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Research Article

The Hyperspherical Functions of a Derivative

Nenad Cakic,[1] Duško Letic,[2] and Branko Davidovic[3]

[1] *Faculty of Electrical Engineering, University of Belgrade, 11000 Belgrade, Serbia*
[2] *Technical Faculty M. Pupin, 23000 Zrenjanin, Serbia*
[3] *Technical High School, 34000 Kragujevac, Serbia*

Correspondence should be addressed to Nenad Cakic, cakic@etf.rs

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We present the results of theoretical research of the generalized hypersherical function (HS) by generalizing two known functions related to the sphere hypersurface and hypervolume and the recurrent relation between them. By introducing two-dimensional degrees of freedom k and n (and the third, radius r), we develop the derivative functions for all three arguments and the possibilities of their use. The symbolical evolution, numerical experiment, and graphical presentation of functions are realized using the Mathcad Professional and Mathematica softwares.

1. Introduction

The hyperspherical function (HS) is a hypothetical function related to multidimensional space and generalization of the sphere geometry. This function is primarily formed on the basis of the interpolating power of the gamma function. It belongs to the group of special functions, so its testing, besides gamma, is performed on the basis of the related functions, such as Γ-gamma, ψ-psi, β-beta, erf, and so forth. Its most significant value is in its generalizing from discrete to continuous [1]. In addition, we move from the field of natural integers of the geometry sphere dimensions—degrees of freedom, to the set of real and nonintegral values, thus, we obtain the prerequisites for a more concise analysis of this function. In this paper the analysis is focused on the infinitesimal calculus application of the hyperspherical function that is given in its generalized form. For the development of hyperspherical and other functions of the multidimensional objects, see: Bishop and Whitlock [2], Collins [3], Conway [4], Dodd and Coll [5], Hinton [6], Hocking and Young [1], Manning [7], Maunder [8], Neville [9], Rohrmann and Santos [10], Rucker [11], Maeda et al. [12], Sloane [13], Sommerville [14], Weels [15], and others; see [16–22]. Nowadays, the research of the hypersperical functions is represented both in Euclid's and Riemann's geometry and topology (Riemann's and Poincare's sphere, multidimensional potentials, theory of fluids, nuclear (atomic) physics, hyperspherical black holes, etc.)

2. The Hyperspherical Functions of a Derivative

2.1. The Hyperspherical Funcional Matrix

The former results, as it is known, [5, 23], give the functions of the hyperspherical surface ($n = 2$) and volume ($n = 3$) therefore, we have

$$\begin{aligned} \mathrm{HS}(k,2,r) &= \frac{2\sqrt{\pi^k} r^{k-1}}{\Gamma(k/2)} = \frac{\partial}{\partial r}\mathrm{HS}(k,3,r), \\ \mathrm{HS}(k,3,r) &= \int_0^r \mathrm{HS}(k,2,r)dr = \frac{\sqrt{\pi^k} r^k}{\Gamma(k/2+1)}. \end{aligned} \tag{2.1}$$

In general, we have

$$\begin{aligned} \mathrm{HS}(k,n,r) &= \int_0^r \mathrm{HS}(k,n-1,r)dr = \iint_0^r \cdots \int_0^r S(k,r)dr\, \underset{n-2}{dr \cdots dr} \\ &= \iint_0^r \cdots \int_0^r \frac{2\pi^{k/2} r^{k-1}}{\Gamma(k/2)} dr\, \underset{n-2}{dr \cdots dr}. \end{aligned} \tag{2.2}$$

Thus, we give the definition of the hyperspherical function [24].

Definition 2.1. The hyperspherical function with two degrees of freedom k and n is defined as

$$\mathrm{HS}(k,n,r) = \frac{2\sqrt{\pi^k} r^{k+n-3}\Gamma(k)}{\Gamma(k+n-2)\Gamma(k/2)} \quad (k,n \in \Re). \tag{2.3}$$

This is a function of three variables and two degrees of freedom k and n and the radius of the hypersphere r. In real spherical entities (Figure 1) such as, diameter, circumference or cycle surface; followed by sphere surface and volume, only the variable r exists.

By keeping the property that the derivatives on the radius r generate new functions (the HS matrix columns), we perform "movements" to lower or higher degrees of freedom, from the starting nth, on the basis of the following recurrent relations:

$$\frac{\partial}{\partial r}\mathrm{HS}(k,n,r) = \mathrm{HS}(k,n-1,r), \qquad \mathrm{HS}(k,n+1,r) = \int_0^r \mathrm{HS}(k,n,r)dr. \tag{2.4}$$

This property is fundamental and hypothetically also holds also for objects out of this submatrix of six elements (Figures 1 and 2). The derivation example (right) shows that, we have obtained the zeroth ($n = 0$) degree of freedom, if we perform the derivation of the nth

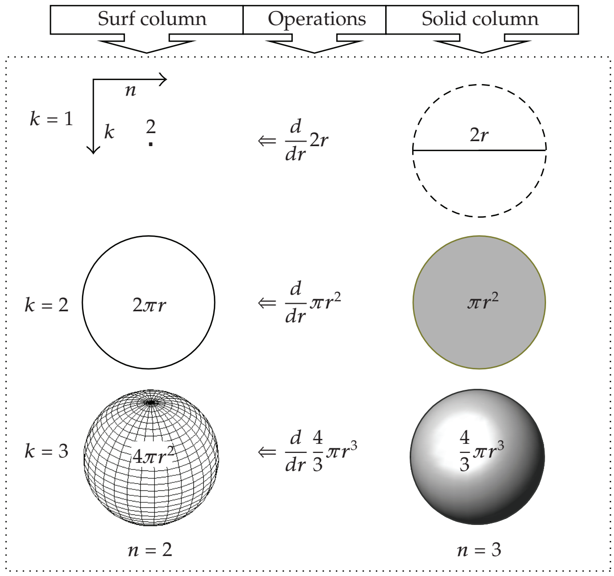

Figure 1: Moving through the vector of real surfaces left: by deducting one degree of freedom k of the surface sphere, we obtain circumference, and for two (degrees), we obtain the point 2. Moving through the vector of real solids (right column), that is, by deduction of one degree of freedom k from the solid sphere, we obtain a circle (disk), and for two (degrees), we get a line segment or diameter.

degree. With the defined value of the derivative, the HS function is also valid in the complex domain of the hyperspherical matrix. Namely, the following expressions hold:

$$\frac{\partial^n}{\partial r^n}\mathrm{HS}(k,n,r)=\mathrm{HS}(k,0,r)\quad\text{or}\quad\frac{\partial^{2n}}{\partial r^{2n}}\mathrm{HS}(k,n,r)=\mathrm{HS}(k,-n,r). \tag{2.5}$$

On the basis of the general hyperspherical function, we obtain the appropriate matrix $M_{k\times n}$ $(k,n\in\mathfrak{R})$, where we also give the concrete values for the selected submatrix 9×9.

For example if $n=2$ and $k\in N$, we obtain the following relation (for $k=\overline{0,5}$):

$$\frac{\partial^4}{\partial r^4}\mathrm{HS}(k,2,r)=\mathrm{HS}(k,-2,r),\quad\text{respectively,}\quad\frac{\partial^4}{\partial r^4}\begin{bmatrix}0\\2\\2\pi r\\4\pi r^2\\2\pi^2 r^3\\\dfrac{8\pi^2 r^4}{3}\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\\0\\64\pi^2\end{bmatrix}. \tag{2.6}$$

The matrix $[M]_{k\times n}$ has the property that every vector of the ith column is a derivative with respect to the radius of the next vector in the sequence according to Figure 2. This recursion property originates from the starting assumptions (2.1). Therefore, we have the following theorem.

Theorem 2.2. *For the columns of the matrix* $[M]_{k\times n}$*, the following equality holds:*

$$[M]^{\langle n-1\rangle} = \frac{\partial}{\partial r}[M]^{\langle n\rangle}. \tag{2.7}$$

For example, for two adjacent columns of the matrix $[M]^{\langle n-1\rangle}$ and $[M]^{\langle n\rangle}$, one obtains the recurrent vectors

$$[M]^{\langle n-1\rangle} = \frac{\partial}{\partial r}\begin{bmatrix} \dfrac{r^{n-3}}{\Gamma(n-2)} \\ \dfrac{2r^{n-2}}{\Gamma(n-1)} \\ \dfrac{2\pi r^{n-1}}{\Gamma(n)} \\ \dfrac{8\pi r^{n}}{\Gamma(n+1)} \\ \vdots \\ \dfrac{2\sqrt{\pi^k}r^{k+n-3}\Gamma(k)}{\Gamma(k+n-2)\Gamma(k/2)} \end{bmatrix} = \begin{bmatrix} \dfrac{r^{n-4}}{\Gamma(n-3)} \\ \dfrac{2r^{n-3}}{\Gamma(n-2)} \\ \dfrac{2\pi r^{n-2}}{\Gamma(n-1)} \\ \dfrac{8\pi r^{n-1}}{\Gamma(n)} \\ \vdots \\ \dfrac{2\sqrt{\pi^k}r^{k+n-4}\Gamma(k)}{\Gamma(k+n-1)\Gamma(k/2)} \end{bmatrix} \quad (n \in \Re). \tag{2.8}$$

One obtains interesting results on the basis of horizontal (n) or vertical (k) degrees of freedom. Therefore, one obtains the following:

$$\left.\frac{2\sqrt{\pi^k}r^k}{k\Gamma(k/2)}\right|_{k=3} \vee \left.\frac{8\pi r^n}{\Gamma(n+1)}\right|_{n=3} \vee \left.\frac{2\sqrt{\pi^k}r^{k+n-3}\Gamma(k)}{\Gamma(k+n-2)\Gamma(k/2)}\right|_{k=3\wedge n=3} \Longrightarrow \frac{4}{3}\pi r^3. \tag{2.9}$$

Consequently, this property is fundamental, because it leads to the same result on the basis of two special formulas, or using only one general.

2.2. The Analysis of the Recurrent Potential Function of the Type z^υ

The hyperspherical function, besides the gamma functions also contains the potential component r^{k+n-3}. The generalized equation of the hth derivative of the graded expression z^υ is [25]

$$\frac{\partial^h z^\upsilon}{\partial z^h} = \frac{\Gamma(\upsilon+1)}{\Gamma(\upsilon-h+1)}z^{\upsilon-h} \quad (-\upsilon \notin N). \tag{2.10}$$

Namely, it is known that the exponent with the basis of the HS function radius is $\upsilon = k+n-3$. Now the mth derivative of the hyperspherical function with respect to the radius is defined by the relation

$$\frac{\partial^m}{\partial r^m}\mathrm{HS}(k,n,r) = \frac{2\sqrt{\pi^k}\Gamma(k)}{\Gamma(k+n-2)\Gamma(k/2)} \cdot \frac{\partial^m r^{k+n-3}}{\partial r^m}. \tag{2.11}$$

$k \backslash n$	−2	−1	0	1	2	3	4	5	6
−3	$\frac{1260}{\pi^2 r^8}$	$-\frac{180}{\pi^2 r^7}$	$\frac{30}{\pi^2 r^6}$	$-\frac{6}{\pi^2 r^5}$	$\frac{3}{2\pi^2 r^4}$	$-\frac{1}{2\pi^2 r^3}$	$\frac{1}{4\pi^2 r^2}$	$-\frac{1}{4\pi^2 r}$	Undef.
−2	0	0	0	0	0	0	0	$-\frac{1}{2\pi}$	$-\frac{r}{2\pi}$
−1	$-\frac{120}{\pi r^6}$	$\frac{24}{\pi r^5}$	$-\frac{6}{\pi r^4}$	$\frac{2}{\pi r^3}$	$-\frac{1}{\pi r^2}$	$\frac{1}{\pi r}$	Undef.	Undef.	Undef.
0	0	0	0	0	0	1	r	$\frac{r^2}{2}$	$\frac{r^3}{6}$
1	0	0	0	0	2	$2r$	r^2	$\frac{r^3}{3}$	$\frac{r^4}{12}$
2	0	0	0	2π	$2\pi r$	πr^2	$\frac{\pi r^3}{3}$	$\frac{\pi r^4}{12}$	$\frac{\pi r^5}{60}$
3	0	0	8π	$8\pi r$	$4\pi r^2$	$\frac{4\pi r^3}{3}$	$\frac{\pi r^4}{3}$	$\frac{\pi r^5}{15}$	$\frac{\pi r^6}{90}$
4	0	$12\pi^2$	$12\pi^2 r$	$6\pi^2 r^2$	$2\pi^2 r^3$	$\frac{\pi^2 r^4}{2}$	$\frac{\pi^2 r^5}{10}$	$\frac{\pi^2 r^6}{60}$	$\frac{\pi^2 r^6}{420}$
5	$64\pi^2$	$64\pi^2 r$	$32\pi^2 r^2$	$\frac{32\pi^2 r^3}{3}$	$\frac{8\pi^2 r^4}{3}$	$\frac{8\pi^2 r^5}{15}$	$\frac{4\pi^2 r^6}{45}$	$\frac{4\pi^2 r^7}{315}$	$\frac{\pi^2 r^8}{630}$

(n/k; $n \in N$; $k \in N$)

Figure 2: The submatrix $\mathrm{HS}(k, n, r)$ of the function for $k \in -3, -2, \ldots, 5$ and $n \in -2, -1, \ldots, 6$ with six highlighted characteristic functions (undef. are undefined values, predominately singular $\pm\infty$ of this function value).

Therefore, using (2.10), the partial derivative in that case is

$$\frac{\partial^m r^{k+n-3}}{\partial r^m} = \frac{r^{k-m+n-3}\Gamma(k+n-2)}{\Gamma(k-m+n-2)}. \tag{2.12}$$

After some transforming, we obtain the form

$$\frac{\partial^m}{\partial r^m}\mathrm{HS}(k, n, m, r) = \frac{2r^{k-m+n-3}\sqrt{\pi^k}\,\Gamma(k)}{\Gamma(k-m+n-2)\Gamma(k/2)}. \tag{2.13}$$

Using the known property $\Gamma(z+1) = z\Gamma(z)$, we have the following theorem.

Theorem 2.3. *The mth derivative of the hyperspherical function with respect to the radius r is*

$$\frac{\partial^m}{\partial r^m} HS(k,n,m,r) = \frac{r^{k-m+n-3}\sqrt{\pi^k}\Gamma(k+1)}{\Gamma(k-m+n-2)\Gamma(k/2+1)}. \tag{2.14}$$

The above equation (2.14) is recurrent by nature, and with it, we obtain every derivative (for $+m$) and integral (for $-m$), depending on what position (n) we want to perform these operations. That way we can define and, in appropriate cases, use a unique operator that unites the operations of differentiating and integrating with respect to the radius of the HS function.

Definition 2.4. The unique operator that unites the operations of differentiating and integrating with respect to the radius of the hyperspherical function is given by:

$$\frac{\partial^{\pm m}}{\partial r^{\pm m}} \mathrm{HS} \equiv D^{\pm m}\mathrm{HS}, \tag{2.15}$$

where $\partial/\partial r \equiv D$.

Symbolically, the integrating (more exactly, the operator $\partial^{-m}/\partial r^{-m}$) can be written in the form

$$\frac{\partial^{-m}}{\partial r^{-m}} \mathrm{HS}(k,n,m,r) = \underbrace{\iint_0^r \cdots \int_0^r \mathrm{HS}(k,n,r)dr\, dr \cdots dr}_{m} = \frac{2r^{k+m+n-3}\sqrt{\pi^k}\Gamma(k)}{\Gamma(k+m+n-2)\Gamma(k/2)}. \tag{2.16}$$

Because of the known characteristics of the gamma function, the value of the differential or integral degree m, does not have to be an integer, as, for example, with classical differentiating (integrating).

2.3. The Derivatives of the Dichotomic HS Functions

We now consider the case in which the hyperspherical function, with the added variable m, can be separated on even and odd members. We form the derivatives of the function with odd members by substituting the Legendre's equivalent in the function $\mathrm{HS}(k,n,m,r)$. Consequently, for these members, we consider the following transformation:

$$\frac{2\sqrt{\pi^k} r^{k-m+n-3}\Gamma(k)}{\Gamma(k-m+n-2)\Gamma(k/2)} \xrightarrow{\Gamma(k/2)=2^{1-k}\sqrt{\pi}(k-1)!/((k-1)/2)!} \frac{2^k \pi^{(k-1)/2} r^{k-m+n-3}}{(k-m+n-3)!}\left(\frac{k-1}{2}\right)!. \tag{2.17}$$

From this, we obtain the function of the mth derivative for odd members (index 1).

Theorem 2.5. *The mth derivative for odd members of the hyperspherical function with respect to r is*

$$\frac{\partial^m}{\partial r^m} HS_1(k,n,m,r) = \frac{2^k \pi^{(k-1)/2} r^{k-m+n-3}}{(k-m+n-3)!}\left(\frac{k-1}{2}\right)!. \tag{2.18}$$

The second, complementary function, with even members (index 2), is obtained on the basis of the second transformation

$$\frac{2\sqrt{\pi^k}r^{k-m+n-3}\Gamma(k+1)}{\Gamma(k-m+n-2)\Gamma(k/2+1)} \xrightarrow{\Gamma(k/2+1)=(k/2)\Gamma(k/2)} \frac{2\sqrt{\pi^k}r^{k-m+n-3}(k-1)!}{(k-m+n-3)!(k/2-1)!}. \tag{2.19}$$

Thus, we form the second, dichotomous function.

Theorem 2.6. *The mth derivative for even members of the hyperspherical function with respect to r is*

$$\frac{\partial^m}{\partial r^m}HS_2(k,n,m,r) = \frac{2\sqrt{\pi^k}r^{k-m+n-3}(k-1)!}{(k-m+n-3)!(k/2-1)!}. \tag{2.20}$$

Taking into consideration that the dichotomous functions as well as their adequate functional series are complementary, by differentiating the hyperspherical function the same characteristic is retained, but for all that we must perform a substitution of variables, for odd members, $k = 2b+1$, and for even ones $k = 2b$. So, the values of the complementary members, are, respectively.

Corollary 2.7. *One has*

$$\begin{aligned}\frac{\partial^m}{\partial r^m}HS_1(k,n,m,r) &= \frac{\partial^m}{\partial r^m}HS(2b+1,n,m,r) = \frac{2^{2b+1}\pi^b r^{2b-m+n-2}b!}{(2b-m+n-2)!},\\ \frac{\partial^m}{\partial r^m}HS_2(k,n,m,r) &= \frac{\partial^m}{\partial r^m}HS(2b,n,m,r) = \frac{\pi^b r^{2b-m+n-3}(2b)!}{(2b-m+n-3)!b!}.\end{aligned} \tag{2.21}$$

Examples 1. (i) *m is an integer*: The derivative degree m can be integer, noninteger or consequently, from the field of real numbers. So, for example, we give its following representative values, which give the same result. In this case, we obtain the second derivative of the HS function related to the degree of freedom, $n = 5$, as

$$\begin{aligned}\frac{\partial^2}{\partial r^2}\mathrm{HS}_1(k,5,2,r) &= \frac{\pi^b(2r)^{2b+1}b!}{(2b+1)!},\\ \frac{\partial^2}{\partial r^2}\mathrm{HS}_2(k,5,2,r) &= \frac{\pi^b r^{2b}}{b!},\end{aligned} \tag{2.22}$$

and, in another case, by integrating the HS function, when $n = 2$ (surfhypersphere)

$$\begin{aligned}\frac{\partial^{-1}}{\partial r^{-1}}\mathrm{HS}_1(k,2,-1,r) &= \frac{\pi^b(2r)^{2b+1}b!}{(2b+1)!},\\ \frac{\partial^{-1}}{\partial r^{-1}}\mathrm{HS}_2(k,2,-1,r) &= \frac{\pi^b r^{2b}}{b!}.\end{aligned} \tag{2.23}$$

(ii) *m is noninteger (fractional)*: With the use of noninteger (fractional) degrees of the derivative, for example, $m = \pm 1/2$, starting with, for example, fixed degrees of freedom $n = 7/2$ and $n = 5/2$, we obtain the same results as with the procedure of integer differentiating/integrating (2.22) and (2.23). We obtain, respectively

$$\frac{\partial^{1/2}}{\partial r^{1/2}} \mathrm{HS}_1\left(k, \frac{7}{2}, \frac{1}{2}, r\right) = \frac{\pi^b (2r)^{2b+1} b!}{(2b+1)!},$$
$$\frac{\partial^{1/2}}{\partial r^{1/2}} \mathrm{HS}_2\left(k, \frac{7}{2}, \frac{1}{2}, r\right) = \frac{\pi^b r^{2b}}{b!}, \tag{2.24}$$

and with noninteger integrating ($m < 0$)

$$\frac{\partial^{-1/2}}{\partial r^{-1/2}} \mathrm{HS}_1\left(k, \frac{5}{2}, -\frac{1}{2}, r\right) = \frac{\pi^b (2r)^{2b+1} b!}{(2b+1)!},$$
$$\frac{\partial^{-1/2}}{\partial r^{-1/2}} \mathrm{HS}_2\left(k, \frac{5}{2}, -\frac{1}{2}, r\right) = \frac{\pi^b r^{2b}}{b!}. \tag{2.25}$$

The values (pairs) of the expressions (2.22), (2.23), (2.24), and (2.25) match the dichotomous values of the hyperspherical function for $n = 3$ therefore, we have

$$\mathrm{HS}_1(2b+1, 3, r) = \frac{\pi^b (2r)^{2b+1} b!}{(2b+1)!},$$
$$\mathrm{HS}_2(2b, 3, r) = \frac{\pi^b r^{2b}}{b!}. \tag{2.26}$$

By integrating the above complementary functions, we obtain the hyperspherical functions series, that describes the solid geometrical entities ($n = 3 \wedge k \in N$) in the form

$$\sum_{k=0}^{\infty} \mathrm{HS}(k, 3, r) = \sum_{k=0}^{\infty} \frac{\sqrt{\pi^k} r^k}{\Gamma(k/2+1)} = \sum_{b=0}^{\infty} \frac{\pi^b (2r)^{2b+1} b!}{(2b+1)!} + \sum_{b=0}^{\infty} \frac{\pi^b r^{2b}}{b!}$$
$$= e^{\pi r^2} \operatorname{erf}\left(r\sqrt{\pi}\right) + e^{\pi r^2} = e^{\pi r^2} \left[1 + \operatorname{erf}\left(r\sqrt{\pi}\right)\right] = e^{\pi r^2} \operatorname{erfc}\left(r\sqrt{\pi}\right), \tag{2.27}$$

which is Mittag-Leffler's [26], that is, Freden's solution (1993) for the solid hyperspherical function series [27]. In the previous expression complementary error functions originated from the expression $\operatorname{erf}(z) + \operatorname{erfc}(z) = 1$. By generalizing (2.4), we obtain the following differential equation:

$$\frac{\partial^n}{\partial r^n} \sum_{k=0}^{\infty} \mathrm{HS}(k, n, r) - \sum_{k=0}^{\infty} \mathrm{HS}(k, 0, r) = 0. \tag{2.28}$$

which describes the relation between the columns of the hyperspherical matrix for $n \in N$.

2.4. The Gradient of the Hyperspherical Function

Taking into consideration its differentiability and multidimensionality, we can apply the gradient on the hyperspherical function. As this function has three variables k, n, and r, we obtain the solution of gradient functions $\nabla_{k,n,r}$ in the following form:

$$\nabla_{k,n,r}\{\mathrm{HS}(k,n,r)\} = \begin{bmatrix} \dfrac{\partial}{\partial k}\mathrm{HS}(k,n,r) \\ \dfrac{\partial}{\partial n}\mathrm{HS}(k,n,r) \\ \dfrac{\partial}{\partial r}\mathrm{HS}(k,n,r) \end{bmatrix} = \mathrm{HS}(k,n,r)\begin{bmatrix} \ln 2r\sqrt{\pi} + \dfrac{1}{2}\psi_0\left(\dfrac{k+1}{2}\right) - \psi_0(k+n-2) \\ \ln r - \psi_0(k+n-2) \\ \dfrac{1}{r}(k+n-3) \end{bmatrix}, \tag{2.29}$$

where $\psi_0(z) = (d/dz)\ln\Gamma(z)$ is digamma function.

This function has a special use in determining the extreme values of the contour HS functions.

2.5. The Contour Graphics of the Derivatives of the Hyperspherical Functions

The derivative functions $hs1 = \partial\mathrm{HS}/\partial k = \nabla_k\{\mathrm{HS}\}$ and $sh1 = \partial\mathrm{HS}/\partial n = \nabla_n\{\mathrm{HS}\}$, also belong to the family of the hyperspherical functions. According to the degrees of freedom k and n, we can split them into two groups of functions.

Definition 2.8. The function of the first derivative with respect to the degree of freedom k is

$$hs1(k,n,r) = \frac{\partial}{\partial k}\mathrm{HS}(k,n,r) = \mathrm{HS}(k,n,r)\left[\ln 2r\sqrt{\pi} + \frac{1}{2}\psi_0\left(\frac{k+1}{2}\right) - \psi_0(k+n-2)\right]. \tag{2.30}$$

The function $hs1(k,n,r)$ has a special use in determining the maximum of the hyperspherical functions, for example, for a unit radius and the domain $k \in \Re$ (Figure 3). Here, on the basis of the known criteria, for each member of the derivative functions family we determine the maximum of the HS function by equaling its derivative with zero. Other than the maximum, there is the "optimal" value k_0. Consequently

$$hs1(k,n,r) = 0, \quad \frac{\partial^2}{\partial k^2}\mathrm{HS}(k,n,r) > 0 \Longrightarrow \max \mathrm{HS}(k,n,r) \wedge k_0. \tag{2.31}$$

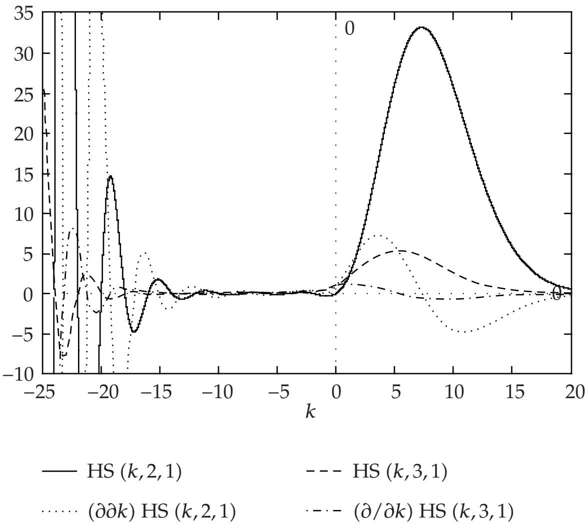

Figure 3: The parallel presentation of the orginal hyperspherical functions (for $n = 2$ and $n = 3$) of the unit radius, and corresponding functions of their first derivatives for $-25 \le k \le 20$.

Example 2.9. For the volume hyperspherical function with the radius $r = 1$, the derivative function is after transforming [24]

$$hs1(k,3,1) = \frac{\mathrm{HS}(k,3,1)}{2}\left[\ln\pi - \psi_0\left(\frac{k}{2}+1\right)\right] = \frac{\sqrt{\pi^k}}{k\Gamma(k/2)}(\ln\pi + \gamma - H_{k/2}), \tag{2.32}$$

where $H_{k/2}$ is a harmonious number. This number is obtained on the basis of the sum [28]

$$H_k = \sum_{j=1}^{k}\frac{1}{j} \quad \text{or} \quad H_{k/2} = \sum_{j=1}^{k/2}\frac{1}{j} = 1 + \frac{1}{2} + \cdots + \frac{k}{2}. \tag{2.33}$$

The relation between the harmonious number and *psi* (digamma function) is the following:

$$H_k = \gamma + \psi_0(k+1), \quad \text{respectively,} \; H_{k/2} = \gamma + \psi_0\left(\frac{k}{2}+1\right). \tag{2.34}$$

The graphical representation of the original $\mathrm{HS}(k,3,1)$ and its derivative are given in Figure 4.

2.6. The Extreme Values of the HS Function from the Degree of Freedom k Point of View

On the basis of the above expressions (2.31) and (2.32), we define the greatest volume for the "optimal" dimension k_0. We find that $k_0 \approx 5.2569$, and the corresponding volume is $\max \mathrm{HS}(k,3,1) = \mathrm{HS}(k_0,3,1) \approx 5.2777$. For other hyperspherical functions, we can also

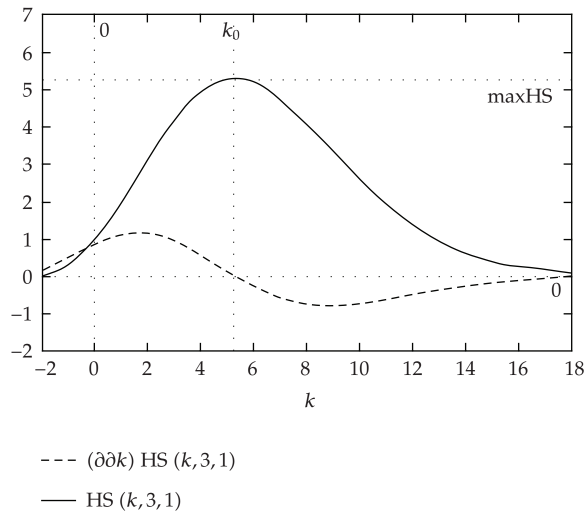

Figure 4: The contour function of the hyperspherical solid $\mathrm{HS}(3, n, 1)$ with the maximum value max HS and optimal degree of freedom k_0 (for $-2 \le k \le 18$).

Table 1

Dimension n	Dimension k_0	max HS	Error ε
0	10.70042116863070	1871.481745773695	~0
1	9.377568346255670	237.5057633480618	$-1.055 \cdot 10^{-13}$
2	7.256946404860572	33.16119448496195	$7.363 \cdot 10^{-15}$
3	5.256946404860572	5.277768021113411	$3.516 \cdot 10^{-15}$
4	2.506537984190255	1.054723774914521	~0
5	0	1/2	0
⋮	⋮	⋮	⋮
n	0	$1/(n-3)!$	0

determine the maximum values on the numerical bases (Table 1). However, for the degrees of freedom $n \ge 5$, the maximum is a function of its factorial.

So, we have

$$\lim_{k \to 0} \mathrm{HS}(k, 5, 1) = \frac{1}{2}, \qquad \lim_{k \to 0} \mathrm{HS}(k, 6, 1) = \frac{1}{6}, \qquad \lim_{k \to 0} \mathrm{HS}(k, 7, 1) = \frac{1}{24}, \tag{2.35}$$

and in general

$$\lim_{k \to 0} \mathrm{HS}(k, n, 1) = \frac{1}{\Gamma(n-2)} = \frac{1}{(n-3)!}, \qquad \lim_{k \to 0^+} \mathrm{HS}(k, n, r) = \frac{r^{n-3}}{(n-3)!}. \tag{2.36}$$

On the following surface 3D graphic $\mathrm{HS}(k, n, 1)$ (Figure 5) we notice the extreme, that is, the maximum value of this function with the 0th freedom degree $n = 0$, which is a global maximum in the domain $k, n \in N$ as well.

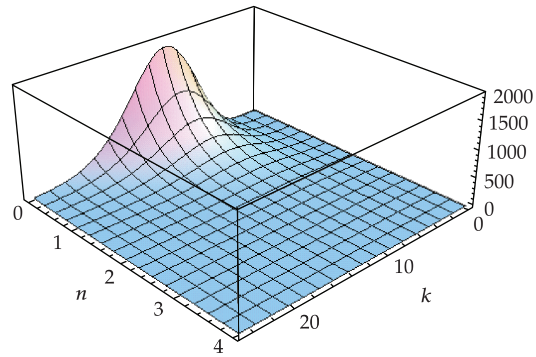

Figure 5: The graphic of the hyperspherical function with the constant radius $r = 1$ and for the degrees of freedom $0 < n \leq 4$ and $0 < k \leq 25$.

2.7. The Extreme Values of the HS Function from the Degree of Freedom n Point of View

Definition 2.10. The function of the first derivative with respect to the degree of freedom n, is

$$sh1(k,n,r) = \frac{\partial}{\partial n}\mathrm{HS}(k,n,r) = \mathrm{HS}(k,n,r)\left[\ln r - \psi_0(k+n-2)\right]. \tag{2.37}$$

The function $sh1(k,n,r)$ is in fact the general solution of the HS function of the degree of freedom n.

On the basis of the known differentiating procedure, besides the local maximum of the contour HS functions, we obtain the corresponding optimal values n_0 as well. Consequently

$$sh1(k,n,r) = 0, \quad \frac{\partial^2}{\partial n^2}\mathrm{HS}(k,n,r) > 0 \Longrightarrow \max\ \mathrm{HS}(k,n,r) = \mathrm{HS}(k,n_0,r) \wedge n_0. \tag{2.38}$$

These functions can have a significant role with further research of hyperspherical functions. By calculating the extreme and optimal values of the HS function, we obtain Table 2.

We can conclude that the following optimal degrees of freedom n_0 and k_0

$$n_0 = 0, \quad 6\pi^5 < k_0 < \frac{128}{21}\pi^5 \tag{2.39}$$

generate a global value maximum of the hyperspherical function, taking only the domain of the natural numbers $(k, n \in N)$.

As in the previous analysis, we determine the value of k_0, the optimal vector is

$$\begin{bmatrix} k_0 \\ n_0 \end{bmatrix} = \begin{bmatrix} 10.7004211686307 \\ 0 \end{bmatrix} \Longrightarrow \max\ \mathrm{HS}(k,n,1) = \mathrm{HS}(k_0,n_0,1) \approx 1871.481745773695. \tag{2.40}$$

The graphical representation of the original $\mathrm{HS}(3,n,1)$ and its derivate are given in Figure 6.

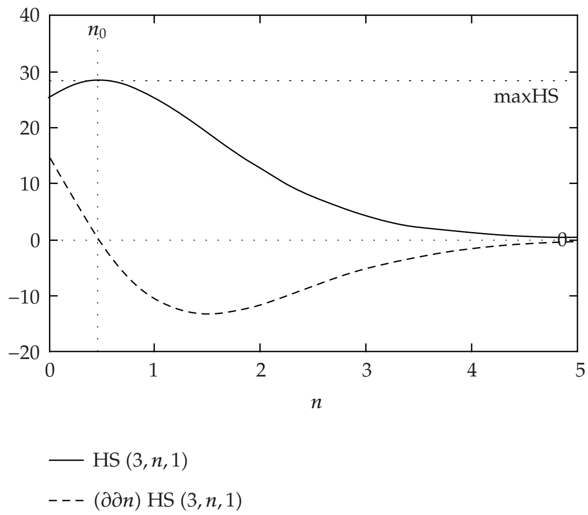

Figure 6: The contour function of the hypersolid $\mathrm{HS}(3, n, 1)$ with the local maximum max HS ≈ 28.379 and the optimal degree of freedom $n_0 \approx 0.4616$, for $k \in N$.

Table 2

Dimension k	Dimension n^*	max HS	Error ε
0	3.461632144968362	1.129173885450133	$1.08 \cdot 10^{-16}$
1	2.461632144968362	2.258347770900258	$1.086 \cdot 10^{-15}$
2	1.461632144968362	7.094808766311162	~ 0
3	0.461632144968362	28.37923506524463	$2.665 \cdot 10^{-15}$
4	0	$12\pi^2$	0
5	0	$32\pi^2$	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$
10	0	$6\pi^5 \approx 1836.18109$	0
11	0	$\frac{128}{21}\pi^5 \approx 1865.26284$	0
12	0	$\frac{11}{6}\pi^6 \approx 1762.54685$	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$
∞	0	$\lim_{k\to\infty} \mathrm{HS}(k, 0, 1) = 0$	0

2.8. The Derivatives of the Hyperspherical Functions of the Higher Order with Respect to the Argument k

The derivatives of the hyperspherical functions with respect to any argument retain the characteristics similar the original HS function. This is especially noticeable in the graphical representation. In addition, it would be interesting to test the regularities in the structure of this function of the higher derivation degree. In the following, we give, analytically and graphically, the derivative functions on argument k (the more complex case) and n (the simpler case).

The second derivative of the hyperspherical function with respect to argumen k is:

$$\begin{aligned} hs2(k,n,r) &= \frac{\partial^2}{\partial k^2}\mathrm{HS}(k,n,r) \\ &= \mathrm{HS}(k,n,r)\left\{\psi_1(k)-\psi_1(k+n-2)+\left[\psi_0(k+n-2)-\frac{1}{2}\psi_0\left(\frac{k+1}{2}\right)-\ln 2r\sqrt{\pi}\right]^2\frac{1}{4}\psi_1\left(\frac{k}{2}\right)\right\}. \end{aligned} \tag{2.41}$$

Here: $\psi_0(z) = (d/dz)\ln\Gamma(z)$ digamma and $\psi_1(z) = (d^2/dz^2)\ln\Gamma(z)$ trigamma function. The concrete value of this derivative for the selected parameters is the following:

$$\begin{aligned} &\frac{\partial^2}{\partial k^2}\mathrm{HS}(3,3,1) \\ &\quad= \frac{2\pi}{9}\left[(3\gamma-8)\ln\pi+\gamma\left(\frac{3\gamma}{2}-8\right)+6\left(\gamma+\ln 2\pi-\frac{8}{3}\right)\ln 2+\frac{3}{4}\left(2\ln^2\pi-\pi^2\right)+\frac{52}{3}\right]. \end{aligned} \tag{2.42}$$

The mixed derivative of the hyperspherical function is

$$\begin{aligned} \frac{\partial^2}{\partial k\partial n}\mathrm{HS}(k,n,r) = \mathrm{HS}(k,n,r)\ &\left\{\psi_0(k+n-2)\ \left[\psi_0(k+n-2)-\frac{1}{2}\psi_0\left(\frac{k+1}{2}\right)-\ln 2r^2\sqrt{\pi}\right]\right. \\ &\left.+\ln r\left[\ln 2r\sqrt{\pi}+\frac{1}{2}\psi_0\left(\frac{k+1}{2}\right)\right]-\psi_1(k+n-2)\right\}. \end{aligned} \tag{2.43}$$

Example 2.11. After introducing some values of k and n it follows that

$$\frac{\partial^2}{\partial k\partial n}\mathrm{HS}(3,3,1) = \frac{\pi}{9}\left[(\ln 4\pi+\gamma)(6\gamma+11)-2\left(\pi^2+8\gamma\right)+\frac{137}{3}\right]. \tag{2.44}$$

The fifth derivative of the hyperspheric function on argument n and for its unique radius is

$$\begin{aligned} \frac{\partial^5}{\partial n^5}\mathrm{HS}(k,n,r) = -\ \mathrm{HS}(k,n,r)\Big\{&\big(\ \psi_0(k+n-2)-\ln r\big)^5-10\big(\psi_0(k+n-2)-\ln r\big)^3\psi_1(k+n-2) \\ &+10\big(\psi_0(k+n-2)-\ln r\big)^2\psi_2(k+n-2)+15\big(\psi_0(k+n-2)-\ln r\big)^2 \\ &\times\psi_1^2(k+n-2)-5\psi_3(k+n-2)\big(\psi_0(k+n-2)-\ln r\big) \\ &-\ 10\psi_1(k+n-2)\psi_2(\upsilon+1)+\psi_4(k+n-2)\Big\}. \end{aligned} \tag{2.45}$$

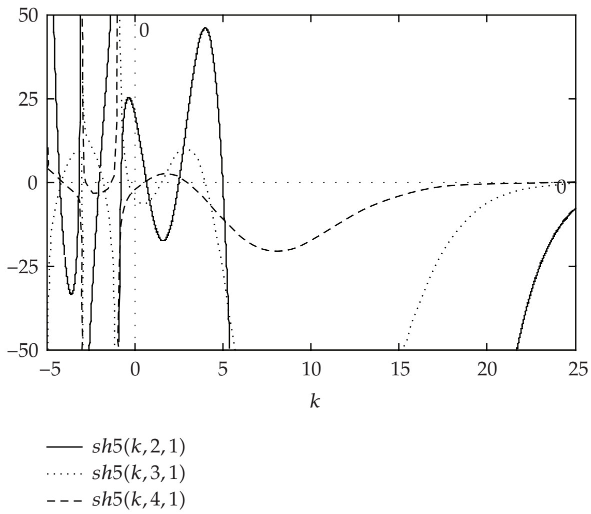

Figure 7: The graphics of the fifth derivative of the hyperspherical functions with respect to the argument n.

Namely, the fifth derivative values of this function for degrees of freedom $k = 0$, $n = 0$ are

$$\frac{\partial^5}{\partial k^5}\mathrm{HS}(0,0,1) = sh5(0,0,1) = 40\ (2\gamma - 3)\zeta(3) + 10\gamma\left[\pi^2(3-\gamma) + 6\gamma\left(1-\gamma\left(1-\frac{\gamma}{6}\right)\right)\right] - \pi^2(10-\zeta(2)). \tag{2.46}$$

Generally, the hyperspherical function of the mth derivative on the argument n can be defined on the basis of the product for poligamma (psi) of the polynominal of the mth degree and the hyperspherical function. Its general form is the following:

$$shm(k,n,r) = \frac{\partial^m}{\partial n^m}\mathrm{HS}(k,n,r) = \mathrm{HS}(k,n,r)\cdot f^m\{\psi_{(m-1)}(k+n-2), r\}. \tag{2.47}$$

So, for example, the functions of the fifth derivative are in Figure 7.

3. Conclusion

The research of the hyperspherical function leaves the following problems unsolved: continual dimensions which include noninteger values, the zeroth dimension, the domain of the dimensions less than zero, consequently the domain of the more complex degrees of freedom, and so forth. In any case, the negative dimensions matrix demands special structuring and deeper mathematical analysis. The hyperspheres present one part of the multigeometrical objects. It is known that a sphere is the most symmetrical geometrical object. Some other objects have such characteristics but not entirely. Namely, a cube is symmetrical on orthogonal coordinates, if its centre is in the intersection of space diagonals, and the axes of the coordinate system are parallel to cube edges. Some other polyhedrons have this characteristic, too. A cube, because of orthogonality, retains the characteristic that its derivatives of the solid-entity functions create the function of surfentity. The objects that do not have the characteristic that recurrence is created on the basis of the derivative function, must introduce new degrees of freedom depending on the object complexity. The next object that has some characteristics similar to sphere is cylinder (this similarity is caused by the equality of the cylinder height and diameter). Its symmetry is dual and is not unambigous on all coordinate axes like the sphere is. These characteristics are the fundamental assumptions for defining the recurrent relations that would determine the relation between functions of some entities and both on rows and columns of a cube, that is, cylinder. The considered analysis of the multidimensional space and the formula of this geometry lead into the conclusion of its complexity and the relations with special functions and the other geometry fields. Special problems not analysed in this paper, refer to zeros and singular values of this function, and some ot them can be seen in the matrix (Figure 2).

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