < r\}$. Furthermore, assume that $A : K \to K$ is a completely continuous map, and $Ax \neq x$ for $x \in \partial K_r = \{x \in K : \|x\| = r\}$. Then, one has the following conclusions:*

(1) *if $\|x\| \le \|Ax\|$ for $x \in \partial K_r$, then $i(A, K_r, K) = 0$;*

(2) *if $\|x\| \ge \|Ax\|$ for $x \in \partial K_r$, then $i(A, K_r, K) = 1$.*

2. Main Results

In the following, we will denote by $C[0,1]$ the space of all continuous functions $x : [0,1] \to R$. This is a Banach space when it is furnished with usual sup-norm $\|x\| := \sup\{|x(s)| : s \in [0,1]\}$.

By [14], the Green's function for the three-point boundary-value problem

$$-x'' = 0, \quad x(0) = 0, \quad x(1) = x(\eta) \tag{2.1}$$

is given by

$$G(t,s) = \begin{cases} t, & t \le s \le \eta, \\ s, & s \le t, \ s \le \eta, \\ \dfrac{1-s}{1-\eta} t, & t \le s, \ s \ge \eta, \\ s + \dfrac{\eta - s}{1-\eta} t, & \eta \le s \le t. \end{cases} \tag{2.2}$$

From the Green's function $G(t,s)$, we have that a function x is a solution of the boundary value problem (1.2) if and only if it satisfies

$$x(t) = \lambda \int_0^1 G(t,s) f(s, x(s)) ds, \quad t \in [0,1]. \tag{2.3}$$

Lemma 2.1. *Suppose that $G(t,s)$ is defined as above. Then we have the following results:*

(1) $0 \le G(t,s) \le G(s,s),\ 0 \le t,\ s \le 1$,

(2) $G(t,s) \ge \eta t G(s,s),\ 0 \le t,\ s \le 1$.

Proof. It is easy to see that (1) holds. To show that (2) holds, we distinguish four cases.

(i) If $t \le s \le \eta$, then

$$G(t,s) = t \ge \eta t s = \eta t G(s,s). \tag{2.4}$$

(ii) If $s \le t$ and $s \le \eta$, then

$$G(t,s) = s \ge \eta t s = \eta t G(s,s). \tag{2.5}$$

(iii) If $t \le s$ and $s \ge \eta$, then

$$G(t,s) = \frac{1-s}{1-\eta} t \ge \eta s t \frac{1-s}{1-\eta} = \eta t G(s,s). \tag{2.6}$$

(iv) Finally, if $\eta \le s \le t$, then

$$\begin{aligned} G(t,s) &= s - \frac{s-\eta}{1-\eta} t \ge s - \frac{s-\eta}{1-\eta} = \frac{\eta(1-s)}{1-\eta} \\ &\ge t s \frac{\eta(1-s)}{1-\eta} = \eta t \frac{s(1-s)}{1-\eta} = \eta t G(s,s). \end{aligned} \tag{2.7}$$

□

Remark 2.2. If $s \le \eta$ and $s \ge \eta$, then $G(s,s) = s$ and $G(s,s) = s(1-s)/(1-\eta)$, respectively.

Define

$$P = \left\{ u \in C[0,1] : u(t) \ge 0,\quad \min_{t \in [\eta/2,1]} u(t) \ge \frac{\eta^2}{2} \|u\| \right\}. \tag{2.8}$$

Obviously, P is a cone in the Banach space $C[0,1]$.

Define an operator $A : P \to C[0,1]$ as follows:

$$(Ax)(t) := \lambda \int_0^1 G(t,s) f(s, x(s)) ds, \quad t \in [0,1]. \tag{2.9}$$

It is easy to know that fixed points of A are solutions of the BVP (1.2).

Now, we can state and prove our main results.

Lemma 2.3. *$A : P \to P$ is completely continuous.*

Proof. For any $x \in P$, by Lemma 2.1 (1), we have $(Ax)(t) \geq 0$, for each $t \in [0,1]$. It follows from Lemma 2.1 that

$$\begin{aligned}\min_{t\in[\eta/2,1]}(Ax)(t) &= \lambda \min_{t\in[\eta/2,1]}\int_0^1 G(t,s)f(s,x(s))ds \geq \frac{\lambda\eta^2}{2}\int_0^1 G(s,s)f(s,x(s))ds \\ &\geq \frac{\lambda\eta^2}{2}\int_0^1 G(t,s)f(s,x(s))ds.\end{aligned} \tag{2.10}$$

Hence, $\min_{t\in[\eta/2,1]}(Ax)(t) \geq (\eta^2/2)\|Ax\|$, which implies $AP \subset P$. Moreover, it is easy to check that $A : P \to P$ is completely continuous. □

By simple calculation, we obtain that

$$\int_{\eta/2}^1 G\left(\frac{\eta}{2},s\right)ds = \int_{\eta/2}^{\eta}\frac{\eta}{2}ds + \int_{\eta}^1 \frac{1-s}{1-\eta}\frac{\eta}{2}ds = \frac{\eta^2}{4} + \frac{\eta(1-\eta)}{4} = \frac{\eta}{4}. \tag{2.11}$$

Lemma 2.4. *Suppose that there exists a positive constant $r > 0$ such that*

$$(H1) \quad f(t,x) \geq \frac{4r}{\eta} \qquad on\ \left[\frac{\eta}{2},1\right] \times \left[\frac{\eta^2}{2}r, r\right] \tag{2.12}$$

holds. If $\lambda > 1$, then

$$i(A, P_r, P) = 0. \tag{2.13}$$

Proof. For $x \in \partial P_r$, it follows from the definition of the cone that

$$\frac{\eta^2}{2}r = \frac{\eta^2}{2}\|x\| \leq \min_{t\in[\eta/2,1]} x(t) \leq x(t) \leq \|x\| = r, \quad t \in \left[\frac{\eta}{2},1\right], \tag{2.14}$$

which implies

$$\frac{\eta^2}{2}r \leq x \leq r, \quad t \in \left[\frac{\eta}{2},1\right]. \tag{2.15}$$

Thus, we have by (H1) and (2.11) that

$$\begin{aligned}(Ax)\left(\frac{\eta}{2}\right) &= \lambda\int_0^1 G\left(\frac{\eta}{2},s\right)f(s,x(s))ds > \int_{\eta/2}^1 G\left(\frac{\eta}{2},s\right)f(s,x(s))ds \\ &\geq \frac{4r}{\eta}\int_{\eta/2}^1 G\left(\frac{\eta}{2},s\right)ds = r = \|x\|\end{aligned} \tag{2.16}$$

since $x \in \partial P_r$. This shows that

$$\|Ax\| > \|x\|, \quad \forall x \in \partial P_r. \tag{2.17}$$

It is obvious that $Ax \neq x$ for $x \in \partial P_r$. Therefore, by Lemma 1.1 (1), we conclude that $i(A, P_r, P) = 0$. □

Lemma 2.5. *Suppose that there exists a positive constant $m > 0$ such that*

$$(H2) \quad f(t,x) \leq p(t)g(x) \quad \text{on } [0,1] \times [0,m], \tag{2.18}$$

where $p \in C([0,1],[0,+\infty))$ and $g \in C([0,+\infty),[0,+\infty))$. If

$$\lambda < \frac{1}{2p_1}\left(\int_0^m \frac{ds}{\sqrt{H(m)-H(s)}}\right)^2, \tag{2.19}$$

where $H(u) := \int_0^u g(s)ds$ and $p_1 = \max_{t\in[0,1]} p(t) > 0$, then

$$i(A, P_m, P) = 1. \tag{2.20}$$

Proof. First, we claim that

$$Ax \neq \mu x, \quad \text{for } x \in \partial P_m,\ \mu \geq 1. \tag{2.21}$$

Suppose to the contrary that there exist $x \in \partial P_m$ and $\mu_0 \geq 1$ such that

$$Ax = \mu_0 x, \quad \text{for } x \in \partial P_m. \tag{2.22}$$

It is clear that (2.22) is equivalent to

$$x''(t) + \frac{\lambda}{\mu_0} f(t,x) = 0. \tag{2.23}$$

Since $x \in C[0,1]$ and $x(\eta) = x(1)$, it follows that there exists a $\xi \in (\eta,1)$ such that $x'(\xi) = 0$. From $x'' \leq 0$ on $(0,1)$, we see that $x(\xi) = \|x\| := m > 0$, $x'(t) \geq 0$ on $(0,\xi)$, and $x'(t) \leq 0$ on $(\xi,1)$. By (2.18) and (2.23), we have

$$x''(t) = -\frac{\lambda}{\mu_0} f(t,x(t)) \geq -\frac{\lambda}{\mu_0} p(t)g(x(t)), \quad t \in [0,1]. \tag{2.24}$$

Multiplying (2.24) by x' and then integrating from t to ξ ($t \in [0,\xi)$), we get from $x'(\xi) = 0$ that

$$-\frac{1}{2}\left(x'(t)\right)^2 \geq -\frac{\lambda}{\mu_0}\int_t^{\xi} p(s)g(x(s))x'(s)ds, \quad t \in [0,\xi), \tag{2.25}$$

that is

$$(x'(t))^2 \leq \frac{2\lambda}{\mu_0}\int_t^{\xi} p(s)g(x(s))x'(s)ds \leq \frac{2\lambda p_1}{\mu_0}\int_{x(t)}^{x(\xi)} g(u)du, \quad t \in [0,\xi), \tag{2.26}$$

which implies that

$$0 \leq x'(t) \leq \sqrt{\frac{2\lambda p_1}{\mu_0}(H(m) - H(x(t)))}, \quad t \in [0,\xi). \tag{2.27}$$

Thus,

$$\int_0^m \frac{ds}{\sqrt{(2\lambda p_1/\mu_0)(H(m) - H(s))}} \leq \int_0^{\xi} dt = \xi \leq 1. \tag{2.28}$$

Hence, we obtain from (2.19) and (2.28) that

$$1 \geq \sqrt{\frac{\mu_0}{2\lambda p_1}}\int_0^m \frac{ds}{\sqrt{H(m) - H(s)}} \geq \sqrt{\frac{1}{2\lambda p_1}}\int_0^m \frac{ds}{\sqrt{H(m) - H(s)}} > 1. \tag{2.29}$$

This contradiction implies that (2.21) holds. By (2.21), we have $\|Ax\| \leq \|x\|$ for $x \in \partial P_m$, and $Ax \neq x$ for $x \in \partial P_m$. Thus, by Lemma 1.1 (2), we obtain

$$i(A, P_m, P) = 1. \tag{2.30}$$

□

For convenience, let

$$\text{(H3)} \quad 2p_1 < \left(\int_0^m \frac{ds}{\sqrt{H(m) - H(s)}}\right)^2. \tag{2.31}$$

Theorem 2.6. *Assume that there exist two distinct positive constants r, m such that (H1)–(H3) hold. If*

$$1 < \lambda < \frac{1}{2p_1}\left(\int_0^m \frac{ds}{\sqrt{H(m) - H(s)}}\right)^2, \tag{2.32}$$

then BVP (1.2) *has at least one positive solution.*

Proof. From Lemmas 2.4 and 2.5, and the property of the fixed point index, we can easily get that the operator A has a fixed point in $\overline{P_m} \setminus P_r$ $(m > r)$ or in $\overline{P_r} \setminus P_m$ $(r > m)$. Therefore, BVP (1.2) has at least one positive solution. □

3. An Example

To illustrate our results we present the following example.

Example 3.1. Consider the following boundary value problem

$$\begin{gathered} x''(t) + \frac{\lambda}{2} e^t (1 + |\sin(xt)|) x^2(t) = 0, \quad 0 < t < 1, \\ x(0) = 0, \qquad x(1) = x\left(\frac{1}{2}\right). \end{gathered} \tag{3.1}$$

Let $f(t, x) = (1/2)e^t(1 + |\sin(xt)|)x^2$ and $\eta = 1/2$. Choosing $m = 1/3,\ r = 800$, we have

$$f(t, x) \le p(t) g(x), \quad \text{on } [0, 1] \times \left[0, \frac{1}{3}\right], \tag{3.2}$$

where $p(t) = e^t$ and $g(x) = x^2$. Thus, $p_1 = e$, $H(u) = (1/3)u^3$, and

$$\left(\int_0^m \frac{ds}{\sqrt{H(m) - H(s)}}\right)^2 = \left(\int_0^{1/3} \frac{ds}{\sqrt{1/81 - s^3/3}}\right)^2 = 6.3701 > 5.4366 = 2p_1. \tag{3.3}$$

Hence, (H2) and (H3) hold. Moreover, we get

$$\begin{aligned} f(t, x) &\ge \frac{1}{2} e^t x^2 \quad \left(\text{on } \left[\frac{\eta}{2}, 1\right] \times \left[\frac{\eta^2}{2} r, r\right]\right) \\ &\ge \frac{1}{2} e^{1/4} 100^2 \quad \left(\text{on } \left[\frac{1}{4}, 1\right] \times [100, 800]\right) \\ &= 6420 > 6400 = 8r = \frac{4r}{\eta} \quad \left(\text{on } \left[\frac{1}{4}, 1\right] \times [100, 800]\right), \end{aligned} \tag{3.4}$$

which implies that (H1) holds. Therefore, it follows from Theorem 2.6 that BVP (3.1) has at least one positive solution if

$$1 < \lambda < \frac{1}{2p_1} \left(\int_0^m \frac{ds}{\sqrt{H(m) - H(s)}}\right)^2 = \frac{6.3701}{5.4366} = 1.1717. \tag{3.5}$$

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