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Research Article

An Existence and Uniqueness Result for a Bending Beam Equation without Growth Restriction

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We discuss the solvability of the fourth-order boundary value problem $u^{(4)} = f(t, u, u'')$, $0 \le t \le 1$, $u(0) = u(1) = u''(0) = u''(1) = 0$, which models a statically bending elastic beam whose two ends are simply supported, where $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous. Under a condition allowing that $f(t, u, v)$ is superlinear in u and v, we obtain an existence and uniqueness result. Our discussion is based on the Leray-Schauder fixed point theorem.

1. Introduction and Main Results

In this paper we deal with the existence of a solution of the fourth-order ordinary differential equation boundary value problem (BVP)

$$\begin{gathered} u^{(4)}(t) = f\left(t, u(t), u''(t)\right), \quad 0 < t < 1, \\ u(0) = u(1) = u''(0) = u''(1) = 0, \end{gathered} \tag{1.1}$$

where $f : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous. This problem models deformations of an elastic beam in the equilibrium state, whose ends are simply supported. Owing to its importance in physics, the solvability of this problem has been studied by many authors; see [1–14].

In [1], Aftabizadeh showed the existence of a solution to BVP(1.1) under the restriction that f is a bounded function. In [2, Theorem 1], Yang extended Aftabizadeh's result and showed the existence for BVP(1.1) under the growth condition of the form

$$\left| f(t, u, v) \right| \le a|u| + b|v| + c, \tag{1.2}$$

where a, b, and c are positive constants such that $(a/\pi^4) + (b/\pi^2) < 1$.

In [6], del Pino and Manásevich further extended the result of Yang and obtained the following existence theorem.

Theorem A. *Assume that there is a pair* $(\alpha, \beta) \in \mathbb{R}^2$ *satisfying*

$$\frac{\alpha}{(k\pi)^4} + \frac{\beta}{(k\pi)^2} \neq 1, \quad \forall k \in \mathbb{N} \tag{1.3}$$

and that there are positive constants a, b, *and* c *such that*

$$a \max_{k\in\mathbb{N}} \frac{1}{\left|k^4\pi^4 - \alpha - k^2\pi^2\beta\right|} + b \max_{k\in\mathbb{N}} \frac{k^2\pi^2}{\left|k^4\pi^4 - \alpha - k^2\pi^2\beta\right|} < 1, \tag{1.4}$$

and f *satisfies the growth condition*

$$\left|f(t,u,v) - (\alpha u - \beta v)\right| \leq a|u| + b|v| + c, \quad \forall t \in [0,1],\ u,v \in \mathbb{R}. \tag{1.5}$$

Then the BVP(1.1) *possesses at least one solution.*

Obviously, the result of Yang follows from Theorem A by just setting $(\alpha, \beta) = (0,0)$. Conditions (1.3)–(1.5) concern a nonresonance condition involving the two-parameter linear eigenvalue problem (LEVP)

$$\begin{gathered} u^{(4)}(t) + \beta u''(t) - \alpha u(t) = 0, \\ u(0) = u(1) = u''(0) = u''(1) = 0. \end{gathered} \tag{1.6}$$

In [6] it was shown that (α, β) is an eigenvalue pair of LEVP(1.6) if and only if $(\alpha/(k\pi)^4) + (\beta/(k\pi)^2) = 1$ for some $k \in \mathbb{N}$. Hence, for $k \in \mathbb{N}$ the straight line

$$L_k = \left\{ (\alpha, \beta) \mid \frac{\alpha}{(k\pi)^4} + \frac{\beta}{(k\pi)^2} = 1 \right\} \tag{1.7}$$

is called an eigenline of LEVP(1.6). Conditions (1.3)-(1.4) trivially imply that

$$\frac{a + bk^2\pi^2}{\left|k^4\pi^4 - \alpha - k^2\pi^2\beta\right|} < 1, \quad \forall k \in \mathbb{N}. \tag{1.8}$$

It is easy to prove that condition (1.8) is equivalent to the fact that the rectangle

$$R(\alpha, \beta; a, b) = [\alpha - a, \alpha + a] \times [\beta - b, \beta + b] \tag{1.9}$$

does not intersect any of the eigenline L_k of LEVP(1.6). In [6], del Pino and Manásevich conjecture that Theorem A is also valid if (1.8) is replaced by (1.4). Particularly, in the case that the partial derivatives f_u and f_v exist, the conjecture means that if for large $|u| + |v|$ the pair

$$\left(f_u(t,u,v), -f_v(t,u,v)\right) \tag{1.10}$$

lies in a certain rectangle $R(\alpha, \beta; a, b)$ which does not intersect any of the eigenline L_k of LEVP(1.6); then BVP(1.1) is solvable. But they could not prove the conjecture.

Recently, the present author [11] has partly answered this conjecture and shows that if the rectangle $R(\alpha, \beta; a, b)$ is replaced by the circle

$$\overline{B}(\alpha, \beta; r) = \left\{(x,y)(x-\alpha)^2 + (y-\beta)^2 \le r^2\right\}, \tag{1.11}$$

the conjecture is correct. In other words, the following result is obtained.

Theorem B. *Assume that f has partial derivatives f_u and f_v in $[0,1] \times \mathbb{R} \times \mathbb{R}$. If there exists a circle $\overline{B}(\alpha, \beta; r)$, which does not intersect any of the eigenline L_k of LEVP(1.6), such that*

$$\left(f_u(t,u,v), -f_v(t,u,v)\right) \in \overline{B}(\alpha, \beta; r) \tag{1.12}$$

for large $|u| + |v|$, then the BVP(1.1) has at least one solution.

See [11, Theorem 2 and Corollary 2]. Condition (1.12) means that f is linear growth on u and v. If f is not linear growth on u or v, Theorem B is invalid.

In this paper, we will extend Theorem B to the case that the circle $\overline{B}(\alpha, \beta; r)$ is replaced by an unbounded domain. Let $\varepsilon \in (0, \pi^6)$ be a positive constant; then we will use the parabolic sector

$$D_\varepsilon = \left\{(x,y) \in \mathbb{R}^2 \mid y \le -\frac{x^2}{4(\pi^6 - \varepsilon)}\right\} \tag{1.13}$$

to substitute the the circle $\overline{B}(\alpha, \beta; r)$ in Theorem B. Noting that D_ε is contained in the parabolic sector

$$D_0 = \left\{(x,y) \in \mathbb{R}^2 \mid y \le -\frac{x^2}{4\pi^6}\right\} \tag{1.14}$$

and D_0 only contacts the first eigenline L_1 at $(2\pi^4, \ -\pi^2)$, we see that D_ε does not intersect any of the eigenline L_k. Our new result is as follows.

Theorem 1.1. *Assume that f has partial derivatives f_u and f_v in $[0,1] \times \mathbb{R} \times \mathbb{R}$. If there is a positive constant $\varepsilon \in (0, \pi^6)$ such that*

$$\left(f_u(t,u,v), -f_v(t,u,v)\right) \in D_\varepsilon, \tag{1.15}$$

then the BVP(1.1) has a unique solution.

In Theorem 1.1, Condition (1.15) allows $f(t,u,v)$ to be superlinear in u and v, and an example will be showed at the end of the paper. The proof of Theorem 1.1 is based on Leray-Schauder fixed point theorem and a differential inequality, which will be given in the next section.

2. Proof of the Main Results

Let $I = [0,1]$ and $H = L^2(I)$ be the usual Hilbert space with the interior product $(u,v) = \int_0^1 u(t)v(t)dt$ and the norm $\|u\|_2 = (\int_0^1 |u(t)|^2 dt)^{1/2}$. For $m \in \mathbb{N}$, let $W^{m,2}(I)$ be the usual Sobolev space with the norm $\|u\|_{m,2} = \sqrt{\sum_{i=0}^{m} \|u^{(i)}\|_2^2}$. $u \in W^{m,2}(I)$ which means that $u \in C^{m-1}(I)$, $u^{(m-1)}(t)$ is absolutely continuous on I and $u^{(m)} \in L^2(I)$.

Given $h \in L^2(I)$, we consider the linear fourth-order boundary value problem (LBVP)

$$\begin{gathered} u^{(4)}(t) = h(t), \quad t \in I, \\ u(0) = u(1) = u''(0) = u''(1) = 0. \end{gathered} \tag{2.1}$$

Let $G(t,s)$ be Green's function to the second-order linear boundary value problem

$$-u'' = 0, \qquad u(0) = u(1) = 0, \tag{2.2}$$

which is explicitly expressed by

$$G(t,s) = \begin{cases} t(1-s), & 0 \le t \le s \le 1, \\ s(1-t), & 0 \le s \le t \le 1. \end{cases} \tag{2.3}$$

For every given $h \in L^2(I)$, it is easy to verify that the LBVP(2.1) has a unique solution $u \in W^{4,2}(I)$ in Carathéodory sense, which is given by

$$u(t) = \iint_0^1 G(t,\tau)G(\tau,s)h(s)ds\, d\tau := Sh(t). \tag{2.4}$$

If $h \in C(I)$, the solution is in $C^4(I)$, and it is a classical solution. Moreover, the solution operator of LBVP(2.1), $S : L^2(I) \to W^{4,2}(I)$ is a linearly bounded operator. By the compactness of the Sobolev embedding $W^{4,2}(I) \hookrightarrow C^2(I)$, we see that $S : L^2(I) \to C^2(I)$ is a completely continuous operator. Hence the restriction $S : C(I) \to C^2(I)$ is completely continuous.

Lemma 2.1. *For every $h \in H$, the unique solution of LBVP(2.1) $u = Sh \in W^{4,2}(I)$ satisfies the inequalities*

$$\pi^6 \|u\|_2^2 \le \pi^4 \|u'\|_2^2 \le \pi^2 \|u''\|_2^2 \le \|u'''\|_2^2. \tag{2.5}$$

Proof. Since sine system $\{\sin k\pi t \mid k \in \mathbb{N}\}$ is a complete orthogonal system of $L^2(I)$, every $h \in L^2(I)$ can be expressed by the Fourier series expansion

$$h(t) = \sum_{k=1}^{\infty} h_k \sin k\pi t, \tag{2.6}$$

where $h_k = 2\int_0^1 h(s)\sin k\pi s\, ds$, $k = 1, 2, \ldots$, and the Parseval equality

$$\|h\|_2^2 = \frac{1}{2}\sum_{k=1}^{\infty} |h_k|^2 \tag{2.7}$$

holds. Let $u = Sh$, then $u \in W^{4,2}(I)$ is the unique solution of LBVP(2.1), and u, u'', and $u^{(4)}$ can be expressed by the Fourier series expansion of the sine system. Since $u^{(4)} = h$, by the integral formula of Fourier coefficient, we obtain that

$$\begin{aligned} u(t) &= \sum_{k=1}^{\infty} \frac{h_k}{k^4\pi^4} \sin k\pi t, \\ u''(t) &= -\sum_{k=1}^{\infty} \frac{h_k}{k^2\pi^2} \sin k\pi t. \end{aligned} \tag{2.8}$$

On the other hand, since cosine system $\{\cos k\pi t \mid k = 0, 1, 2, \ldots\}$ is another complete orthogonal system of $L^2(I)$, every $v \in L^2(I)$ can be expressed by the cosine series expansion

$$v(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos k\pi t, \tag{2.9}$$

where $a_k = 2\int_0^1 h(s)\cos k\pi s\, ds$, $k = 0, 1, 2, \ldots$. For the above $u = Sh$, by the integral formula of the coefficient of cosine series, we obtain the cosine series expansions of u' and u''':

$$\begin{aligned} u'(t) &= \sum_{k=1}^{\infty} \frac{h_k}{k^3\pi^3} \cos k\pi t, \\ u'''(t) &= -\sum_{k=1}^{\infty} \frac{h_k}{k\pi} \cos k\pi t. \end{aligned} \tag{2.10}$$

By (2.8)–(2.10) and Parseval equality, we have that

$$\|u\|_2^2 = \frac{1}{2}\sum_{k=1}^{\infty}\left|\frac{h_k}{k^4\pi^4}\right|^2 \le \frac{1}{2\pi^2}\sum_{k=1}^{\infty}\left|\frac{h_k}{k^3\pi^3}\right|^2 = \frac{1}{\pi^2}\|u'\|_2^2,$$
$$\|u'\|_2^2 = \frac{1}{2}\sum_{k=1}^{\infty}\left|\frac{h_k}{k^3\pi^3}\right|^2 \le \frac{1}{2\pi^2}\sum_{k=1}^{\infty}\left|\frac{h_k}{k^2\pi^2}\right|^2 = \frac{1}{\pi^2}\|u''\|_2^2, \tag{2.11}$$
$$\|u''\|_2^2 = \frac{1}{2}\sum_{k=1}^{\infty}\left|\frac{h_k}{k^2\pi^2}\right|^2 \le \frac{1}{2\pi^2}\sum_{k=1}^{\infty}\left|\frac{h_k}{k\pi}\right|^2 = \frac{1}{\pi^2}\|u'''\|_2^2.$$

This implies that (2.5) holds. □

Proof of Theorem 1.1. We define a mapping $F : C^2(I) \to C(I)$ by

$$F(u)(t) := f(t, u(t), u''(t)), \quad u \in C^2(I). \tag{2.12}$$

By the continuity of f, $F : C^2(I) \to C(I)$ is continuous and it maps every bounded set of $C^2(I)$ into a bounded set of $C(I)$. Hence, the composite mapping $S \circ F : C^2(I) \to C^2(I)$ is completely continuous. By the definition of the solution operator S of LBVP(2.1), the solution of BVP(1.1) is equivalent to the fixed point of $S\circ F$. We first use the Leray-Schauder fixed point theorem [15] to show that $S \circ F$ has a fixed point. For this, we consider the homotopic family of the operator equations

$$u = \lambda(S \circ F)(u), \quad 0 < \lambda < 1. \tag{2.13}$$

We need to prove that the set of the solutions of (2.13) is bounded in $C^2(I)$.

Let $u \in C^2(I)$ be a solution of (2.13) for $\lambda \in (0,1)$. Set $h = \lambda F(u)$. Since $h \in C(I)$, by the definition of S, $u = Sh \in C^4(I)$ is the unique solution of LBVP(2.1). Hence u satisfies the differential equation

$$\begin{gathered} u^{(4)}(t) = \lambda f(t, u(t), u''(t)), \quad 0 \le t \le 1, \\ u(0) = u(1) = u''(0) = u''(1) = 0. \end{gathered} \tag{2.14}$$

Set $M = \max_{t\in I}|f(t,0,0)|$. Multiplying the first formula of (2.14) by $-u''(t)$ and by the theorem of differential mean value, we have

$$\begin{aligned} -u^{(4)}u'' &= -\lambda f(t,u,u'')u'' \\ &= -\lambda\left[f(t,u,u'') - f(t,0,0)\right]u'' - \lambda f(t,0,0)u'' \\ &= -\lambda f_u(t,\xi,\eta)\, uu'' - \lambda f_v(t,\xi,\eta)(u'')^2 - \lambda f(t,0,0)u'' \\ &\le \lambda\left(\frac{f_u^2(t,\xi,\eta)}{4(\pi^6-\varepsilon)} - f_v(t,\xi,\eta)\right)(u'')^2 + \lambda\left(\pi^6-\varepsilon\right)u^2 + M|u''|, \end{aligned} \tag{2.15}$$

where $\xi = \theta u$, $\eta = \theta u''$ for some $\theta \in (0,1)$. In the last step of this estimation we use the inequality

$$-f_u(t,\xi,\eta)uu'' \le \frac{f_u^2(t,\xi,\eta)}{4(\pi^6-\varepsilon)}(u'')^2 + \left(\pi^6-\varepsilon\right)u^2 \tag{2.16}$$

which is derived from the inequality $2pq \le p^2+q^2$ by choosing

$$p = -\frac{f_u(t,\xi,\eta)}{2\sqrt{\pi^6-\varepsilon}}u'', \qquad q = \sqrt{\pi^6-\varepsilon}u. \tag{2.17}$$

Since $(f_u(t,\xi,\eta), -f_v(t,\xi,\eta)) \in D_\varepsilon$, it follows that

$$\frac{f_u^2(t,\xi,\eta)}{4(\pi^6-\varepsilon)} - f_v(t,\xi,\eta) \le 0. \tag{2.18}$$

Hence, we obtain that

$$-u^{(4)}u'' \le \lambda\left(\pi^6-\varepsilon\right)u^2 + M|u''| \le \left(\pi^6-\varepsilon\right)u^2 + \frac{\varepsilon}{2\pi^4}(u'')^2 + \frac{\pi^4M^2}{2\varepsilon}, \tag{2.19}$$

in which we use the inequality $pq \le (p^2/2)+(q^2/2)$ for $M|u''|$ by choosing $p = (\sqrt{\varepsilon}/\pi^2)|u''|$ and $q = \pi^2M/\sqrt{\varepsilon}$. Integrating inequality (2.19) on I using integration by parts and Lemma 2.1, we have

$$\begin{aligned}\|u'''\|_2^2 &\le \left(\pi^6-\varepsilon\right)\|u\|_2^2 + \frac{\varepsilon}{2\pi^4}\|u''\|_2^2 + \frac{\pi^4M^2}{2\varepsilon}\\ &\le \frac{\pi^6-\varepsilon}{\pi^6}\|u'''\|_2^2 + \frac{\varepsilon}{2\pi^6}\|u'''\|_2^2 + \frac{\pi^4M^2}{2\varepsilon}\\ &= \left(1-\frac{\varepsilon}{2\pi^6}\right)\|u'''\|_2^2 + \frac{\pi^4M^2}{2\varepsilon},\end{aligned} \tag{2.20}$$

from which it follows that

$$\|u'''\|_2^2 \le \frac{\pi^{10}M^2}{\varepsilon^2}. \tag{2.21}$$

From this and Lemma 2.1, we obtain that

$$\|u\|_{3,2} \le \left(\sum_{i=0}^{3}\left\|u^{(i)}\right\|_2^2\right)^{1/2} \le \left(\frac{1}{\pi^6}+\frac{1}{\pi^4}+\frac{1}{\pi^2}+1\right)^{1/2}\|u'''\|_2 \le \frac{2\pi^5M}{\varepsilon}. \tag{2.22}$$

Hence, by the continuity of the Sobolev embedding $W^{3,2}(I) \hookrightarrow C^2(I)$, we have

$$\|u\|_{C^2(I)} \leq C\|u\|_{3,2} \leq C\frac{2\pi^5 M}{\varepsilon} =: \overline{C}, \tag{2.23}$$

where C is the Sobolev embedding constant. This means that the set of the solutions of (2.13) is bounded in $C^2(I)$. By the Leray-Schauder fixed point theorem [15], $S \circ F$ has a fixed point in $C^2(I)$ which is a solution of BVP(1.1).

Now, let $u_1, u_2 \in C^4(I)$ be two solutions of BVP(1.1). Set $u = u_2 - u_1$ and $h = F(u_2) - F(u_1)$. Then $u = S(F(u_2) - F(u_2)) = Sh$ is the solution of LBVP(2.1), and it satisfies the equation

$$u^{(4)}(t) = f(t, u_2, u_2'') - f(t, u_1, u_2''), \quad t \in I. \tag{2.24}$$

Multiplying this equality by $-(u_2 - u_1)''$ and by the theorem of differential mean value and Condition (1.15), we have that

$$\begin{aligned} -u^{(4)}u'' &= -\left(f(t, u_2, u_2'') - f(t, u_1, u_2'')\right)u'' \\ &= -f_u(t, \xi, \eta)uu'' - f_v(t, \xi, \eta)(u'')^2 \\ &\leq \left(\frac{f_u^2(t, \xi, \eta)}{4(\pi^6 - \varepsilon)} - f_v(t, \xi, \eta)\right)(u'')^2 + \left(\pi^6 - \varepsilon\right)u^2 \\ &\leq \left(\pi^6 - \varepsilon\right)u^2, \end{aligned} \tag{2.25}$$

where $\xi = u_1 + \theta(u_2 - u_1)$, $\eta = u_1'' + \theta(u_2'' - u_1'')$ for some $\theta \in (0,1)$. Integrating this inequality on I and using Lemma 2.1, we obtain that

$$\pi^6\|u\|_2^2 \leq \|u'''\|_2^2 \leq \left(\pi^6 - \varepsilon\right)\|u\|_2^2. \tag{2.26}$$

This implies that $\|u\|_2 = 0$, and hence we have $u_1 = u_2$. Thus BVP(1.1) has only one solution.

The proof of Theorem 1.1 is completed. □

Example 2.2. Consider the fourth-order boundary value problem

$$\begin{gathered} u^{(4)}(t) = a(t)u(t) + u''(t) + \left(u''(t)\right)^3 + \sin \pi t, \quad t \in I, \\ u(0) = u(1) = u''(0) = u''(1) = 0, \end{gathered} \tag{2.27}$$

where $a \in C(I)$. Noting that

$$f(t, u, v) = a(t)u + v + v^3 + \sin \pi t \tag{2.28}$$

is upperlinear growth on v, one can check that all the known results of [1–14] are not applicable to this equation. But, if $\max_{t\in I}|a(t)| < 2\pi^3$, then

$$\frac{f_u^2}{4(\pi^6-\varepsilon)} - f_v = \frac{a^2(t)}{4(\pi^6-\varepsilon)} - 1 - 3v^2 \leq \frac{a^2(t)}{4(\pi^6-\varepsilon)} - 1 = \frac{a^2(t) - 4(\pi^6-\varepsilon)}{4(\pi^6-\varepsilon)} \leq 0 \tag{2.29}$$

for small enough $\varepsilon \in (0, 4\pi^6)$. Hence, Condition (1.15) holds, and by Theorem 1.1, the boundary value problem (2.27) has a unique solution.

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