Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2010, Article ID 418273, 8 pages
doi:10.1155/2010/418273

Research Article

On a Higher-Order Nonlinear Difference Equation

Bratislav D. Iričanin

Faculty of Electrical Engineering, University of Belgrade, Bulevar Kralja Aleksandra 73, 11120 Belgrade, Serbia

Correspondence should be addressed to Bratislav D. Iričanin, iricanin@etf.rs

Received 20 May 2010; Accepted 2 June 2010

Academic Editor: Stevo Stević

This paper shows that all positive solutions of a higher-order nonlinear difference equation are bounded, extending some recent results in the literature.

1. Introduction

There is a considerable interest in studying nonlinear difference equations nowadays; see, for example, [1–40] and numerous references listed therein.

The investigation of the higher-order nonlinear difference equation

$$x_n = A + \frac{x_{n-m}^p}{x_{n-k}^r}, \quad n \in \mathbb{N}_0, \tag{1.1}$$

where $A, r > 0$ and $p \geq 0$, and $k, m \in \mathbb{N}$, $k \neq m$, was suggested by Stević at numerous talks and in papers (see, e.g., [20, 28, 30, 34–38] and the related references therein).

In this paper we show that under some conditions on parameters A, r, and p all positive solutions of the difference equation

$$x_n = A + \frac{x_{n-1}^p}{x_{n-k}^r}, \quad n \in \mathbb{N}_0, \tag{1.2}$$

where $k \in \mathbb{N} \setminus \{1\}$, are bounded. To do this we modify some methods and ideas from Stević's papers [30, 35–37]. Our motivation stems from these four papers.

The reader can find results for some particular cases of (1.2), as well as on some closely related equations treated in, for example, [1, 2, 5–11, 18–20, 26, 30, 33–35, 38, 40].

2. Main Result

Here we investigate the boundedness of the positive solutions to (1.2) for the case $0 < p < (rk^k/(k-1)^{k-1})^{1/k}$. The following result completely describes the boundedness of positive solutions to (1.2) in this case. The result is an extension of one of the main results in [35].

Theorem 2.1. *Assume, $p, r > 0$ and $k \in \mathbb{N} \setminus \{1\}$. Then every positive solution of* (1.2) *is bounded if*

$$0 < p < \left(\frac{rk^k}{(k-1)^{k-1}}\right)^{1/k}. \tag{2.1}$$

Proof. First note that from (1.2) it directly follows that

$$x_n > A, \quad \text{for } n \in \mathbb{N}_0. \tag{2.2}$$

Using (1.2), it follows that

$$\begin{aligned}
x_n &= A + \frac{x_{n-1}^p}{x_{n-k}^r} \\
&= A + \left(\frac{x_{n-1}}{x_{n-k}^{r/p}}\right)^p \\
&= A + \left(\frac{A}{x_{n-k}^{r/p}} + \frac{x_{n-2}^p}{x_{n-k}^{r/p} x_{n-k-1}^r}\right)^p \\
&= A + \left(\frac{A}{x_{n-k}^{r/p}} + \left(\frac{x_{n-2}}{x_{n-k}^{r/p^2} x_{n-k-1}^{r/p}}\right)^p\right)^p \\
&= A + \left(\frac{A}{x_{n-k}^{r/p}} + \left(\frac{A}{x_{n-k}^{r/p^2} x_{n-k-1}^{r/p}} + \frac{x_{n-3}^p}{x_{n-k}^{r/p^2} x_{n-k-1}^{r/p} x_{n-k-2}^r}\right)^p\right)^p \\
&= A + \left(\frac{A}{x_{n-k}^{r/p}} + \left(\frac{A}{x_{n-k}^{r/p^2} x_{n-k-1}^{r/p}} + \left(\frac{x_{n-3}}{x_{n-k}^{r/p^3} x_{n-k-1}^{r/p^2} x_{n-k-2}^{r/p}}\right)^p\right)^p\right)^p.
\end{aligned} \tag{2.3}$$

After k steps we obtain the following formula

$$\begin{aligned} x_n &= A + \Bigg(\frac{A}{x_{n-k}^{r/p}} + \Bigg(\frac{A}{x_{n-k}^{r/p^2} x_{n-k-1}^{r/p}} + \Bigg(\frac{A}{x_{n-k}^{r/p^3} x_{n-k-1}^{r/p^2} x_{n-k-2}^{r/p}} \\ &\quad + \cdots + \Bigg(\frac{A}{x_{n-k}^{r/p^{k-1}} x_{n-k-1}^{r/p^{k-2}} \cdots x_{n-(2k-2)}^{r/p}} + \frac{x_{n-k}^{p}}{x_{n-k}^{r/p^{k-1}} x_{n-k-1}^{r/p^{k-2}} \cdots x_{n-(2k-2)}^{r/p} x_{n-(2k-1)}^{r}}\Bigg)^p \cdots \Bigg)^p \\ &= A + \Bigg(\frac{A}{x_{n-k}^{r/p}} + \Bigg(\frac{A}{x_{n-k}^{r/p^2} x_{n-k-1}^{r/p}} + \Bigg(\frac{A}{x_{n-k}^{r/p^3} x_{n-k-1}^{r/p^2} x_{n-k-2}^{r/p}} \\ &\quad + \cdots + \Bigg(\frac{A}{x_{n-k}^{r/p^{k-1}} x_{n-k-1}^{r/p^{k-2}} \cdots x_{n-(2k-2)}^{r/p}} + \frac{x_{n-k}^{p-(r/p^{k-1})}}{x_{n-k-1}^{r/p^{k-2}} \cdots x_{n-(2k-2)}^{r/p} x_{n-(2k-1)}^{r}}\Bigg)^p \cdots \Bigg)^p. \end{aligned} \tag{2.4}$$

Two subcases can be considered now.

Case 1 ($r \geq p^k$). If $r \geq p^k$, then by (2.2) equality (2.4) implies that

$$\begin{aligned} x_n &< A + \left(\frac{A}{A^{r/p}} + \left(\frac{A}{A^{r/p^2+r/p}} + \left(\frac{A}{A^{r/p^3+r/p^2+r/p}} \right.\right.\right. \\ &\quad \left.\left.\left. + \cdots + \left(\frac{A}{A^{r/p^{k-1}+r/p^{k-2}+\cdots+r/p}} + \frac{1}{A^{r/p^{k-1}+r/p^{k-2}+\cdots+r/p+r-p}}\right)^p \cdots \right)^p \right. \right. < \infty, \end{aligned} \tag{2.5}$$

for $n \geq 2k-1$. This means that (x_n) is a bounded sequence.

Case 2 ($p^k > r$). In this case we have

$$p - \frac{r}{p^{k-1}} > 0. \tag{2.6}$$

From (2.4) and (1.2) we further obtain

$$
\begin{aligned}
x_n &= A + \left(\frac{A}{x_{n-k}^{r/p}} + \left(\frac{A}{x_{n-k}^{r/p^2} x_{n-k-1}^{r/p}} + \left(\frac{A}{x_{n-k}^{r/p^3} x_{n-k-1}^{r/p^2} x_{n-k-2}^{r/p}} \right.\right.\right. \\
&\qquad \left.\left.\left. + \cdots + \left(\frac{A}{x_{n-k}^{r/p^{k-1}} x_{n-k-1}^{r/p^{k-2}} \cdots x_{n-(2k-2)}^{r/p}} + \frac{x_{n-k}^{p-r/p^{k-1}}}{x_{n-k-1}^{r/p^{k-2}} \cdots x_{n-(2k-2)}^{r/p} x_{n-(2k-1)}^{r}} \right)^{p} \cdots \right)^{p} \right. \\
&= A + \left(\frac{A}{x_{n-k}^{r/p}} + \left(\frac{A}{x_{n-k}^{r/p^2} x_{n-k-1}^{r/p}} + \left(\frac{A}{x_{n-k}^{r/p^3} x_{n-k-1}^{r/p^2} x_{n-k-2}^{r/p}} \right.\right.\right. \\
&\qquad \left.\left.\left. + \cdots + \left(\frac{A}{\prod_{j=0}^{k-2} x_{n-k-j}^{z_0^{(j)}}} + \frac{x_{n-k}^{p-z_0^{(0)}}}{\left(\prod_{j=1}^{k-2} x_{n-k-j}^{z_0^{(j)}}\right) x_{n-(2k-1)}^{r}} \right)^{p} \cdots \right)^{p} \right. \\
&= A + \left(\frac{A}{x_{n-k}^{r/p}} + \left(\frac{A}{x_{n-k}^{r/p^2} x_{n-k-1}^{r/p}} + \left(\frac{A}{x_{n-k}^{r/p^3} x_{n-k-1}^{r/p^2} x_{n-k-2}^{r/p}} \right.\right.\right. \\
&\qquad \left.\left.\left. + \cdots + \left(\frac{A}{\prod_{j=0}^{k-2} x_{n-k-j}^{z_0^{(j)}}} + \left(\frac{x_{n-k}}{\left(\prod_{j=1}^{k-2} x_{n-k-j}^{z_0^{(j)}/p-z_0^{(0)}}\right) x_{n-(2k-1)}^{r/(p-z_0^{(0)})}} \right)^{p(p-z_0^{(0)})} \right)^{p} \cdots \right)^{p} \right. \\
&= A + \left(\frac{A}{x_{n-k}^{r/p}} + \left(\frac{A}{x_{n-k}^{r/p^2} x_{n-k-1}^{r/p}} + \left(\frac{A}{x_{n-k}^{r/p^3} x_{n-k-1}^{r/p^2} x_{n-k-2}^{r/p}} \right.\right.\right. \\
&\qquad \left.\left.\left. + \cdots + \left(\frac{A}{\prod_{j=0}^{k-2} x_{n-k-1-j}^{z_1^{(j)}}} + \frac{x_{n-k-1}^{p-z_1^{(0)}}}{\prod_{j=1}^{k-2} x_{n-k-1-j}^{z_1^{(j)}} x_{n-2k}^{r}} \right)^{p(p-z_0^{(0)})} \right)^{p} \cdots \right)^{p} \right. \\
&= \cdots = A + \left(\frac{A}{x_{n-k}^{r/p}} + \left(\frac{A}{x_{n-k}^{r/p^2} x_{n-k-1}^{r/p}} + \left(\frac{A}{x_{n-k}^{r/p^3} x_{n-k-1}^{r/p^2} x_{n-k-2}^{r/p}} \right.\right.\right. \\
&\qquad \left.\left.\left. + \cdots + \left(\frac{A}{\prod_{j=0}^{k-2} x_{n-k-m-j}^{z_m^{(j)}}} + \frac{x_{n-k-m}^{p-z_m^{(0)}}}{\left(\prod_{j=1}^{k-2} x_{n-k-m-j}^{z_m^{(j)}}\right) x_{n-2k+1-m}^{r}} \right)^{p\prod_{i=0}^{m-1}(p-z_i^{(0)})} \right)^{p} \cdots \right)^{p}, \right.
\end{aligned}
\tag{2.7}
$$

for each $k \in \mathbb{N} \setminus \{1\}$ and every $n \geq 2k + m - 1$, where the sequences $(z_m^{(j)})$, $j = 0, 1, \ldots, k-2$, satisfy the system

$$z_{m+1}^{(0)} = \frac{z_m^{(1)}}{p - z_m^{(0)}}, z_{m+1}^{(1)} = \frac{z_m^{(2)}}{p - z_m^{(0)}}, \ldots, z_{m+1}^{(k-3)} = \frac{z_m^{(k-2)}}{p - z_m^{(0)}}, z_{m+1}^{(k-2)} = \frac{r}{p - z_m^{(0)}}, \tag{2.8}$$

and the initial values are given by

$$z_0^{(j)} = rp^{j+1-k}, \quad j = 0, 1, \ldots, k-2. \tag{2.9}$$

Note that $p^k > r$ implies that $z_0^{(0)} < p$. Assume $z_m^{(0)} < p$ for every $m \in \mathbb{N}_0$.

By a direct calculation it follows that $z_0^{(j)} < z_1^{(j)}$, $j = 0, 1, \ldots, k-2$, which, along with (2.8) implies that $(z_m^{(j)})$, $j = 0, 1, \ldots, k-2$, are strictly increasing sequences.

From system (2.8), we have,

$$z_{m+1}^{(0)} = \frac{r}{\left(p - z_m^{(0)}\right)\left(p - z_{m-1}^{(0)}\right) \cdots \left(p - z_{m-k+2}^{(0)}\right)}, \quad m \geq k-2. \tag{2.10}$$

If it were $z_m^{(0)} < p$, $m \in \mathbb{N}_0$, then there was

$$\lim_{m \to \infty} z_m^{(0)} = z \in (0, p]. \tag{2.11}$$

Clearly z is a solution of the equation

$$f(x) = x(p - x)^{k-1} - r = 0. \tag{2.12}$$

Since

$$f(0) = f(p) = -r, \tag{2.13}$$

and

$$f'(x) = (p - x)^{k-2}(p - kx), \tag{2.14}$$

we see that the function f attains its maximum at the point $x = p/k$.

Further, by assumption (2.1) we get

$$f\left(\frac{p}{k}\right) = \frac{(k-1)^{k-1}}{k^k}\left(p^k - r\frac{k^k}{(k-1)^{k-1}}\right) < 0, \tag{2.15}$$

which along with (2.13) implies that (2.12) does not have solutions on $(0, p]$, arriving at a contradiction.

This implies that there is a fixed index $m_0 \in \mathbb{N}$ such that

$$z_{m_0-1}^{(0)} < p, \quad z_{m_0}^{(0)} \geq p. \tag{2.16}$$

From this, inequality (2.2), and identity (2.7) with $m = m_0$, it follows that

$$\begin{aligned} x_n &= A + \left(\frac{A}{x_{n-k}^{r/p}} + \left(\frac{A}{x_{n-k}^{r/p^2} x_{n-k-1}^{r/p}} + \left(\frac{A}{x_{n-k}^{r/p^3} x_{n-k-1}^{r/p^2} x_{n-k-2}^{r/p}} \right.\right.\right. \\ &\quad \left.\left.\left. + \cdots + \left(\frac{A}{\prod_{j=0}^{k-2} x_{n-k-m_0-j}^{z_{m_0}^{(j)}}} + \frac{x_{n-k-m_0}^{p-z_{m_0}^{(0)}}}{\left(\prod_{j=1}^{k-2} x_{n-k-m_0-j}^{z_{m_0}^{(j)}}\right) x_{n-2k+1-m_0}^{r}} \right)^{p\prod_{i=0}^{m_0-1}(p-z_i^{(0)})} \right)^{p} \cdots \right)^{p} \\ &\leq A + \left(\frac{A}{A^{r/p}} + \left(\frac{A}{A^{r/p^2+r/p}} + \left(\frac{A}{A^{r/p^3+r/p^2+r/p}} \right.\right.\right. \\ &\quad \left.\left.\left. + \cdots + \left(\frac{A}{A^{\sum_{j=0}^{k-2} z_{m_0}^{(j)}}} + \frac{1}{A^{r+z_{m_0}^{(0)}-p+\sum_{j=1}^{k-2} z_{m_0}^{(j)}}} \right)^{p\prod_{i=0}^{m_0-1}(p-z_i^{(0)})} \right)^{p} \cdots \right)^{p} < \infty \end{aligned} \tag{2.17}$$

for $n \geq 2k + m_0 - 1$.

From (2.17) the boundedness of the sequence (x_n) directly follows, as desired. □

Acknowledgment

The research was partially supported by the Serbian Ministry of Science, through The Mathematical Institute of SASA, Belgrade, Project no. 144013.

References

[1] A. M. Amleh, E. A. Grove, G. Ladas, and D. A. Georgiou, "On the recursive sequence $y_{n+1} = \alpha + (y_{n-1}/y_n)$," *Journal of Mathematical Analysis and Applications*, vol. 233, no. 2, pp. 790–798, 1999.

[2] L. Berg, "On the asymptotics of nonlinear difference equations," *Zeitschrift für Analysis und Ihre Anwendungen*, vol. 21, no. 4, pp. 1061–1074, 2002.

[3] L. Berg, "Inclusion theorems for non-linear difference equations with applications," *Journal of Difference Equations and Applications*, vol. 10, no. 4, pp. 399–408, 2004.

[4] L. Berg, "Nonlinear difference equations with periodic solutions," *Rostocker Mathematisches Kolloquium*, no. 61, pp. 13–20, 2006.

[5] R. DeVault, C. Kent, and W. Kosmala, "On the recursive sequence $x_{n+1} = p + (x_{n-k}/x_n)$," *Journal of Difference Equations and Applications*, vol. 9, no. 8, pp. 721–730, 2003.

[6] H. M. El-Owaidy, A. M. Ahmed, and M. S. Mousa, "On asymptotic behaviour of the difference equation $x_{n+1} = \alpha + (x^p_{n-1}/x^p_n)$," *Journal of Applied Mathematics & Computing*, vol. 12, no. 1-2, pp. 31–37, 2003.

[7] J. Feuer, "On the behavior of solutions of $x_{n+1} = p + (x_{n-1}/x_n)$," *Applicable Analysis*, vol. 83, no. 6, pp. 599–606, 2004.

[8] E. A. Grove and G. Ladas, *Periodicities in Nonlinear Difference Equations*, vol. 4 of *Advances in Discrete Mathematics and Applications*, Chapman & Hall/CRC, Boca Raton, Fla, USA, 2005.

[9] L. Gutnik and S. Stević, "On the behaviour of the solutions of a second-order difference equation," *Discrete Dynamics in Nature and Society*, Article ID 27562, 14 pages, 2007.

[10] A. E. Hamza, "On the recursive sequence $x_{n+1} = \alpha + (x_{n-1}/x_n)$," *Journal of Mathematical Analysis and Applications*, vol. 322, no. 2, pp. 668–674, 2006.

[11] V. L. Kocić and G. Ladas, *Global Behavior of Nonlinear Difference Equations of Higher Order with Applications*, vol. 256 of *Mathematics and Its Applications*, Kluwer Academic Publisher, Dordrecht, The Netherlands, 1993.

[12] S. Stević, "A global convergence results with applications to periodic solutions," *Indian Journal of Pure and Applied Mathematics*, vol. 33, no. 1, pp. 45–53, 2002.

[13] S. Stević, "A note on the difference equation $x_{n+1} = \sum_{i=0}^{k}(a_i/x^{p_i}_{n-i})$," *Journal of Difference Equations and Applications*, vol. 8, no. 7, pp. 641–647, 2002.

[14] S. Stević, "On the recursive sequence $x_{n+1} = x_{n-1}/g(x_n)$," *Taiwanese Journal of Mathematics*, vol. 6, no. 3, pp. 405–414, 2002.

[15] S. Stević, "The recursive sequence $x_{n+1} = g(x_n, x_{n-1})/(A + x_n)$," *Applied Mathematics Letters of Rapid Publication*, vol. 15, no. 3, pp. 305–308, 2002.

[16] S. Stević, "Asymptotic behavior of a nonlinear difference equation," *Indian Journal of Pure and Applied Mathematics*, vol. 34, no. 12, pp. 1681–1687, 2003.

[17] S. Stević, "On the recursive sequence $x_{n+1} = A/\prod_{i=0}^{k} x_{n-i} + 1/\prod_{j=k+2}^{2(k+1)} x_n - j$," *Taiwanese Journal of Mathematics*, vol. 7, no. 2, pp. 249–259, 2003.

[18] S. Stević, "On the recursive sequence $x_{n+1} = \alpha_n + (x_{n-1}/x_n)$ II," *Dynamics of Continuous, Discrete & Impulsive Systems*, vol. 10, no. 6, pp. 911–916, 2003.

[19] S. Stević, "On the recursive sequence $x_{n+1} = \alpha_n + (x_{n-1}/x_n)$," *International Journal of Mathematical Sciences*, vol. 2, no. 2, pp. 237–243, 2003.

[20] S. Stević, "On the recursive sequence $x_{n+1} = \alpha_n + (x^p_{n-1}/x^p_n)$," *Journal of Applied Mathematics & Computing*, vol. 18, no. 1-2, pp. 229–234, 2005.

[21] S. Stević, "On the recursive sequence $x_{n+1} = (\alpha + \beta x_{n-k})/f(x_n, \dots, x_{n-k+1})$," *Taiwanese Journal of Mathematics*, vol. 9, no. 4, pp. 583–593, 2005.

[22] S. Stević, "A short proof of the Cushing-Henson conjecture," *Discrete Dynamics in Nature and Society*, vol. 2006, Article ID 37264, 5 pages, 2006.

[23] S. Stević, "Global stability and asymptotics of some classes of rational difference equations," *Journal of Mathematical Analysis and Applications*, vol. 316, no. 1, pp. 60–68, 2006.

[24] S. Stević, "On positive solutions of a $(k+1)$-th order difference equation," *Applied Mathematics Letters of Rapid Publication*, vol. 19, no. 5, pp. 427–431, 2006.

[25] S. Stević, "On the recursive sequence $x_{n+1} = ax^p_{n-2m+1}/(b + cx^{p-1}_{n-2k})$," *Journal of Applied Mathematics & Computing*, vol. 21, no. 1-2, pp. 223–232, 2006.

[26] S. Stević, "Asymptotics of some classes of higher-order difference equations," *Discrete Dynamics in Nature and Society*, vol. 2007, Article ID 56813, 20 pages, 2007.

[27] S. Stević, "Asymptotic periodicity of a higher-order difference equation," *Discrete Dynamics in Nature and Society*, Article ID 13737, 9 pages, 2007.

[28] S. Stević, "Boundedness character of a max-type difference equation," in *Proceedings of the Conference in Honour of Allan Peterson, Book of Abstracts*, p. 28, Novacella, Italy, July 2007.

[29] S. Stević, "Existence of nontrivial solutions of a rational difference equation," *Applied Mathematics Letters*, vol. 20, no. 1, pp. 28–31, 2007.

[30] S. Stević, "On the recursive sequence $x_{n+1} = A + (x^p_n/x^r_{n-1})$," *Discrete Dynamics in Nature and Society*, vol. 2007, Article ID 40963, 9 pages, 2007.

[31] S. Stević, "On the recursive sequence $x_n = 1+(\sum_{i=1}^{k} a_i x_{n-p_i})/(\sum_{j=1}^{m} \beta_j x_{n-q_i})$," *Discrete Dynamics in Nature and Society*, Article ID 39404, 7 pages, 2007.

[32] S. Stević, "Boundedness and global stability of a higher-order difference equation," *Journal of Difference Equations and Applications*, vol. 14, no. 10-11, pp. 1035–1044, 2008.

[33] S. Stević, "On the difference equation $x_{n+1} = a + (x_{n-1}/x_n)$," *Computers & Mathematics with Applications*, vol. 56, no. 5, pp. 1159–1171, 2008.

[34] S. Stević, "On the recursive sequence $x_{n+1} = \max\{c, x_n^p/x_{n-1}^p\}$," *Applied Mathematics Letters of Rapid Publication*, vol. 21, no. 8, pp. 791–796, 2008.

[35] S. Stević, "Boundedness character of a class of difference equations," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 70, no. 2, pp. 839–848, 2009.

[36] S. Stević, "Boundedness character of a fourth order nonlinear difference equation," *Chaos, Solitons and Fractals*, vol. 40, no. 5, pp. 2364–2369, 2009.

[37] S. Stević, "Boundedness character of two classes of third-order difference equations," *Journal of Difference Equations and Applications*, vol. 15, no. 11-12, pp. 1193–1209, 2009.

[38] S. Stević, "On a class of higher-order difference equations," *Chaos, Solitons and Fractals*, vol. 42, no. 1, pp. 138–145, 2009.

[39] S. Stević, "Global stability of some symmetric difference equations," *Applied Mathematics and Computation*, vol. 216, no. 1, pp. 179–186, 2010.

[40] X.-X. Yan, W.-T. Li, and Z. Zhao, "On the recursive sequence $x_{n+1} = \alpha - (x_n/x_{n-1})$," *Journal of Applied Mathematics & Computing*, vol. 17, no. 1-2, pp. 269–282, 2005.