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Research Article

Nearly Ring Homomorphisms and Nearly Ring Derivations on Non-Archimedean Banach Algebras

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We prove the generalized Hyers-Ulam stability of homomorphisms and derivations on non-Archimedean Banach algebras. Moreover, we prove the superstability of homomorphisms on unital non-Archimedean Banach algebras and we investigate the superstability of derivations in non-Archimedean Banach algebras with bounded approximate identity.

1. Introduction and Preliminaries

In 1897, Hensel [1] has introduced a normed space which does not have the Archimedean property.

During the last three decades theory of non-Archimedean spaces has gained the interest of physicists for their research in particular in problems coming from quantum physics, p-adic strings, and superstrings [2]. Although many results in the classical normed space theory have a non-Archimedean counterpart, their proofs are essentially different and require an entirely new kind of intuition [3–9].

Let $\mathbb{K}$ be a field. A non-Archimedean absolute value on $\mathbb{K}$ is a function $|\cdot| : \mathbb{K} \to \mathbb{R}$ such that for any $a, b \in \mathbb{K}$ we have

(i) $|a| \geq 0$ and equality holds if and only if $a = 0$,

(ii) $|ab| = |a||b|$,

(iii) $|a + b| \leq \max\{|a|, |b|\}$.

Condition (iii) is called the strict triangle inequality. By (ii), we have $|1| = |-1| = 1$. Thus, by induction, it follows from (iii) that $|n| \leq 1$ for each integer n. We always assume in addition that $|\cdot|$ is non trivial, that is, that there is an $a_0 \in \mathbb{K}$ such that $|a_0| \notin \{0, 1\}$.

Let X be a linear space over a scalar field $\mathbb{K}$ with a non-Archimedean nontrivial valuation $|\cdot|$. A function $\|\cdot\| : X \to \mathbb{R}$ is a non-Archimedean norm (valuation) if it satisfies the following conditions:

(NA1) $\|x\| = 0$ if and only if $x = 0$;

(NA2) $\|rx\| = |r|\|x\|$ for all $r \in \mathbb{K}$ and $x \in X$;

(NA3) the strong triangle inequality (ultrametric), namely,

$$\|x + y\| \leq \max\{\|x\|, \|y\|\} \quad (x, y \in X). \tag{1.1}$$

Then $(X, \|\cdot\|)$ is called a non-Archimedean space.

It follows from (NA3) that

$$\|x_m - x_l\| \leq \max\{\|x_{j+1} - x_j\| : l \leq j \leq m-1\} \quad (m > l), \tag{1.2}$$

therefore a sequence $\{x_m\}$ is Cauchy in X if and only if $\{x_{m+1} - x_m\}$ converges to zero in a non-Archimedean space. By a complete non-Archimedean space we mean one in which every Cauchy sequence is convergent. A non-Archimedean Banach algebra is a complete non-Archimedean algebra $\mathcal{A}$ which satisfies $\|ab\| \leq \|a\|\|b\|$ for all $a, b \in \mathcal{A}$. For more detailed definitions of non-Archimedean Banach algebras, we can refer to [10].

The first stability problem concerning group homomorphisms was raised by Ulam [11] in 1960 and affirmatively solved by Hyers [12]. Perhaps Aoki was the first author who has generalized the theorem of Hyers (see [13]).

T. M. Rassias [14] provided a generalization of Hyers' Theorem which allows the *Cauchy difference to be unbounded.*

Theorem 1.1 (T. M. Rassias). *Let $f : E \to E'$ be a mapping from a normed vector space E into a Banach space E' subject to the inequality*

$$\|f(x + y) - f(x) - f(y)\| \leq \epsilon(\|x\|^p + \|y\|^p) \tag{1.3}$$

for all $x, y \in E$, where ϵ and p are constants with $\epsilon > 0$ and $p < 1$. Then the limit

$$L(x) = \lim_{n \to \infty} \frac{f(2^n x)}{2^n} \tag{1.4}$$

exists for all $x \in E$ and $L : E \to E'$ is the unique additive mapping which satisfies

$$\|f(x) - L(x)\| \leq \frac{2\epsilon}{2 - 2^p}\|x\|^p \tag{1.5}$$

for all $x \in E$. Also, if for each $x \in E$ the mapping $f(tx)$ is continuous in $t \in \mathbb{R}$, then L is $\mathbb{R}$-linear.

Moreover, Bourgin [15] and Găvruţa [16] have considered the stability problem with unbounded Cauchy differences (see also [17–27]).

On the other hand, J. M. Rassias [28–33] considered the Cauchy difference controlled by a product of different powers of norm. However, there was a singular case; for this singularity a counterexample was given by Găvruţa [34]. This stability phenomenon is called the Ulam-Găvruta-Rassias stability (see also [35]).

Theorem 1.2 (J. M. Rassias [28]). *Let X be a real normed linear space and Y a real complete normed linear space. Assume that $f : X \to Y$ is an approximately additive mapping for which there exist constants $\theta \geq 0$ and $p, q \in \mathbb{R}$ such that $r = p + q \neq 1$ and f satisfies the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \theta \|x\|^p \|y\|^q \tag{1.6}$$

for all $x, y \in X$. Then there exists a unique additive mapping $L : X \to Y$ satisfying

$$\|f(x) - L(x)\| \leq \frac{\theta}{|2^r - 2|} \|x\|^r \tag{1.7}$$

for all $x \in X$. If, in addition, $f : X \to Y$ is a mapping such that the transformation $t \mapsto f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$, then L is an $\mathbb{R}$-linear mapping.

Very recently, Ravi et al. [36] in the inequality (1.6) replaced the bound by a mixed one involving the product and sum of powers of norms, that is, $\theta\{\|x\|^p \|y\|^p + (\|x\|^{2p} + \|y\|^{2p})\}$.

For more details about the results concerning such problems and mixed product-sum stability (J. M.-Rassias Stability) the reader is referred to [37–49].

Khodaei and T. M. Rassias [50] have established the general solution and investigated the Hyers-Ulam-Rassias stability of the following n-dimensional additive functional equation:

$$\begin{aligned} &\sum_{k=2}^{n} \left(\sum_{i_1=2}^{k} \sum_{i_2=i_1+1}^{k+1} \cdots \sum_{i_{n-k+1}=i_{n-k}+1}^{n} \right) f\left(\sum_{i=1, i \neq i_1, \ldots, i_{n-k+1}}^{n} a_i x_i - \sum_{r=1}^{n-k+1} a_{i_r} x_{i_r} \right) \\ &\quad + f\left(\sum_{i=1}^{n} a_i x_i \right) \\ &\quad = 2^{n-1} a_1 f(x_1), \end{aligned} \tag{1.8}$$

where $a_1, \ldots, a_n \in \mathbb{Z} - \{0\}$ with $a_1 \neq \pm 1$.

In this paper, we investigate the Hyers-Ulam stability of homomorphisms and derivations associated with functional equation (1.8).

2. Main Results

Before taking up the main subject, for a given $f : \mathcal{A} \to \mathcal{B}$ between vector spaces, we define the difference operator

$$Df(x_1,\ldots,x_n) := \sum_{k=2}^{n}\left(\sum_{i_1=2}^{k}\sum_{i_2=i_1+1}^{k+1}\cdots\sum_{i_{n-k+1}=i_{n-k}+1}^{n}\right) f\left(\sum_{i=1, i\neq i_1,\ldots,i_{n-k+1}}^{n} a_i x_i - \sum_{r=1}^{n-k+1} a_{i_r} x_{i_r}\right) + f\left(\sum_{i=1}^{n} a_i x_i\right) - 2^{n-1} a_1 f(x_1). \tag{2.1}$$

Theorem 2.1. *Let $\mathcal{A}, \mathcal{B}$ be two non-Archimedean Banach algebras and let $\psi : \mathcal{A}^n \to [0,\infty), \phi : \mathcal{A}^2 \to [0,\infty)$ be functions such that*

$$\lim_{m\to\infty}\frac{1}{|a_1|^m}\psi(a_1^m x_1,\ldots,a_1^m x_n) = \lim_{k\to\infty}\frac{1}{k}\phi(kx,y) = 0 \tag{2.2}$$

for all $x_1,\ldots,x_n \in \mathcal{A}$, and the limit

$$\widetilde{\psi}(x) := \lim_{m\to\infty}\max\left\{\frac{1}{|a_1|^{\ell}}\psi\left(a_1^{\ell}x,0,\ldots,0\right) : 0 \le \ell < m\right\} \tag{2.3}$$

exists and $\lim_{k\to\infty}(1/k)\widetilde{\psi}(kx) = 0$ for all $x \in \mathcal{A}$. Suppose that $f : \mathcal{A} \to \mathcal{B}$ is a function satisfying

$$\|Df(x_1,\ldots,x_n)\| \le \psi(x_1,\ldots,x_n), \qquad \|f(xy) - f(x)f(y)\| \le \phi(x,y) \tag{2.4}$$

for all $x_1,\ldots,x_n,x,y \in \mathcal{A}$. Then there exists a ring homomorphism $H : \mathcal{A} \to \mathcal{B}$ such that

$$\|f(x) - H(x)\| \le \frac{1}{|2^{n-1}a_1|}\widetilde{\psi}(x) \tag{2.5}$$

for all $x \in \mathcal{A}$ and

$$H(x)(H(y) - f(y)) = (f(x) - H(x))H(y) = 0 \tag{2.6}$$

for all $x, y \in \mathcal{A}$. Moreover, if

$$\lim_{j\to\infty}\lim_{m\to\infty}\max\left\{\frac{1}{|a_1|^{\ell}}\psi\left(a_1^{\ell}x,0,\ldots,0\right) : j \le \ell < m + j\right\} = 0, \tag{2.7}$$

then H is the unique ring homomorphism satisfying (2.5).

Proof. By [50, Theorem 4.4], there exists an additive function $H : \mathcal{A} \to \mathcal{B}$ which satisfies (2.5). We have

$$H(x) := \lim_{m\to\infty} a_1^m f\left(\frac{x}{a_1^m}\right) \tag{2.8}$$

for all $x \in \mathcal{A}$. Now we show that H is a multiplicative function. It follows from (2.5) that

$$\|f(kx) - H(kx)\| \leq \frac{1}{|2^{n-1}a_1|}\tilde{\varphi}(kx) \tag{2.9}$$

for all $x \in \mathcal{A}$ and all $k \in \mathbb{N}$. On the other hand H is additive then we have

$$\left\|\frac{1}{k}f(kx) - H(x)\right\| \leq \frac{1}{|2^{n-1}a_1|k}\tilde{\varphi}(kx) \tag{2.10}$$

for all $x \in \mathcal{A}$ and all $k \in \mathbb{N}$. If $k \to \infty$, then by (2.3), the right hand side of above inequality tends to zero. It follows that

$$H(x) = \lim_{k\to\infty}\frac{1}{k}f(kx) \tag{2.11}$$

for all $x \in \mathcal{A}$. Applying (2.3), (2.4), and (2.11) we have

$$H(xy) - H(x)f(y) = \lim_{k\to\infty}\frac{1}{k}(f(kxy) - f(kx)f(y)) = 0 \tag{2.12}$$

for all $x, y \in \mathcal{A}$. This means that

$$H(xy) = H(x)f(y) \tag{2.13}$$

for all $x, y \in \mathcal{A}$. From (2.13) and additivity of H we have

$$H(x)H(y) = H(x)\lim_{k\to\infty}\frac{1}{k}f(ky) = \lim_{k\to\infty}\frac{1}{k}(H(x)f(ky)) = \lim_{k\to\infty}\frac{1}{k}H(x(ky)) = H(xy) \tag{2.14}$$

for all $x, y \in \mathcal{A}$. In other words, H is multiplicative. It follows from (2.13) and (2.14) that

$$H(x)(H(y) - f(y)) = 0 \tag{2.15}$$

for all $x, y \in \mathcal{A}$. Similarly, we can show that

$$(f(x) - H(x))H(y) = 0 \tag{2.16}$$

for all $x, y \in \mathcal{A}$. To prove the uniqueness property of H, let $T : \mathcal{A} \to \mathcal{B}$ be another ring homomorphism which satisfies (2.5). Applying (2.11) and (2.5) we have

$$\|H(x) - T(x)\| = \lim_{k\to\infty} \frac{1}{k} \|f(kxy) - T(kx)\| \le \lim_{k\to\infty} \frac{1}{k} \frac{1}{|2^{n-1}a_1|} \widetilde{\varphi}(kx) = 0 \tag{2.17}$$

for all $x \in \mathcal{A}$ which is the desired conclusion. □

Now, we establish the superstability of homomorphisms as follows.

Corollary 2.2. *Let $\mathcal{A}, \mathcal{B}$ be two unital non-Archimedean Banach algebras, and let $\varphi : \mathcal{A}^n \to [0,\infty)$, $\phi : \mathcal{A}^2 \to [0,\infty)$, $f : \mathcal{A} \to \mathcal{B}$ be functions with conditions of Theorem 2.1. Suppose that*

$$\lim_{m\to\infty} a_1^m f\left(\frac{1_{\mathcal{A}}}{a_1^m}\right) = 1_{\mathcal{B}}. \tag{2.18}$$

Then the mapping $f : \mathcal{A} \to \mathcal{B}$ is a ring homomorphism.

Proof. It follows from (2.6) and (2.18) that $f = H$ in Theorem 2.1. Hence, f is a ring homomorphism. □

Corollary 2.3. *Let $\eta : [0,\infty) \to [0,\infty)$ be a function satisfying*

(i) $\eta(|a_1|t) \le \eta(|a_1|)\eta(t)$ *for all* $t \ge 0$;

(ii) $\eta(|a_1|) < |a_1|$;

(iii) $\lim_{k\to\infty}(1/k)\eta(k|a_1|) = 0$.

Suppose that $\varepsilon > 0$, and let $f : \mathcal{A} \to \mathcal{B}$ satisfying

$$\|Df(x_1,\ldots,x_n)\| + \|f(xy) - f(x)f(y)\| \le \varepsilon \,\mathrm{Min}\left\{\sum_{i=1}^{n} \eta(\|x_i\|), \eta(\|x\|)\eta(\|y\|)\right\} \tag{2.19}$$

for all $x_1,\ldots,x_n, x, y \in \mathcal{A}$. Then there exists a unique ring homomorphism $H : \mathcal{A} \to \mathcal{B}$ such that

$$\|f(x) - H(x)\| \le \frac{\varepsilon}{|2^{n-1}a_1|} \eta(\|x\|) \tag{2.20}$$

for all $x \in \mathcal{A}$.

Proof. Defining $\varphi : \mathcal{A}^n \to [0,\infty)$ and $\phi : \mathcal{A}^2 \to [0,\infty)$ by

$$\varphi(x_1,\ldots,x_n) := \varepsilon \sum_{i=1}^{n} \eta(\|x_i\|), \qquad \phi(x,y) := \eta(\|x\|)\eta(\|y\|), \tag{2.21}$$

respectively, we have

$$\lim_{m\to\infty}\frac{1}{|a_1|^m}\varphi\left(a_1^m x_1,\ldots,a_1^m x_n\right)\le\lim_{m\to\infty}\left(\frac{\eta(|a_1|)}{|a_1|}\right)^m\varphi(x_1,\ldots,x_n)=0 \tag{2.22}$$

for all $x_1,\ldots,x_n\in\mathcal{A}$. Hence

$$\begin{gathered}\tilde{\varphi}(x):=\lim_{m\to\infty}\max\left\{\frac{1}{|a_1|^\ell}\varphi\left(a_1^\ell x,0,\ldots,0\right):0\le\ell<m\right\}=\varphi(x,0,\ldots,0),\\ \lim_{j\to\infty}\lim_{m\to\infty}\max\left\{\frac{1}{|a_1|^\ell}\varphi\left(a_1^\ell x,0,\ldots,0\right):j\le\ell<m+j\right\}=\lim_{j\to\infty}\frac{1}{|a_1|^j}\varphi\left(a_1^j x,0,\ldots,0\right)=0\end{gathered} \tag{2.23}$$

for all $x\in\mathcal{A}$. On the other hand

$$\lim_{k\to\infty}\frac{1}{k}\phi(kx,y)=\lim_{k\to\infty}\frac{1}{k}\eta(k\|x\|)\eta(\|y\|)=0 \tag{2.24}$$

for all $x,y\in\mathcal{A}$. The conclusion follows from Theorem 2.1. □

Remark 2.4. The classical example of the function η is the function $\eta(t)=t^p$ for all $t\in[0,\infty)$, where $p>1$ with the further assumption that $|a_1|<1$.

Now, we prove the stability of derivations non-Archimedean Banach algebras by using Theorem 2.1.

Theorem 2.5. *Let $\mathcal{A}$ be a non-Archimedean Banach algebra, and let $\mathcal{X}$ be a non-Archimedean Banach $\mathcal{A}$-module. Let $\varphi:\mathcal{A}^n\to[0,\infty)$, $\phi:\mathcal{A}^2\to[0,\infty)$ be a function such that*

$$\lim_{m\to\infty}\frac{1}{|a_1|^m}\varphi\left(a_1^m x_1,\ldots,a_1^m x_n\right)=\lim_{k\to\infty}\frac{1}{k}\phi(kx,y)=0 \tag{2.25}$$

for all $x_1,\ldots,x_n\in\mathcal{A}$, and the limit

$$\tilde{\varphi}(x):=\lim_{m\to\infty}\max\left\{\frac{1}{|a_1|^\ell}\varphi\left(a_1^\ell x,0,\ldots,0\right):0\le\ell<m\right\} \tag{2.26}$$

exists and $\lim_{k\to\infty}(1/k)\tilde{\varphi}(kx)=0$ for all $x\in\mathcal{A}$. Suppose that $f:\mathcal{A}\to\mathcal{X}$ is a function satisfying

$$\|Df(x_1,\ldots,x_n)\|\le\varphi(x_1,\ldots,x_n),\qquad\|f(xy)-f(x)y-xf(y)\|\le\phi(x,y) \tag{2.27}$$

for all $x_1, \ldots, x_n, x, y \in \mathcal{A}$. *Then there exists a ring derivation* $D : \mathcal{A} \to \mathcal{X}$ *such that*

$$\|f(x) - D(x)\| \le \frac{1}{|2^{n-1}a_1|}\widetilde{\varphi}(x) \tag{2.28}$$

for all $x \in \mathcal{A}$.

Proof. It is easy to see that $\mathcal{X} \oplus_1 \mathcal{A}$ is a non-Archimedean Banach algebra equipped with the product

$$(x_1, a_1)(x_2, a_2) = (x_1 \cdot a_2 + a_1 \cdot x_2, a_1 a_2) \quad (a_1, a_2 \in \mathcal{A}, x_1, x_2 \in \mathcal{X}) \tag{2.29}$$

and with the following ℓ_1-norm:

$$\|(x, a)\| = \|x\| + \|a\| \quad (a \in \mathcal{A}, x \in \mathcal{X}). \tag{2.30}$$

Let us define the mapping $\varphi_f : \mathcal{A} \to \mathcal{X} \oplus_1 \mathcal{A}$ by $a \mapsto (f(a), a)$. It is easy to see that $\varphi_f : \mathcal{A} \to \mathcal{X} \oplus_1 \mathcal{A}$ satisfies the conditions of Theorem 2.1. By Theorem 2.1, there exists a unique ring homomorphism $H : \mathcal{A} \to \mathcal{X} \oplus_1 \mathcal{A}$ such that

$$\|H(a) - \varphi_f(a)\| \le \frac{1}{|2^{n-1}a_1|}\widetilde{\varphi}(a) \quad (a \in \mathcal{A}). \tag{2.31}$$

We define projection maps $\pi_1 : \mathcal{X} \oplus_1 \mathcal{A} \to \mathcal{X}$ and $\pi_2 : \mathcal{X} \oplus_1 \mathcal{A} \to \mathcal{A}$ by $(x, b) \mapsto x$ and $(x, b) \mapsto b$, respectively.

It follows from (2.31) that

$$\|(\pi_2 \circ \varphi_f)(ka) - (\pi_2 \circ H)(ka)\| \le \|\varphi_f(ka) - H(ka)\| \le \frac{1}{|2^{n-1}a_1|}\widetilde{\varphi}(ka) \quad (k \in \mathbb{N}, a \in \mathcal{A}). \tag{2.32}$$

By the additivity of mappings under consideration

$$\begin{gathered}(\pi_2 \circ \varphi)(ka) = k(\pi_2 \circ \varphi)(a), \\ (\pi_2 \circ \varphi_f)(ka) = \pi_2(f(ka), ka) = ka,\end{gathered} \tag{2.33}$$

whence, by (2.32),

$$\|a - (\pi_2 \circ H)(a)\| \le \frac{1}{k}\frac{1}{|2^{n-1}a_1|}\widetilde{\varphi}(ka) \tag{2.34}$$

for all $k \in \mathbb{N}, a \in \mathcal{A}$. By letting k tend to ∞ in (2.34), we obtain by (2.25) that

$$(\pi_2 \circ H)(a) = a \quad (a \in \mathcal{A}). \tag{2.35}$$

Put $D := \pi_1 \circ H$. Then we have

$$\begin{aligned}((\pi_1 \circ H)(ab), ab) &= (\pi_1(H(ab)), \pi_2(H(ab))) = H(ab) = H(a)H(b) \\ &= (\pi_1(H(a)), \pi_2(H(a)))(\pi_1(H(b)), \pi_2(H(b))) \\ &= (\pi_1(H(a)), a)(\pi_1(H(b)), b) \\ &= (a\pi_1(H(b)) + \pi_1(H(a))b, ab)\end{aligned} \tag{2.36}$$

for all $a, b \in \mathcal{A}$. It follows that D is a derivation. On the other hand, by (2.31) we have

$$\left\|D(a) - f(a)\right\| = \left\|\pi_1(H(a)) - \pi_1\left(\varphi_f(a)\right)\right\| \le \left\|H(a) - \varphi_f(a)\right\| \le \frac{1}{|2^{n-1}a_1|}\widetilde{\varphi}(a) \tag{2.37}$$

for all $a \in \mathcal{A}$.

To prove the uniqueness property of D, assume that D^* is another derivation from $\mathcal{A}$ into $\mathcal{X}$ satisfying

$$\left\|D^*(a) - f(a)\right\| \le \frac{1}{|2^{n-1}a_1|}\widetilde{\varphi}(a) \quad (a \in \mathcal{A}). \tag{2.38}$$

Then by (2.25), we have

$$\begin{aligned}\|D(a) - D^*(a)\| &= \lim_{k\to\infty}\frac{1}{k}\|D(ka) - D^*(ka)\| \le \lim_{k\to\infty}\left(\frac{1}{k}\left\|D^*(a) - f(a)\right\| + \frac{1}{k}\left\|D(a) - f(a)\right\|\right) \\ &\le \lim_{k\to\infty}\frac{2}{k}\frac{1}{|2^{n-1}a_1|}\widetilde{\varphi}(ka) \\ &= 0\end{aligned} \tag{2.39}$$

for all $a \in \mathcal{A}$. This means that $D(a) = D^*(a)$ for all $a \in \mathcal{A}$. □

Corollary 2.6. *Let $\eta : [0, \infty) \to [0, \infty)$ be a function satisfying*

(i) $\eta(|a_1|t) \le \eta(|a_1|)\eta(t)$ *for all* $t \ge 0$;

(ii) $\eta(|a_1|) < |a_1|$;

(iii) $\lim_{k\to\infty}(1/k)\eta(k|a_1|) = 0$.

Suppose that $\varepsilon > 0$, and let $f : \mathcal{A} \to \mathcal{X}$ satisfying

$$\left\|Df(x_1, \ldots, x_n)\right\| + \left\|f(xy) - f(x)y - xf(y)\right\| \le \varepsilon \operatorname{Min}\left\{\sum_{i=1}^{n}\eta(\|x_i\|), \eta(\|x\|)\eta(\left\|y\right\|)\right\} \tag{2.40}$$

for all $x_1, \ldots, x_n, x, y \in \mathcal{A}$. *Then there exists a unique ring derivation* $D : \mathcal{A} \to \mathcal{X}$ *such that*

$$\|f(x) - D(x)\| \leq \frac{\varepsilon}{|2^{n-1}a_1|}\eta(\|x\|) \tag{2.41}$$

for all $x \in \mathcal{A}$.

Now, we would like to prove the superstability of derivations on non-Archimedean Banach algebras.

Theorem 2.7. *Let* $\mathcal{A}$ *be a non-Archimedean Banach algebra with bounded approximate identity. Let* $\varphi : \mathcal{A}^n \to [0,\infty)$, $\phi : \mathcal{A}^2 \to [0,\infty)$, $f : \mathcal{A} \to \mathcal{A}$ *be functions satisfying the conditions of Theorem 2.5. Then* $f : \mathcal{A} \to \mathcal{A}$ *is a ring derivation.*

Proof. In the proof of Theorem 2.5, we can see that

$$H(b)\big(H(a) - \varphi_f(a)\big) = \big(H(a) - \varphi_f(a)\big)H(b) = 0 \tag{2.42}$$

for all $a, b \in \mathcal{A}$

$$\begin{aligned}
\big(f(a) - D(a)\big)b &= \pi_1\big(\big(f(a) - D(a)\big)b, 0\big) \\
&= \pi_1\big(\big(f(a) - D(a), 0\big)(D(b), b)\big) \\
&= \pi_1\big(\big(\pi_1\big(H(a) - \varphi_f(a)\big), 0\big)(\pi_1(H(b)), b)\big) \\
&= \pi_1\big(\big(\pi_1\big(H(a) - \varphi_f(a)\big), 0\big)H(b)\big) \\
&= \pi_1\big(\big((\pi_1(H(a)), a) - \big(\pi_1(\varphi_f(a)), a\big)\big)H(b)\big) \\
&= \pi_1(0,0) \quad \big(\text{by } (2.42)\big) \\
&= 0
\end{aligned} \tag{2.43}$$

for all $a, b \in \mathcal{A}$. Since $\mathcal{A}$ has a bounded approximate identity, then by above equation, we have $f(a) = D(a)$ for all $a \in \mathcal{A}$. f is a ring derivation on $\mathcal{A}$. □

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