Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2010, Article ID 472760, 9 pages
doi:10.1155/2010/472760

Research Article

Some Existence Theorems for Nonconvex Variational Inequalities Problems

Narin Petrot

Department of Mathematics, Faculty of Science, Naresuan University, Phitsanulok 65000, Thailand

Correspondence should be addressed to Narin Petrot, narinp@nu.ac.th

Received 9 November 2010; Accepted 18 December 2010

Academic Editor: Roman Dwilewicz

By using nonsmooth analysis knowledge, we provide the conditions for existence solutions of the variational inequalities problems in nonconvex setting. We also show that the strongly monotonic assumption of the mapping may not need for the existence of solutions. Consequently, the results presented in this paper can be viewed as an improvement and refinement of some known results from the literature.

1. Introduction

Variational inequalities theory, which was introduced by Stampacchia [1], provides us with a simple, natural, general, and unified framework to study a wide class of problems arising in pure and applied sciences. The development of variational inequality theory can be viewed as the simultaneous pursuit of two different lines of research. On the one hand, it reveals the fundamental facts on the qualitative aspects of the solutions to important classes of problems. On the other hand, it also enables us to develop highly efficient and powerful new numerical methods for solving, for example, obstacle, unilateral, free, moving, and complex equilibrium problems.

It should be pointed out that almost all the results regarding the existence and iterative schemes for solving variational inequalities and related optimizations problems are being considered in the convexity setting; see [2–5] for examples. Moreover, all the techniques are based on the properties of the projection operator over convex sets, which may not hold in general, when the sets are nonconvex. Notice that the convexity assumption, made by researchers, has been used for guaranteeing the well definedness of the proposed iterative algorithm which depends on the projection mapping. In fact, the convexity assumption may not require for the well definedness of the projection mapping because it may be well defined, even in the nonconvex case (e.g., when the considered set is a closed subset of a finite dimensional space or a compact subset of a Hilbert space, etc.).

The main aim of this paper is intending to consider the conditions for the existence solutions of some variational inequalities problems in nonconvex setting. We will make use of some recent nonsmooth analysis techniques to overcome the difficulties that arise from the nonconvexity. Also, it is worth mentioning that we have considered when the mapping may not satisfy the strongly monotonic assumption. In this sense, our result represents an improvement and refinement of the known results.

2. Preliminaries

Let $\mathscr{H}$ be a real Hilbert space whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. Let C be a nonempty closed subset of $\mathscr{H}$. We denote by $d_C(\cdot)$ the usual distance function to the subset C; that is, $d_C(u) = \inf_{v \in C} \|u - v\|$. Let us recall the following well-known definitions and some auxiliary results of nonlinear convex analysis and nonsmooth analysis.

Definition 2.1. Let $u \in \mathscr{H}$ be a point not lying in C. A point $v \in C$ is called a closest point or a projection of u onto C if $d_C(u) = \|u - v\|$. The set of all such closest points is denoted by $\operatorname{proj}_C(u)$; that is,

$$\operatorname{proj}_C(u) = \{v \in C : d_C(u) = \|u - v\|\}. \tag{2.1}$$

Definition 2.2. Let C be a subset of $\mathscr{H}$. The proximal normal cone to C at x is given by

$$N_C^P(x) = \{z \in \mathscr{H} : \exists \rho > 0; x \in \operatorname{proj}_C(x + \rho z)\}. \tag{2.2}$$

The following characterization of $N_C^P(x)$ can be found in [6].

Lemma 2.3. *Let C be a closed subset of a Hilbert space $\mathscr{H}$. Then,*

$$z \in N_C^P(x) \Longleftrightarrow \exists \sigma > 0, \qquad \langle z, y - x \rangle \le \sigma \|y - x\|^2, \quad \forall y \in C. \tag{2.3}$$

Clarke et al. [7] and Poliquin et al. [8] have introduced and studied a new class of nonconvex sets, which are called uniformly prox-regular sets. This class of uniformly prox-regular sets has played an important part in many nonconvex applications such as optimization, dynamic systems, and differential inclusions.

Definition 2.4. For a given $r \in (0, +\infty]$, a subset C of $\mathscr{H}$ is said to be uniformly prox-regular with respect to r if, for all $\overline{x} \in C$ and for all $0 \ne z \in N_C^P(\overline{x})$, one has

$$\left\langle \frac{z}{\|z\|}, x - \overline{x} \right\rangle \le \frac{1}{2r} \|x - \overline{x}\|^2, \quad \forall x \in C. \tag{2.4}$$

We make the convention $1/r = 0$ for $r = +\infty$.

It is well known that a closed subset of a Hilbert space is convex if and only if it is proximally smooth of radius $r > 0$. Thus, in view of Definition 2.4, for the case of $r = \infty$, the uniform r-prox-regularity C is equivalent to the convexity of C. Moreover, it is clear that the class of uniformly prox-regular sets is sufficiently large to include the class p-convex sets, $C^{1,1}$ submanifolds (possibly with boundary) of $\mathscr{H}$, the images under a $C^{1,1}$ diffeomorphism of convex sets, and many other nonconvex sets; see [6, 8].

Now, let us state the following facts, which summarize some important consequences of the uniform prox-regularity. The proof of this result can be found in [7, 8].

Lemma 2.5. *Let C be a nonempty closed subset of $\mathscr{H}, r \in (0,+\infty]$ and set $C_r := \{x \in \mathscr{H}; d(x,C) < r\}$. If C is uniformly r-uniformly prox-regular, then the following hold:*

(1) *for all $x \in C_r$, $\mathrm{proj}_C(x) \neq \emptyset$,*

(2) *for all $s \in (0,r)$, proj_C is Lipschitz continuous with constant $r/(r-s)$ on C_s,*

(3) *the proximal normal cone is closed as a set-valued mapping.*

In this paper, we are interested in the following classes of nonlinear mappings.

Definition 2.6. A mapping $T : C \to \mathscr{H}$ is said to be

(a) *γ-strongly monotone* if there exists a constant $\gamma > 0$ such that

$$\langle Tx - Ty, x - y\rangle \geq \gamma \|x - y\|^2, \quad \forall x, y \in C, \tag{2.5}$$

(b) *μ-Lipschitz* if there exist a constants $\mu > 0$ such that

$$\|Tx - Ty\| \leq \mu\|x - y\|, \quad \forall x, y \in C. \tag{2.6}$$

3. System of Nonconvex Variational Inequalities Involving Nonmonotone Mapping

Let $\mathscr{H}$ be a real Hilbert space, and let C be a nonempty closed subset of $\mathscr{H}$. In this section, we will consider the following problem: find $x^*, y^* \in C$ such that

$$\begin{aligned} y^* - x^* - \rho T y^* &\in N_C^P(x^*), \\ x^* - y^* - \eta T x^* &\in N_C^P(y^*), \end{aligned} \tag{3.1}$$

where ρ and η are fixed positive real numbers, C is a closed subset of $\mathscr{H}$, and $T : C \to \mathscr{H}$ is a mapping.

The iterative algorithm for finding a solution of the problem (3.1) was considered by Moudafi [9], when C is r-uniformly prox-regular and T is a strongly monotone mapping. He also remarked that two-step models (3.1) for nonlinear variational inequalities are relatively more challenging than the usual variational inequalities since it can be applied to problems arising, especially from complementarity problems, convex quadratic programming, and other variational problems. In this section, we will generalize such result by considering the conditions for existence solution of problem (3.1) when T is not necessary strongly monotone. To do so, we will use the following algorithm as an important tool.

Algorithm 3.1. Let C be an r-uniformly prox-regular subset of $\mathcal{H}$. Assume that $T : C \to \mathcal{H}$ is a nonlinear mapping. Letting x_0 be an arbitrary point in C, we consider the following two-step projection method:

$$y_n = \operatorname{proj}_C[x_n - \eta(Tx_n)],$$
$$x_{n+1} = \operatorname{proj}_C[y_n - \rho(Ty_n)], \tag{3.2}$$

where ρ, η are positive reals number, which were appeared in problem (3.1).

Remark 3.2. The projection algorithm above has been introduced in the convex case, and its convergence was proved see [10]. Observe that (3.2) is well defined provided the projection on C is not empty. Our adaptation of the projection algorithm will be based on Lemma 2.5.

Now we will prove the existence theorems of problem (3.1), when C is a closed uniformly r-prox-regular. Moreover, from now on, the number r will be understood as a finite positive real number (if not specified otherwise). This is because, as we know, if $r = \infty$, then such a set C is nothing but the closed convex set.

We start with an important remark.

Remark 3.3. Let C be a uniformly r-prox-regular closed subset of $\mathcal{H}$. Let $T_1, T_2 : C \to \mathcal{H}$ be such that T_1 is a μ_1-Lipschitz continuous, γ-strongly monotone mapping and T_2 is a μ_2-Lipschitz continuous mapping. If $\xi = r[\mu_1^2 - \gamma\mu_2 - \sqrt{(\mu_1^2 - \gamma\mu_2)^2 - \mu_1^2(\gamma - \mu_2)^2}]/\mu_1^2$, then for each $s \in (0, \xi)$ we have

$$\gamma t_s - \mu_2 > \sqrt{(\mu_1^2 - \mu_2^2)(t_s^2 - 1)}, \tag{3.3}$$

where $t_s = r/(r - s)$.

It is worth to point out that, in Remark 3.3, we have to assume that $\mu_2 < \mu_1$. Thus, from now on, without loss of generality we will always assume that $\mu_2 < \mu_1$.

Theorem 3.4. *Let C be a uniformly r-prox-regular closed subset of a Hilbert space $\mathcal{H}$, and let $T : C \to \mathcal{H}$ be a nonlinear mapping. Let $T_1, T_2 : C \to \mathcal{H}$ be such that T_1 is a μ_1-Lipschitz continuous and γ-strongly monotone mapping, T_2 is a μ_2-Lipschitz continuous mapping. If $T = T_1 + T_2$ and the following conditions are satisfied:*

(a) *$M^{\rho,\eta}\delta_{T(C)} < \xi$, where $\delta_{T(C)} = \sup\{\|u - v\|; u, v \in T(C)\}$;*

(b) *there exists $s \in (M^{\rho,\eta}\delta_{T(C)}, \xi)$ such that*

$$\frac{\gamma t_s - \mu_2}{t_s(\mu_1^2 - \mu_2^2)} - \zeta < \rho, \eta < \min\left\{\frac{\gamma t_s - \mu_2}{t_s(\mu_1^2 - \mu_2^2)} + \zeta, \frac{1}{t_s \mu_2}\right\}, \tag{3.4}$$

where $M^{\rho,\eta} = \max\{\rho, \eta\}$, $t_s = r/(r - s)$, and $\zeta = \sqrt{(t_s\gamma - \mu_2)^2 - (\mu_1^2 - \mu_2^2)(t_s^2 - 1)}/t_s(\mu_1^2 - \mu_2^2)$.

Then the problem (3.1) *has a solution. Moreover, the sequence (x_n, y_n) which is generated by* (3.2) *strongly converges to a solution $(x^*, y^*) \in C \times C$ of the problem* (3.1).

Proof. Firstly, by condition (b), we can easily check that $y_n - \rho T y_n$ and $x_n - \eta T x_n$ belong to the set C_s, for all $n = 1, 2, 3, \ldots$. Thus, from Lemma 2.5 (1), we know that (3.2) is well defined. Consequently, from (3.2) and Lemma 2.5 (2), we have

$$\begin{aligned}\|x_{n+1} - x_n\| &= \|\operatorname{proj}_C(y_n - \rho T y_n) - \operatorname{proj}_C(y_{n-1} - \rho T y_{n-1})\| \\ &\leq t_s \|y_n - y_{n-1} - \rho(T y_n - T y_{n-1})\| \\ &\leq t_s \left[\|y_n - y_{n-1} - \rho(T_1 y_n - T_1 y_{n-1})\| + \rho \|T_2 y_n - T_2 y_{n-1}\|\right].\end{aligned} \tag{3.5}$$

Since the mapping T_1 is γ-strongly monotone and μ_1-Lipschitz continuous, we obtain

$$\begin{aligned}&\left\|y_n - y_{n-1} - \rho(T_1 y_n - T_1 y_{n-1})\right\|^2 \\ &\quad = \left\|y_n - y_{n-1}\right\|^2 - 2\rho\langle y_n - y_{n-1}, T_1 y_n - T_1 y_{n-1}\rangle + \rho^2 \left\|T_1 y_n - T_1 y_{n-1}\right\|^2 \\ &\quad \leq \left\|y_n - y_{n-1}\right\|^2 - 2\rho\gamma \|y_n - y_{n-1}\| + \rho^2 \mu_1^2 \left\|y_n - y_{n-1}\right\|^2 \\ &\quad = \left(1 - 2\rho\gamma + \rho^2\mu_1^2\right) \left\|y_n - y_{n-1}\right\|^2.\end{aligned} \tag{3.6}$$

On the other hand, since T_2 is μ_2-Lipschitz continuous, we have

$$\|T_2 y_n - T_2 y_{n-1}\| \leq \mu_2 \|y_n - y_{n-1}\|. \tag{3.7}$$

Thus, by (3.5), (3.6), and (3.7), we obtain

$$\|x_{n+1} - x_n\| \leq t_s \left[\rho\mu_2 + \sqrt{1 - 2\rho\gamma + \rho^2\mu_1^2}\right] \|y_n - y_{n-1}\|. \tag{3.8}$$

Similarly, we have

$$\begin{aligned}\|y_n - y_{n-1}\| &= \|\operatorname{proj}_C(x_n - \eta T x_n) - \operatorname{proj}_C(x_{n-1} - \eta T x_{n-1})\| \\ &\leq t_s \|x_n - x_{n-1} - \eta(T x_n - T x_{n-1})\| \\ &\leq t_s \left[\|x_n - x_{n-1} - \eta(T_1 x_n - T_1 x_{n-1})\| + \eta \|T_2 x_n - T_2 x_{n-1}\|\right] \\ &\leq t_s \left[\eta\mu_2 + \sqrt{1 - 2\eta\gamma + \eta^2\mu_1^2}\right] \|x_n - x_{n-1}\|.\end{aligned} \tag{3.9}$$

Combining (3.8) and (3.9), we get

$$\|x_{n+1} - x_n\| \leq t_s^2 \theta_\rho \theta_\eta \|x_n - x_{n-1}\|, \tag{3.10}$$

where $\theta_\rho := \rho\mu_2 + \sqrt{1-2\rho\gamma+\rho^2\mu_1^2}$ and $\theta_\eta := \eta\mu_2 + \sqrt{1-2\eta\gamma+\eta^2\mu_1^2}$. Moreover, by (3.4), we know that $t_s\theta_\rho$ and $t_s\theta_\eta$ are elements of the interval $(0,1)$. Thus, from (3.10), it follows that

$$\|x_{n+1}-x_n\| \le \kappa^n\|x_1-x_0\| \tag{3.11}$$

for all $n = 1,2,3,\ldots$, where $\kappa := t_s^2\theta_\rho\theta_\eta$. Hence, for any $m \ge n > 1$, it follows that

$$\|x_m-x_n\| \le \sum_{i=n}^{m-1}\|x_{i+1}-x_i\| \le \sum_{i=n}^{m-1}\kappa^i\|x_1-x_0\| \le \frac{\kappa^n}{1-\kappa}\|x_1-x_0\|. \tag{3.12}$$

Since $\kappa < 1$, it follows that $\kappa^n \to 0$ as $n \to \infty$, and this implies that $\{x_n\} \subset C$ is a Cauchy sequence. Consequently, from (3.9), we also have that $\{y_n\}$ is a Cauchy sequence in C. Thus, by Lemma 2.5 (3), the closedness property of C implies that there exists $(x^*,y^*) \in C\times C$ such that $(x_n,y_n) \to (x^*,y^*)$ as $n \to \infty$.

We claim that $(x^*,y^*) \in C\times C$ is a solution of the problem (3.1). Indeed, by the definition of the proximal normal cone, from (3.2), we have

$$\begin{aligned}(x_n-y_n)-\eta(Tx_n) &\in N_C^P(y_n),\\ (y_n-x_{n+1})-\rho(Ty_n) &\in N_C^P(x_{n+1}).\end{aligned} \tag{3.13}$$

By letting $n \to \infty$, using the closedness property of the proximal cone together with the continuity of T, we have

$$\begin{aligned}x^*-y^*-\eta(Tx^*) &\in N_C^P(y^*),\\ y^*-x^*-\rho(Ty^*) &\in N_C^P(x^*).\end{aligned} \tag{3.14}$$

This completes the proof. □

Immediately, by setting $T_2 = 0$, we have the following result.

Theorem 3.5. *Let C be a uniformly r-prox-regular closed subset of a Hilbert space $\mathcal{H}$. Let $T : C \to \mathcal{H}$ be a μ-Lipschitz continuous and γ-strongly monotone mapping. If the following conditions are satisfied:*

(a) $M^{\rho,\eta}\delta_{T(C)} < \xi$, *where* $\delta_{T(C)} = \sup\{\|u-v\|; u,v \in T(C)\}$;

(b) *there exists* $s \in (M^{\rho,\eta}\delta_{T(C)},\xi)$ *such that*

$$\frac{\gamma}{\mu^2}-\zeta < \rho,\eta < \frac{\gamma}{\mu^2}+\zeta, \tag{3.15}$$

where $\zeta = \sqrt{(t_s\gamma)^2-(\mu_1^2)(t_s^2-1)}/t_s(\mu_1^2)$ *and* $t_s = r/(r-s)$.

Then the problem (3.1) *has a solution. Moreover, the sequence* (x_n,y_n) *which is generated by* (3.2) *strongly converges to a solution* $(x^*,y^*) \in C\times C$ *of the problem* (3.1).

In view of proving Theorem 3.4, we can obtain the following result, which contains a recent result presented by Moudafi [9] as a special case.

Theorem 3.6. *Let C be a uniformly r-prox-regular closed subset of a Hilbert space $\mathscr{H}$, and let $T : C \to \mathscr{H}$ be a mapping. Let $T_1, T_2 : C \to \mathscr{H}$ be such that T_1 is a μ_1-Lipschitz continuous and γ-strongly monotone mapping, T_2 is a μ_2-Lipschitz continuous mapping. If $T = T_1 + T_2$ and there exists $s \in (0, \xi)$ such that*

$$\frac{\gamma t_s - \mu_2}{t_s(\mu_1^2 - \mu_2^2)} - \zeta < \rho < min\left\{\frac{\gamma t_s - \mu_2}{t_s(\mu_1^2 - \mu_2^2)} + \zeta, \frac{1}{t_s\mu_2}, \frac{s}{1 + \|Ty_n\|}\right\},$$
$$\frac{\gamma t_s - \mu_2}{t_s(\mu_1^2 - \mu_2^2)} - \zeta < \eta < min\left\{\frac{\gamma t_s - \mu_2}{t_s(\mu_1^2 - \mu_2^2)} + \zeta, \frac{1}{t_s\mu_2}, \frac{s}{1 + \|Tx_n\|}\right\} \tag{3.16}$$

for all $n = 1, 2, 3, \ldots$, where $t_s = r/(r - s)$, $\zeta = \sqrt{(t_s\gamma - \mu_2)^2 - (\mu_1^2 - \mu_2^2)(t_s^2 - 1)}/t_s(\mu_1^2 - \mu_2^2)$ and the sequence (x_n, y_n) was generated by (3.2), *then the sequence (x_n, y_n) strongly converges to a solution $(x^*, y^*) \in C \times C$ of the problem* (3.1).

Remark 3.7. (i) An inspection of Theorem 3.6 shows that the sequences $\{Tx_n\}$ and $\{Ty_n\}$ are bounded.

(ii) By setting $T_2 =: 0$, we see that Theorem 3.6 reduces to a result presented by Moudafi [9].

Remark 3.8. If C is a convex set, by the definition of the proximal normal cone, we can reformulate (3.1) as follows: find $x^*, y^* \in C \times C$ such that

$$\langle \rho T(y^*) + x^* - y^*, x - x^* \rangle \geq 0, \quad \forall x \in C,$$
$$\langle \eta T(x^*) + y^* - x^*, x - y^* \rangle \geq 0, \quad \forall x \in C. \tag{3.17}$$

The problem (3.17) was introduced and studied by Verma [10], when T is a strong monotone mapping. Hence, Theorem 3.4 extends and improves the results presented by Verma [10]. For further recent results related to the problem (3.17), see also [2, 3, 5, 11–13].

4. Further Results

By using the techniques as in Theorem 3.4, we can also obtain an existence theorem of the following problem: find $x^* \in C$ such that

$$-Tx^* \in N_C^P(x^*). \tag{4.1}$$

The problem of type (4.1) was studied by Noor [14] but in a finite dimension Hilbert space setting. In this section, we intend to consider the problem (4.1) in an infinite dimension Hilbert space. To do this, the following remark is useful.

Remark 4.1. Let $T : C \to C$ be a γ-strongly monotone and μ-Lipschitz continuous mapping. Then, the function $f : (1, \mu^2/(\mu^2-\gamma^2)) \to (0,\infty)$ which is defined by

$$f(t) = \frac{\sqrt{t^2(\gamma^2-\mu^2)+\mu^2}}{t\mu^2}, \quad \forall t \in \left(1, \frac{\mu^2}{\mu^2-\gamma^2}\right), \tag{4.2}$$

is a continuous decreasing function on its domain.

We now close this section by proving an existence theorem to the problem (4.1) in a nonconvex infinite dimensional setting.

Theorem 4.2. *Let C be a uniformly r-prox-regular closed subset of a Hilbert space $\mathscr{H}$, and let $T : C \to \mathscr{H}$ be a γ-strongly monotone and μ-Lipschitz continuous mapping. If $0 < \delta_{T(C)} \leq \gamma r$, then the problem* (4.1) *has a solution.*

Proof. Firstly, by using an elementary calculation, we know that the function $h : [1, \mu^2/(\mu^2-\gamma^2)) \to (0,\infty)$ which is defined by

$$h(t) = \frac{r(t-1)}{t\delta_{T(C)}} + f(t), \quad \forall t \in \left[1, \frac{\mu^2}{\mu^2-\gamma^2}\right), \tag{4.3}$$

is a continuous increasing function on $[1, \sqrt{(\mu^2r^2-\delta^2_{T(C)})/r^2(\mu^2-\gamma^2)}]$. Moreover, we see that the net $\{t_s\}_{s\in(0,r)}$ which is defined by $t_s =: r/(r-s)$ converges to 1 as $s \downarrow 0$. Using these observations, together with the fact that $h(t) \downarrow \gamma/\mu^2$ as $t \downarrow 1$, we can find $s^* \in (0, r(r^2\gamma^2 - \delta^2_{T(C)})/(\mu^2r^2-\delta^2_{T(C)}))$ such that $\mu^2 h(t_{s^*}) > \gamma$. It is worth to notice that, from the choice of s^*, we have $\gamma/\mu^2 - f(t_{s^*}) < s^*/\delta_{T(C)}$.

Now, we choose a fixed positive real number ρ such that

$$\frac{\gamma}{\mu^2} - f(t_{s^*}) < \rho < min\left\{\frac{\gamma}{\mu^2} + f(t_{s^*}), \frac{s^*}{\delta_{T(C)}}\right\}. \tag{4.4}$$

Next, let us start with an element $x_0 \in C$ and use an induction process to obtain a sequence $\{x_n\} \subset C$ satisfying

$$x_{n+1} = \mathrm{proj}_C(x_n - \rho T x_n), \quad \forall n = 0,1,2,\ldots. \tag{4.5}$$

Note that, because of the choice of ρ, we can easily check that $x_n - \rho T x_n \in C_{s^*}$ for all $n = 1,2,3,\ldots$. Following the proof of Theorem 3.4, we know that $\{x_n\}$ is a Cauchy sequence in C. If $x_n \to x^*$ as $n \to \infty$, the closedness property of the proximal cone together with the continuity of T, from (4.5), we see that x^* is a solution of the problem (4.1). This completes the proof. □

Remark 4.3. Theorems 3.4, 3.5, and 4.2 not only give the conditions for the existence solution of the problems (3.1) and (4.1), respectively, but also provide the algorithm to find such solutions for any initial vector $x_0 \in C$.

Acknowledgments

This study was supported by the Commission on Higher Education and the Thailand Research Fund (Project no. MRG5380247).

References

[1] G. Stampacchia, "Formes bilinéaires coercitives sur les ensembles convexes," *Comptes Rendus de l'Académie des Sciences*, vol. 258, pp. 4413–4416, 1964.
[2] Y. J. Cho and X. Qin, "Systems of generalized nonlinear variational inequalities and its projection methods," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 69, no. 12, pp. 4443–4451, 2008.
[3] Y. J. Cho, S. M. Kang, and X. Qin, "On systems of generalized nonlinear variational inequalities in Banach spaces," *Applied Mathematics and Computation*, vol. 206, no. 1, pp. 214–220, 2008.
[4] N. Petrot, "Existence and algorithm of solutions for general set-valued Noor variational inequalities with relaxed (μ, ν)-cocoercive operators in Hilbert spaces," *Journal of Applied Mathematics and Computing*, vol. 32, no. 2, pp. 393–404, 2010.
[5] S. Suantai and N. Petrot, "Existence and stability of iterative algorithms for the system of nonlinear quasi-mixed equilibrium problems," *Applied Mathematics Letters*, vol. 24, no. 3, pp. 308–313, 2011.
[6] F. H. Clarke, Y. S. Ledyaev, R. J. Stern, and P. R. Wolenski, *Nonsmooth Analysis and Control Theory*, vol. 178 of *Graduate Texts in Mathematics*, Springer, New York, NY, USA, 1998.
[7] F. H. Clarke, R. J. Stern, and P. R. Wolenski, "Proximal smoothness and the lower-C^2 property," *Journal of Convex Analysis*, vol. 2, no. 1-2, pp. 117–144, 1995.
[8] R. A. Poliquin, R. T. Rockafellar, and L. Thibault, "Local differentiability of distance functions," *Transactions of the American Mathematical Society*, vol. 352, no. 11, pp. 5231–5249, 2000.
[9] A. Moudafi, "Projection methods for a system of nonconvex variational inequalities," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 71, no. 1-2, pp. 517–520, 2009.
[10] R. U. Verma, "General convergence analysis for two-step projection methods and applications to variational problems," *Applied Mathematics Letters*, vol. 18, no. 11, pp. 1286–1292, 2005.
[11] R. P. Agarwal and R. U. Verma, "General system of $A(\eta)$-maximal relaxed monotone variational inclusion problems based on generalized hybrid algorithms," *Communications in Nonlinear Science and Numerical Simulation*, vol. 15, no. 2, pp. 238–251, 2010.
[12] M. A. Noor, "On a system of general mixed variational inequalities," *Optimization Letters*, vol. 3, no. 3, pp. 437–451, 2009.
[13] N. Petrot, "A resolvent operator technique for approximate solving of generalized system mixed variational inequality and fixed point problems," *Applied Mathematics Letters*, vol. 23, no. 4, pp. 440–445, 2010.
[14] M. A. Noor, "Iterative schemes for nonconvex variational inequalities," *Journal of Optimization Theory and Applications*, vol. 121, no. 2, pp. 385–395, 2004.