Research Article

Fractional Differential Equations in Terms of Comparison Results and Lyapunov Stability with Initial Time Difference

Coşkun Yakar

Department of Mathematics, Gebze Institute of Technology, Gebze-Kocaeli 141-41400, Turkey

Correspondence should be addressed to Coşkun Yakar, cyakar@gyte.edu.tr

The qualitative behavior of a perturbed fractional-order differential equation with Caputo's derivative that differs in initial position and initial time with respect to the unperturbed fractional-order differential equation with Caputo's derivative has been investigated. We compare the classical notion of stability to the notion of initial time difference stability for fractional-order differential equations in Caputo's sense. We present a comparison result which again gives the null solution a central role in the comparison fractional-order differential equation when establishing initial time difference stability of the perturbed fractional-order differential equation with respect to the unperturbed fractional-order differential equation.

1. Introduction

We have investigated that the stability of perturbed solution with respect to unperturbed solution with initial time difference of the nonlinear differential equations of fractional-order. The differential operators are taken in the Riemann-Liouville and Caputo's sense and the initial conditions are specified according to Caputo's suggestion [1], thus allowing for interpretation in a physically meaningful way [2].

Lyapunov's second method is a standard technique used in the study of the qualitative behavior of differential systems [3–6] along with a comparison result [2, 7] that allows the prediction of behavior of a differential system when the behavior of the null solution of a comparison system is known. However, there has been difficulty with this approach when trying to apply it to unperturbed fractional differential systems [2, 6, 8] and associated perturbed fractional differential systems with an initial time difference. The difficulty arises because there is a significant difference between initial time difference (ITD) stability [7, 9–16] and the classical notion of stability for fractional differential systems [2, 6]. The classical

notions of stability [2–6, 8, 17] are with respect to the null solution, but ITD stability [7, 9–16] is with respect to the unperturbed fractional-order differential system where the perturbed fractional-order differential system and the unperturbed fractional-order differential system differ both in initial position and initial time [7, 9–16].

In this paper, we have dissipated this complexity and have a new comparison result which again gives the null solution a central role in the comparison fractional-order differential system. This result creates many paths for continuing research by direct application and generalization [13, 15, 16].

In Section 2, we present basic definitions and necessary rudimentary material. In Section 3, we discuss and compare the differences between the classical notion of stability and the recent notion of initial time difference (ITD) stability by means of fractional-order differential systems. In Section 4, we have a comparison result in which the stability properties of the null solution of the comparison system imply the corresponding stability properties of the unperturbed fractional-order differential system. In Section 5, we have an other comparison result in which the stability properties of the null solution of the comparison system imply the corresponding (ITD) stability properties of the perturbed fractional-order differential system with respect to the unperturbed fractional-order differential system.

2. Preliminaries

In this section we give relation among the fractional-order derivatives: Caputo, Reimann-Liouville and Grünwald-Letnikov fractional-order derivatives and necessary definition of initial value problems of fractional-order differential equations with these sense.

2.1. Fractional-Order Derivatives: Caputo, Reimann-Liouville and Grünwald-Letnikov

Caputo's and Reimann-Liouville's definitions of fractional derivatives, are namely,

$$^{c}D^{q}x = \frac{1}{\Gamma(1-q)} \int_{\tau_0}^{t} (t-s)^{-q} x'(s) ds, \quad \tau_0 \le t \le T, \tag{2.1}$$

$$D^{q}x = \frac{1}{\Gamma(p)} \left(\frac{d}{dt} \int_{\tau_0}^{t} (t-s)^{p-1} x(s) ds \right), \quad \tau_0 \le t \le T, \tag{2.2}$$

respectively, order of $0 < q < 1$ and $p + q = 1$, where Γ denotes the Gamma function.

The most important advantage for fractional-order differential equations with Caputo's derivative is the initial conditions that are the same form as that of ordinary differential equations with integer derivatives. Another difference is that the Caputo derivative for a constant C is zero, while the Riemann-Liouville fractional derivative for a constant C is not zero but equals to $D^{q}C = C(t-\tau_0)^{-q}/\Gamma(1-q)$. By using (2.1), therefore,

$$^{c}D^{q}x(t) = D^{q}[x(t) - x(\tau_0)],$$

$$^{c}D^{q}x(t) = D^{q}x(t) - \frac{x(\tau_0)}{\Gamma(1-q)} (t-\tau_0)^{-q}. \tag{2.3}$$

In particular, if $x(\tau_0) = 0$, we obtain

$$^{c}D^{q}x(t) = D^{q}x(t). \tag{2.4}$$

Hence, we can see that Caputo's derivative is defined for functions for which Riemann-Liouville fractional-order derivative exists.

Let us write that Grünwald-Letnikov's notion of fractional-order derivative in a convenient form

$$D_0^q x(t) = \lim_{\substack{h \to 0 \\ nh=t-\tau_0}} \frac{1}{h^q} [x(t) - S(x,h,r,q)], \tag{2.5}$$

where $S(x,h,r,q) = \sum_{r=1}^{n} (-1)^{r+1} \binom{q}{r} x(t-rh)$. If we know that $x(t)$ is continuous and $dx(t)/dt$ exist and integrable, then Riemann-Liouville and Grünwald-Letnikov fractional-order derivatives are connected by the relation

$$D^q x(t) = D_0^q x(t) = \frac{x(\tau_0)(t-\tau_0)^{-q}}{\Gamma(1-q)} + \int_{\tau_0}^{t} \frac{(t-s)^{-q}}{\Gamma(1-q)} \frac{d}{ds} x(s)\, ds. \tag{2.6}$$

Using (2.3) implies that we have the following relations among the Caputo, Riemann-Liouville and Grünwald-Letnikov fractional derivatives

$$^{c}D^q x(t) = D^q[x(t) - x(\tau_0)] = D_0^q[x(t) - x(\tau_0)] = \frac{1}{\Gamma(1-q)} \int_{\tau_0}^{t} (t-s)^{-q} \frac{dx(s)}{ds}\, ds. \tag{2.7}$$

The foregoing equivalent expressions are very useful in the study of qualitative properties of solutions of fractional differential equations.

2.2. Existence of Euler Solution

We consider the initial value problem of the fractional-order differential equation with Reimann-Liouville's derivative

$$D^q x = f(t,x), \quad x(t)(t-t_0)^{1-q}\big|_{t=t_0} = x^0 \quad \text{for } t \geq t_0, \; t_0 \in \mathbb{R}_+, \tag{2.8}$$

where f is any function from $[t_0, T] \times \mathbb{R}^n \to \mathbb{R}^n$. Let

$$\pi = [t_0, t_1, \ldots, t_N] \tag{2.9}$$

be a partition of $[t_0, T]$.

Consider the interval $[t_0, t_1]$ and observe that the right hand side of the initial value problem of fractional-order differential equation with Reimann-Liouville's derivative

$$D^q x = f\left(t_0, x^0\right), \quad x(t)(t - t_0)^{1-q}\big|_{t=t_0} = x^0 \quad \text{for } t \geq t_0 \tag{2.10}$$

on $[t_0, t_1]$ is constant.

Therefore, the initial value problem has a unique solution of (2.10) of the fractional-order differential equation with Reimann-Liouville's derivative given by

$$x(t) = \frac{x^0 (t - t_0)^{q-1}}{\Gamma(q)} + f\left(t_0, x^0\right) \frac{(t - t_0)^q}{\Gamma(1 + q)}, \quad t \in [t_0, t_1]. \tag{2.11}$$

Define the node $x_1 = x(t_1)$ and iterate next by considering on $[t_1, t_2]$ the initial value problem

$$D^q x = f(t_1, x_1), \quad x_1(t)(t - t_1)^{1-q}\big|_{t=t_1} = x_1^0 \quad \text{for } t \geq t_1. \tag{2.12}$$

The next node is $x_2 = x(t_2)$ and we proceed this way till an arc $x_\pi = x_\pi(t)$ has been defined on all $[t_0, T]$. Let us employ the notation x_π to emphasize the role played by the particular partition π in determining x_π which is the Euler curved arc corresponding to the partition π. The diameter of the partition π is given by

$$\mu_\pi = \max[t_i - t_{i-1} : 1 \leq i \leq N]. \tag{2.13}$$

Definition 2.1. An Euler solution is any curved arc $x = x(t)$ which is the uniform limit of Euler curved arcs x_π, corresponding to some sequence π_j such that $\pi_j \to 0$, which means the convergence of the diameter $\mu_{\pi_j} \to 0$ as $j \to \infty$.

Now, we can give the following result on existence of an Euler solution of the initial value problem of fractional-order differential equation with Reimann-Liouville's derivative for (2.8).

Theorem 2.2. *Assume that*

(i) $\|f(t, x)\| \leq g(t, \|x\|), (t, x) \in [t_0, T] \times \mathbb{R}^n$, *where* $g \in C[[t_0, T] \times \mathbb{R}_+, \mathbb{R}_+], g(t, u)$ *is nondecreasing in* (t, u);

(ii) *The maximal solution* $r(t) = r(t, t_0, u^0)$ *of the fractional-order scalar differential equation with Reimann-Liouville's derivative*

$$D^q u = g(t, u), u(t)(t - t_0)^{1-q}\big|_{t=t_0} = u^0 \geq 0 \quad \text{for } t \geq t_0, \ t_0 \in \mathbb{R}_+ \tag{2.14}$$

exists on $[t_0, T]$.

Then

(a) *there exists at least one Euler solution* $x(t) = x(t, t_0, x^0)$ *to the initial value problem* (2.8), *which satisfies a Hölder condition;*

(b) *any Euler solution $x(t)$ of* (2.8) *satisfies the relation*

$$\left\| x(t) - x^0(t) \right\| \le r\left(t, t_0, u^0\right) - u^0, \quad t \in [t_0, T], \tag{2.15}$$

where $u^0 = \|x^0\|$ and $x^0(t) = x^0(t - t_0)^{q-1}/\Gamma(q)$.

For proof of Theorem 2.2, please see in [6].

If $f(t, x)$ in (2.8) is assumed to be continuous, then $x(t) = x(t, t_0, x^0)$, an Euler solution, is actually a solution of the initial value problem (2.8).

Theorem 2.3. *Under the assumptions of Theorem 2.2 and if we suppose that $f \in C[[t_0, t_0 + T] \times \mathbb{R}^n, \mathbb{R}^n]$, then $x(t)$ is a solution of initial value problem* (2.8).

For proof Theorem 2.3, please see in [6].

2.3. Fractional-Order Differential Equations with Caputo's Derivative

Consider the initial value problems of the fractional-order differential equations with Caputo's derivative

$$^cD^qx = f(t, x), \quad x(t_0) = x_0 \quad \text{for } t \ge t_0, \ t_0 \in \mathbb{R}_+, \tag{2.16}$$

$$^cD^qx = f(t, x), \quad x(\tau_0) = y_0 \quad \text{for } t \ge \tau_0, \ \tau_0 \in \mathbb{R}_+, \tag{2.17}$$

where $x_0 = \lim_{t \to t_0} D^{q-1}x(t)$ and $y_0 = \lim_{t \to \tau_0} D^{q-1}x(t)$ exist, and the perturbed system of initial value problem of the fractional-order differential equation with Caputo's derivative of (2.17)

$$^cD^qy = F(t, y), \quad y(\tau_0) = y_0 \quad \text{for} \ \ t \ge \tau_0 \ge t_0, \tag{2.18}$$

where $y_0 = \lim_{t \to \tau_0} D^{q-1}y(t)$, exists, and $f, F \in C[[t_0, \tau_0 + T] \times \mathbb{R}^n, \mathbb{R}^n]$; satisfy a local Lipschitz condition on the set $\mathbb{R}_+ \times S\rho$, $S\rho = [x \in \mathbb{R}^n : \|x\| < \rho < \infty]$ and $f(t, 0) = 0$ for $t \ge 0$. In particular, $F(t, y) = f(t, y) + R(t, y)$, we have a special case of (2.18) and $R(t, y)$ is said to be perturbation term.

Corollary 2.4. *Let $0 < q < 1$, and $f : (t_0, t_0+T] \times S\rho \to \mathbb{R}$ a function such that $f(t, x) \in L(t_0, t_0+T)$ for any $x \in S\rho$. If $x(t) \in L(t_0, t_0+T)$, then $x(t)$ satisfies a.e. the initial value problems of the fractional-order differential equations with Reimann-Liouville's derivative* (2.19) *if, and only if, $x(t)$ satisfies a.e. the Volterra fractional-order integral equation* (2.20).

For proof of Corollary 2.4, please see in [2].

We assume that we have sufficient conditions to the existence and uniqueness of solutions through (t_0, x_0) and (τ_0, y_0). If $f \in C[[t_0, t_0 + T] \times \mathbb{R}^n, \mathbb{R}^n]$ and $x(t)$ is the solution of

$$D^qx = f(t, x), \quad x(t)(t - t_0)^{1-q}|_{t=t_0} = x^0 \quad \text{for } t \ge t_0, \ t_0 \in \mathbb{R}_+, \tag{2.19}$$

where $D^q x$ is the Reimann-Liouville fractional-order derivative of x as in (2.2), then it also satisfies the Volterra fractional-order integral equation

$$x(t) = \frac{x^0(t - t_0)^{q-1}}{\Gamma(q)} + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t - s)^{q-1} f(s, x(s)) ds, \quad t_0 \le t \le t_0 + T \qquad (2.20)$$

that is, every solution of (2.20) is also a solution of (2.19); for detail please see [2].

We will only give the basic existence and uniqueness result with the Lipschitz condition by using contraction mapping theorem and a weighted norm with Mittag-Leffler function in [6].

Theorem 2.5. *Assume that*

(i) $f \in C[R, \mathbb{R}^n]$ *and bounded by M on R, where $R = [(t, x) : t_0 \le t \le t_0 + T, \|x - x_0\| \le b]$;*

(ii) $\|f(t, x) - f(t, y)\| \le L\|x - y\|, L > 0, (t, x) \in R$, *where the inequalities are componentwise.*

Then there exists a unique solution $x(t) = x(t, t_0, x_0)$ on $[t_0, t_0 + \alpha]$ for the initial value problem of the fractional-order differential equation with Caputo's derivative of (2.16), where $\alpha = \min[T, (b\Gamma(q + 1)/M)^{1/q}]$.

For proof of Theorem 2.5, please see in [6].

2.4. Stability Criteria with ITD and Lyapunov-Like Function

Before we can establish our comparison theorem and Lyapunov stability criteria for initial time difference, we need to introduce the following definitions of ITD stability and Lyapunov-like functions.

Definition 2.6. The solution $y(t, \tau_0, y_0)$ of the initial value problems of fractional-order differential equation with Caputo's derivative of (2.18) through (τ_0, y_0) is said to be initial time difference stable with respect to the solution $\tilde{x}(t, \tau_0, x_0) = x(t - \eta, t_0, x_0)$, where $x(t, t_0, x_0)$ is any solution of the initial value problems of fractional-order differential equation with Caputo's derivative of (2.16) for $t \ge \tau_0, \tau_0 \in \mathbb{R}_+$ and $\eta = \tau_0 - t_0$ if and only if given any $\epsilon > 0$ there exist $\delta_1 = \delta_1(\epsilon, \tau_0) > 0$ and $\delta_2 = \delta_2(\epsilon, \tau_0) > 0$, such that

$$\|y(t, \tau_0, y_0) - x(t - \eta, t_0, x_0)\| < \epsilon, \quad \text{whenever } \|y_0 - x_0\| < \delta_1, \ |\tau_0 - t_0| < \delta_2 \text{ for } t \ge \tau_0. \tag{2.21}$$

If δ_1, δ_2 are independent of τ_0, then the solution $y(t, \tau_0, y_0)$ of the initial value problems of fractional-order differential equation with Caputo's derivative of (2.18) is initial time difference uniformly stable with respect to the fractional solution $x(t - \eta, t_0, x_0)$. If the solution of initial value problems of fractional-order differential equation with Caputo's derivative of $y(t, \tau_0, y_0)$ of the fractional system (2.18) through (τ_0, y_0) is initial time difference stable and there exist $\gamma_1(\tau_0) > 0$ and $\gamma_2(\tau_0) > 0$ such that

$$\lim_{t \to \infty} \|y(t, \tau_0, y_0) - x(t - \eta, t_0, x_0)\| = 0 \tag{2.22}$$

for all $y(t, \tau_0, y_0)$ and $x(t - \eta, t_0, x_0)$ with $\|y_0 - x_0\| < \gamma_1$ and $|\tau_0 - t_0| < \gamma_2$ for $t \geq \tau_0$, then it is said to be initial time difference asymptotically stable with respect to the fractional solution $x(t - \eta, t_0, x_0)$. It is initial time difference uniformly asymptotically stable with respect to the fractional solution $x(t - \eta, t_0, x_0)$ if γ_1 and γ_2 are independent of τ_0.

Definition 2.7. A function $\phi(r)$ is said to belong to the class $\mathcal{K}$ if $\phi \in C[(0, \rho), \mathbb{R}_+]$, $\phi(0) = 0$, and $\phi(r)$ is strictly monotone increasing in r.

Definition 2.8. For any Lyapunov-like function $V(t, x) \in C[\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}_+]$, we define the fractional-order Dini derivatives in Caputo's sense ${}^c D_+^q V(t, x)$ and ${}^c D_-^q V(t, x)$ as follows

$$ {}^c D_+^q V(t, x) = \lim_{h \to 0^+} \sup \frac{1}{h^q} [V(t, x) - V(t - h, x - h^q f(t, x))], \tag{2.23} $$

$$ {}^c D_-^q V(t, x) = \lim_{h \to 0^-} \inf \frac{1}{h^q} [V(t, x) - V(t - h, x - h^q f(t, x))] \tag{2.24} $$

for $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^n$.

Definition 2.9. For a real-valued function $V(t, x) \in C[\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}_+]$, we define the generalized fractional-order derivatives (Dini-like derivatives) in Caputo's sense ${}^c_* D_+^q V(t, y - \tilde{x})$ and ${}^c_* D_-^q V(t, y - \tilde{x})$ as follows

$$ {}^c_* D_+^q V(t, y - \tilde{x}) = \lim_{h \to 0^+} \sup \frac{1}{h^q} \left[V(t, y - \tilde{x}) - V\left(t - h, y - \tilde{x} - h^q \left(F(t, y) - \tilde{f}(t, \tilde{x})\right)\right) \right], \tag{2.25} $$

$$ {}^c_* D_-^q V(t, y - \tilde{x}) = \lim_{h \to 0^-} \inf \frac{1}{h^q} \left[V(t, y - \tilde{x}) - V\left(t - h, y - \tilde{x} - h^q \left(F(t, y) - \tilde{f}(t, \tilde{x})\right)\right) \right] \tag{2.26} $$

for $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^n$.

3. Comparing Fractional Stability with Fractional (ITD) Stability

3.1. Fractional Classical Notion of Stability

Let $x(t, t_0, x_0)$ and $x(t, t_0, y_0)$ be any solutions of the initial value problems of fractional-order differential equations with Caputo's derivative of (2.16) and of (3.1), respectively,

$$ {}^c D^q x = f(t, x), \quad x(t_0) = y_0 \quad \text{for } t \geq t_0, \ t_0 \in \mathbb{R}_+, \tag{3.1} $$

where $f \in C[[t_0, T] \times \mathbb{R}^n, \mathbb{R}^n]$.

Assume that $f(t, 0) = 0, t \in \mathbb{R}_+$ so that $x = 0$ is a null solution of fractional-order differential equation with Caputo's derivative of (3.1) through $(t_0, 0)$. Now, we can state the well-known definitions concerning the stability of the null solution.

Definition 3.1. The null solution $x = 0$ of fractional-order differential equation with Caputo's derivative of (3.1) is said to be stable if and only if for each $\epsilon > 0$ and for all $t_0 \in \mathbb{R}_+$, there exists a positive function $\delta = \delta(\epsilon, t_0)$ that is continuous in t_0 for each ϵ such that

$$\|x_0\| \leq \delta \text{ implies } \|x(t, t_0, x_0)\| < \epsilon \quad \text{for } t \geq t_0. \tag{3.2}$$

If δ is independent of t_0, then the null solution $x = 0$ of initial value problems of fractional-order differential equation with Caputo's derivative of (3.1) is said to be uniformly stable.

Definition 3.2. The solution $x(t, t_0, y_0)$ of initial value problems of fractional-order differential equation with Caputo's derivative of (3.1) through (t_0, y_0) is said to be stable with respect to the solution $x(t, t_0, x_0)$ of fractional-order differential equation with Caputo's derivative of (3.1) for $t \geq t_0 \in \mathbb{R}_+$ if and only if given any $\epsilon > 0$ there exists a positive function $\delta = \delta(\epsilon, t_0)$ that is continuous in t_0 for each ϵ such that

$$\|y_0 - x_0\| \leq \delta \text{ implies } \|x(t, t_0, y_0) - x(t, t_0, x_0)\| < \epsilon \quad \text{for } t \geq t_0. \tag{3.3}$$

If δ is independent of t_0, then the solution of the fractional-order differential equation with Caputo's derivative of (3.1) is uniformly stable with respect to the solution $x(t, t_0, x_0)$ of (2.16).

We remark that for the purpose of studying the classical stability of a given solution $x(t, t_0, y_0)$ of the initial value problem of fractional-order differential equation with Caputo's derivative of (3.1), it is convenient to make a change of variable. Let $x(t, t_0, x_0)$ and $x(t, t_0, y_0)$ be the unique solutions of the fractional-order differential equations with Caputo's derivative (2.16) and (3.1), respectively, and set $z(t, t_0, y_0 - x_0) = x(t, t_0, y_0) - x(t, t_0, x_0)$ for $t \geq t_0$. Then

$$^cD^q z(t, t_0, y_0 - x_0) = {}^cD^q x(t, t_0, y_0) - {}^cD^q x(t, t_0, x_0), \tag{3.4}$$

$$^cD^q z(t, t_0, y_0 - x_0) = f(t, z(t, t_0, y_0 - x_0) + x(t, t_0, x_0)) - f(t, x(t, t_0, x_0)),$$

$$^cD^q z(t, t_0, y_0 - x_0) = \widetilde{f}(t, z(t, t_0, y_0 - x_0)). \tag{3.5}$$

It is easy to observe that $z(t, t_0, y_0 - x_0) \equiv 0$ is a solution of the transformed initial value problems of the fractional differential equation with Caputo's derivative if $y_0 - x_0 = 0$ which implies $f(t, 0) = 0$. Since $^cD^q z(t) = 0$ and $z(t, t_0, 0) = 0$ is the null solution, the solution of $x(t, t_0, x_0)$ initial value problems of the fractional-order differential equation with Caputo's derivative of (2.16) corresponds to the identically null solution of $^cD^q z = \widetilde{f}(t, z)$, where $\widetilde{f}(t, z(t, t_0, y_0 - x_0)) = f(t, z(t, t_0, y_0 - x_0) + x(t, t_0, x_0)) - f(t, x(t, t_0, x_0))$. Hence, we can always assume, without any loss of generality, that $x(t, t_0, x_0) \equiv 0$ is the null solution of the given fractional-order differential equation with Caputo's derivative of (3.1) and we can limit our study of stability to that of the null solution [2, 6, 8]. However, it is impossible to do the same transformation for fractional (ITD) stability which we deal with it.

3.2. New Notion of Fractional (ITD) Stability

Let $x(t, \tau_0, y_0)$ be a fractional solution of (2.17) and $\tilde{x}(t, \tau_0, x_0) = x(t-\eta, t_0, x_0)$, where $x(t, t_0, x_0)$ is any solution of initial value problems of the fractional-order differential equations with Caputo's derivative of the system (2.16) for $t \geq \tau_0 \geq 0$. Let us make a transformation similar to that in (3.5). Set $z(t, \tau_0, y_0 - x_0) = x(t, \tau_0, y_0) - x(t - \eta, t_0, x_0)$ for $t \geq \tau_0$. Then

$$
{}^cD^q z(t, \tau_0, y_0 - x_0) = {}^cD^q x(t, \tau_0, y_0) - {}^cD^q x(t - \eta, t_0, x_0),
$$

$$
{}^cD^q z(t, \tau_0, y_0 - x_0) = \tilde{f}(\eta; t, z(t, \tau_0, y_0 - x_0)).
$$

(3.6)

One can observe that even if $y_0 = x_0, z(t, \tau_0, 0)$ is not zero and is not the null solution of the initial value problems of transformed fractional-order differential equation with Caputo's derivative and the solution. $x(t - \eta, t_0, x_0)$ does not correspond to the identically zero solution of ${}^cD^q z = \tilde{f}(\eta; t, z)$. Therefore, we cannot use stability properties of the fractional-order differential equation with Caputo's derivative of null solution in order to find fractional (ITD) stability properties using this approach.

4. A Fractional Comparison Result

In our earlier work and in the work of others [4–6], the differences between the classical notion of fractional stability and fractional ITD stability did not allow the use of the behavior of the null solution in our fractional ITD stability analysis. The main result presented in this section resolves those difficulties with a new approach that allows the use of the fractional stability of the null solution of the comparison system to predict the Caputo's fractional stability properties of $y(t, \tau_0, y_0)$ the solution of fractional-order differential equation with Caputo's derivative of (2.18) with respect to $\tilde{x}(t) = x(t - \eta, t_0, x_0)$, where $x(t, t_0, x_0)$ is any solution of the fractional-order differential equation with Caputo's derivative of (2.16).

Let $0 < q < 1$ and $p + q = 1$. The function space is denoted by $C_p[[\tau_0, T], \mathbb{R}]$ as follows:

$$
C_p[[\tau_0, T], \mathbb{R}] = \left[u \in C[(\tau_0, T], \mathbb{R}], \ (t - \tau_0)^{1-q} u(t) \in C[[\tau_0, T], \mathbb{R}] \right].
$$

(4.1)

The Riemann-Liouville fractional derivative is defined by

$$
D^q x(t) = \frac{x(\tau_0)(t - \tau_0)^{-q}}{\Gamma(1 - q)} + \int_{\tau_0}^t \frac{(t - s)^{-q}}{\Gamma(1 - q)} \frac{d}{ds} x(s) \, ds.
$$

(4.2)

Now, we will prove the following comparison result.

Theorem 4.1. *Assume that $m \in C_p[[\tau_0, T], \mathbb{R}_+]$ is locally Hölder continuous, $g \in C_p[[\tau_0, T] \times \mathbb{R}_+, \mathbb{R}]$ and*

$$
D^q m(t) \leq g(t, m(t)), \quad \tau_0 \leq t \leq T.
$$

(4.3)

Let $r(t)$ be the maximal solution of the initial value problem of fractional-order scalar differential equation with Riemann-Liouville's derivative

$$D^q u(t) = g(t, u(t)), \quad u(t)(t - \tau_0)^{1-q}|_{t=\tau_0} = u^0 \geq 0 \tag{4.4}$$

existing on $[\tau_0, T]$ such that $m^0 \leq u^0$, where $m^0 = m(t)(t - \tau_0)^{1-q}|_{t=\tau_0}$. Then we have

$$m(t) \leq r(t), \quad \tau_0 \leq t \leq T. \tag{4.5}$$

Proof. In view of the definition of the maximal solution $r(t)$ of the fractional-order differential equation with Riemann-Liouville's derivative of (4.4), it is enough to prove that

$$m(t) < u(t, \epsilon), \quad \tau_0 \leq t \leq T, \tag{4.6}$$

where $u(t, \epsilon)$ is any solution of the initial value problem of fractional-order scalar differential equation with Riemann-Liouville's derivative

$$D^q u = g(t, u) + \epsilon \text{ with initial value } u^0 + \epsilon, \quad \epsilon > 0. \tag{4.7}$$

Now it follows from (4.7) that

$$D^q u(t, \epsilon) > g(t, u(t, \epsilon)). \tag{4.8}$$

Then by applying the comparison result [2, Theorem 2.1 in page 23], we get (4.6) and since $\lim_{\epsilon \to 0} u(t, \epsilon) = r(t)$ uniformly on each compact set $\tau_0 \leq t \leq T_0 < T$. Hence,

$$m(t) \leq \lim_{\epsilon \to 0} u(t, \epsilon) = r(t), \quad \tau_0 \leq t \leq T. \tag{4.9}$$

The proof is complete. □

Theorem 4.2. *Assume that $f \in C[[\tau_0, T] \times S(\rho), \mathbb{R}^n]$ and $[x, f(t, x)]_+ \leq g(t, \|x\|), (t, x) \in [\tau_0, T] \times S(\rho)$, where $[x, f(t, x)]_+ = \lim_{h \to 0^+} \sup(1/h^q)[\|x\| - \|x - h^q f(t, x)\|]$ and $g \in C[[\tau_0, T] \times \mathbb{R}_+, \mathbb{R}], g(t, 0) \equiv 0$. Then the stability properties of the trivial solution of the comparison initial value problem of fractional-order differential equation with Caputo's derivative*

$$^c D^q u = g(t, u(t)), \quad u(\tau_0) = u_0 \geq 0 \tag{4.10}$$

imply the corresponding stability results of the solution $x(t, \tau_0, y_0)$ of the initial value problem of the fractional-order differential equation with Caputo's derivative of (2.17), *respectively.*

Proof. Let $u \equiv 0$ of (4.10) be stable. Then given $0 < \epsilon < \rho$ and $\tau_0 \geq 0$, there exists for a $\delta = \delta(\epsilon, \tau_0) > 0$ with the property that

$$0 \leq u_0 < \delta \text{ implies } u(t, \tau_0, u_0) < \epsilon, \quad t \geq \tau_0. \tag{4.11}$$

We claim that the trivial solution of (2.17) is stable for these ϵ and δ. If this was false, then there would exists a solution $x(t)$ of (2.17) and $t_1 > \tau_0$ such that

$$\|x(t_1)\| = \epsilon, \quad \|x(t)\| \leq \epsilon \quad \text{for } \tau_0 \leq t \leq t_1. \tag{4.12}$$

For $[\tau_0, t_1]$, we set $m(t) = \|x(t)\|$ and choose $\|y_0\| \leq u_0$. Then we have

$$x(t) - S(x, h, r, q) = h^q f(t, x(t)) + \epsilon(h^q) \tag{4.13}$$

which shows that $S(x, h, r, q) = x(t) - h^q f(t, x(t)) - \epsilon(h^q)$ where $\epsilon(h^q)/h^q \to 0$ as $h \to 0^+$. Now, by using the fractional-order Dini derivatives in Caputo's sense $^c D_+^q m(t)$ in Definition 2.8 as in (2.23), we have

$$^c D_+^q m(t) = \lim_{h \to 0^+} \sup \frac{1}{h^q} \left[\|x(t)\| - \|x(t) - h^q f(t, x(t))\| \right] \leq g(t, m(t)) \tag{4.14}$$

for $\tau_0 \leq t \leq t_1$ and $m(\tau_0) = \|y_0\|$.

This yields by comparison Theorem 4.1, the estimate

$$\|x(t, \tau_0, y_0)\| \leq r(t, \tau_0, \|y_0\|), \quad \tau_0 \leq t \leq t_1, \tag{4.15}$$

where $r(t, \tau_0, u_0)$ is the maximal solution of (4.4). At $t = t_1$, we therefore arrive at the following contradiction

$$\epsilon = \|x(t_1, \tau_0, y_0)\| \leq r(t_1, \tau_0, \|y_0\|) < \epsilon. \tag{4.16}$$

Therefore, this do justify our claim. Hence the trivial solution of initial value problem of the fractional-order differential equation with Caputo's derivative of (2.17) is stable.

Next suppose that $u \equiv 0$ is asymptotically stable. Since this implies by definition of the stability of $u \equiv 0$, the stability of the trivial solution of (4.10) is the foregoing argument. This means the inequality (4.15) holds for all $\tau_0 \leq t$ and hence, it is clear, by hypothesis, that if $\|x_0\| < \delta_0$, then $\lim_{t \to \infty} \|x(t, \tau_0, x_0)\| = 0$. The proof is therefore complete. $\quad\square$

5. An Initial Time Difference Fractional Comparison Result

In this section, we have an other comparison result in which the stability properties of the null solution of the comparison system imply the corresponding initial time difference stability properties of the perturbed fractional-order differential system in Caputo's sense with respect to the unperturbed fractional-order differential system in Caputo's sense.

Theorem 5.1. *Let $f, F \in C[[t_0, T] \times \mathbb{R}^n, \mathbb{R}^n]$, and let*

$$G(t, r) = \max_{\tilde{x}, y \in \overline{B}(x_0; r)} \left\| F(t, y) - \tilde{f}(t, \tilde{x}) \right\|, \tag{5.1}$$

where $G(t, r) \in C[\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+]$ and $\overline{B}$ is closed ball with center at x_0 and radius r. Assume that $r^*(t, \tau_0, \|y_0 - x_0\|)$ is the maximal solution of initial value problem of fractional-order differential equation with Caputo's derivative $^cD^q u = G(t, u), u(\tau_0) = \|y_0 - x_0\|$ through $(\tau_0, \|y_0 - x_0\|)$. $\tilde{x}(t, \tau_0, x_0) = x(t - \eta, t_0, x_0)$, where $x(t, t_0, x_0)$ is any solution of fractional-order differential equation of (2.16) for $t \geq \tau_0 \geq 0, t_0 \in \mathbb{R}_+$, and $\eta = \tau_0 - t_0$, and $y(t, \tau_0, y_0)$ is the solution of fractional-order differential equation (2.18) with Caputo's derivatives. Then

$$\|y(t, \tau_0, y_0) - x(t - \eta, t_0, x_0)\| \leq r^*(t, \tau_0, \|y_0 - x_0\|) \quad \text{for } t \geq \tau_0. \tag{5.2}$$

Proof. Let $m(t) = \|y(t, \tau_0, y_0) - \tilde{x}(t, \tau_0, x_0)\|$ for $t \geq \tau_0$. Then by using the fractional-order Dini derivatives in Caputo's sense $^cD_+^q m(t)$ in Definition 2.8 as in (2.23), we obtain

$$
\begin{aligned}
^cD_+^q m(t) &= {}^cD_+^q \left[\|y(t, \tau_0, y_0) - \tilde{x}(t, \tau_0, x_0)\| \right] \\
&\leq \left\| F(t, y(t, \tau_0, y_0)) - \tilde{f}(t, \tilde{x}(t, \tau_0, x_0)) \right\| \\
&\leq \max_{\tilde{x}, y \in \overline{B}(x_0; m(t))} \left\| F(t, y) - \tilde{f}(t, \tilde{x}) \right\| \\
&= G(t, m(t)).
\end{aligned}
\tag{5.3}
$$

Therefore, $^cD_+^q m(t) \leq G(t, m(t)), m(\tau_0) = \|y_0 - x_0\|$ and by using a standard result in [6], we get $m(t) \leq r^*(t, \tau_0, \|y_0 - x_0\|)$. $\qquad \square$

Now, we can formulate the comparison results via Lyapunov-like functions.

Theorem 5.2. *Let $V(t, z) \in C[\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}_+]$ and $V(t, z)$ be locally Lipschitzian in z. Assume that the generalized fractional-order derivatives (Dini-like derivatives) in Caputo's sense*

$$^c_*D_+^q V(t, y - \tilde{x}) = \lim_{h \to 0^+} \sup \frac{1}{h^q} \left[V(t, y - \tilde{x}) - V\left(t - h, y - \tilde{x} - h^q\left(F(t, y) - \tilde{f}(t, \tilde{x})\right)\right) \right] \tag{5.4}$$

satisfies $^c_*D_+^q V(t, y - \tilde{x}) \leq G(t, V(t, y - \tilde{x}))$ with $(t, \tilde{x}), (t, y) \in \mathbb{R}_+ \times \mathbb{R}^n$, where $G(t, u) \in C[\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}]$. Let $r(t) = r(t, \tau_0, u_0)$ be the maximal solution of the fractional-order differential equation $^cD^q u = G(t, u), u(\tau_0) = u_0 \geq 0$, for $t \geq t_0$. If $\tilde{x}(t) = x(t - \eta, t_0, x_0)$, where $x(t, t_0, x_0)$ is any solution of the system (2.16) for $t \geq \tau_0 \geq 0, t_0 \in \mathbb{R}_+$ and $\eta = \tau_0 - t_0$ and $y(t) = y(t, \tau_0, y_0)$ is any solution of (2.18) for $t \geq \tau_0$ such that $V(\tau_0, y_0 - x_0) \leq u_0$, then $V(t, y(t) - \tilde{x}(t)) \leq r(t)$ for $t \geq \tau_0$.

Proof. Let $\tilde{x}(t) = x(t - \eta, t_0, x_0)$, where $x(t, t_0, x_0)$ is any solution of the system fractional-order differential equation (2.16) for $t \geq \tau_0 \geq 0, t_0 \in \mathbb{R}_+$, and $\eta = \tau_0 - t_0$ and $y(t) = y(t, \tau_0, y_0)$ is any solution of fractional-order differential equation (2.18) for $t \geq \tau_0$ such that $V(\tau_0, y_0 - x_0) \leq u_0$ holds. Define $m(t) = V(t, y(t) - \tilde{x}(t))$ for $t \geq \tau_0$ so that $m(t_0) \leq u_0$. Then for small enough $h > 0$, we get

$$m(t) - m(t - h) = V(t, y(t) - \tilde{x}(t)) - V(t - h, y(t - h) - \tilde{x}(t - h)). \tag{5.5}$$

Since V is locally Lipschitzian in $\tilde{x}$ and fractional-order Dini derivatives in Caputo's sense $^cD_+^q m(t)$

$$
\begin{aligned}
V(t, y(t) - \tilde{x}(t)) &- V(t - h, S(y - \tilde{x}, h, r, q)) \\
&\leq V(t, y(t) - \tilde{x}(t)) - V\left(t - h, (y(t) - \tilde{x}(t)) - h^q\left(F(t, y) - \tilde{f}(t, \tilde{x})\right)\right) \\
&+ V\left(t - h, (y(t) - \tilde{x}(t)) - h^q\left(F(t, y) - \tilde{f}(t, \tilde{x})\right)\right) - V(t - h, S(y - \tilde{x}, h, r, q)),
\end{aligned}
\tag{5.6}
$$

where $L > 0$ is the Lipschitz constant, $S(y - \tilde{x}, h, r, q) = y - \tilde{x} - h^q[(F(t, y) - \tilde{f}(t, \tilde{x}))] - [\epsilon_1(h^q) - \epsilon_2(h^q))]$ and ϵ_1 and ϵ_2 are error terms

$$
\begin{aligned}
^cD_+^q m(t) &\leq \lim_{h \to 0^+} \sup \frac{1}{h^q}\left[V(t, y(t) - \tilde{x}(t)) - V\left(t - h, (y(t) - \tilde{x}(t)) - h^q\left(F(t, y) - \tilde{f}(t, \tilde{x})\right)\right)\right] \\
&+ \lim_{h \to 0^+} \sup \frac{1}{h^q}\left[V\left(t - h, (y(t) - \tilde{x}(t)) - h^q\left(F(t, y) - \tilde{f}(t, \tilde{x})\right)\right) - V(t - h, S(y - \tilde{x}, h, r, q))\right],
\end{aligned}
$$

$$
\begin{aligned}
^cD_+^q m(t) &\leq \lim_{h \to 0^+} \sup \frac{1}{h^q}[L(\epsilon_1(h^q) - \epsilon_2(h^q))] \\
&+ \lim_{h \to 0^+} \sup \frac{1}{h^q}\left[V(t, y(t) - \tilde{x}(t)) - V(t - h, (y(t) - \tilde{x}(t))) - h^q\left(F(t, y) - \tilde{f}(t, \tilde{x})\right)\right],
\end{aligned}
\tag{5.7}
$$

where $\lim_{h \to 0^+} \sup L[(\epsilon_1(h^q) - \epsilon_2(h^q))/h^q] \to 0$. Since $^c_*D_+^q V(t, y - \tilde{x})$ is the generalized fractional-order derivatives (Dini-like derivatives) in Caputo's sense, we have

$$
\begin{aligned}
^cD_+^q m(t) &\leq {}^c_*D_+^q V(t, y(t) - \tilde{x}(t)) \leq G(t, V(t, y(t) - \tilde{x}(t))) = G(t, V(t, m(t))), \\
^cD_+^q m(t) &\leq G(t, V(t, m(t))), \qquad m(\tau_0) = V(\tau_0, y(\tau_0) - \tilde{x}(\tau_0)) \leq u_0.
\end{aligned}
\tag{5.8}
$$

$\square$

By using Theorem 5.1, this implies

$$
m(t) = V(t, y(t) - \tilde{x}(t)) \leq r(t, \tau_0, \|y_0 - x_0\|),
\tag{5.9}
$$

where r is the maximal solution of $^cD^q u = G(t, u), u(\tau_0) = u_0 \geq 0$ for $t \geq t_0$.

Now, we present the main comparison result that yields knowledge of initial time difference fractional stability properties if we know the stability properties of the null solution of the fractional comparison system.

Theorem 5.3. *Assume that*

(i) *let $V(t, z) \in C[\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}_+]$ be locally Lipschitzian in z, positive definite and decrescent where the fractional-order Dini derivatives in Caputo's sense $^cD_+^q m(t)$*

$$
^cD_+^q m(t) \leq \lim_{h \to 0^+} \sup \frac{1}{h^q}\left[V(t, y(t) - \tilde{x}(t)) - V\left(t - h, (y - \tilde{x}) - h^q\left(F(t, y) - \tilde{f}(t, \tilde{x})\right)\right)\right]
\tag{5.10}
$$

satisfies ${}^c_* D^q_+ V(t, y - \tilde{x}) \leq G(t, V(t, y - \tilde{x}))$ for $(t, \tilde{x})$ and $(t, y) \in \mathbb{R}_+ \times \mathbb{R}^n$, where $G(t, u) \in C[\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}]$ and the generalized fractional-order (Dini-like) derivatives in Caputo's sense ${}^c_* D^q_+ V(t, x)$,

$$a(\|x\|) \leq V(t, x) \leq b(\|x\|), \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}^n, a, b \in \mathcal{K}. \tag{5.11}$$

(ii) Let $r(t) = r(t, \tau_0, u_0)$ be the maximal solution of the fractional-order differential equation with Caputo's derivative

$$^c D^q u = G(t, u), \quad u(\tau_0) = u_0 \geq 0 \quad for \ t \geq \tau_0. \tag{5.12}$$

Then the stability properties of the null solution of the fractional-order differential system with Caputo's derivative (5.12) with $G(t, 0) = 0$ imply the corresponding stability properties of $y(t, \tau_0, y_0)$ any solution of fractional-order differential system with Caputo's derivative (2.18) with respect to $\tilde{x}(t, \tau_0, x_0) = x(t - \eta, t_0, x_0)$, where $x(t, t_0, x_0)$ is any solution of fractional-order differential system with Caputo's derivative of (2.16).

Proof. Let $\tilde{x}(t) = x(t - \eta, t_0, x_0)$, where $x(t, t_0, x_0)$ is any solution of the fractional-order differential system with Caputo's derivative of (2.16) for $t \geq t_0 \geq 0$, $t_0 \in \mathbb{R}_+$, and $\eta = \tau_0 - t_0$ and $y(t) = y(t, \tau_0, y_0)$ is any solution of fractional-order differential system with Caputo's derivative of (2.18) for $t \geq t_0$ such that $V(\tau_0, y_0 - x_0)$ holds. If we define $m(t) = V(t, y(t) - \tilde{x}(t))$ for $t \geq \tau_0$ so that $m(\tau_0) \leq u_0$, then Theorems 5.1 and 5.2 imply that

$$^c D^q_+ m(t) \leq G(V(t, m(t))), \quad m(\tau_0) = V(\tau_0, y(\tau_0) - \tilde{x}(\tau_0)) \leq u_0, \tag{5.13}$$

where the fractional-order Dini derivatives in Caputo's sense ${}^c D^q_+ m(t)$ and the generalized fractional-order (Dini-like) derivatives in Caputo's sense ${}^c_* D^q_+ V(t, x)$ have been used. Thus, $m(t) = V(t, y(t) - \tilde{x}(t)) \leq r(t, \tau_0, \|y_0 - x_0\|)$, where r is the maximal solution of the fractional comparison system of (5.12) with Caputo's derivative. Let the null solution of the fractional comparison system of (5.12) with Caputo's derivative be stable. Given any $\epsilon > 0$, since V is positive definite and $a \in \mathcal{K}$ by (i) we have

$$a(\|y - \tilde{x}\|) \leq V(t, y - \tilde{x}) \quad for \ (t, \tilde{x}), (t, y) \in \mathbb{R}_+ \times \mathbb{R}^n,$$

$$V(t, y(t) - \tilde{x}(t)) \leq r(t, \tau_0, \|y_0 - x_0\|) < a(\epsilon) \quad for \ t \geq \tau_0 \ provided \ that \ \|y_0 - x_0\| < \delta(\epsilon, \tau_0). \tag{5.14}$$

Hence

$$V(t, y(t) - \tilde{x}(t)) \leq r(t, \tau_0, \|y_0 - x_0\|) < a(\epsilon). \tag{5.15}$$

Since a^{-1} exists, we have $\|y(t) - \tilde{x}(t)\| < \epsilon$ and $\|y_0 - x_0\| < \delta(\epsilon, \tau_0)$. Therefore, $y(t, \tau_0, y_0)$ the solution of fractional-order differential equation of (2.18) with Caputo's derivative is stable with respect to $\tilde{x}(t) = x(t - \eta, t_0, x_0)$, where $x(t, t_0, x_0)$ is any solution of the fractional-order

differential equation of (2.16) with Caputo's derivative. If the null solution of the fractional comparison system of (5.12) with Caputo's derivative is asymptotically stable, then

$$a\big(\big\|y(t) - \tilde{x}(t)\big\|\big) \leq r\big(t, \tau_0, \big\|y_0 - x_0\big\|\big) \quad \text{holds for } t \geq \tau_0, \ \big\|y_0 - x_0\big\| \text{ small enough.} \quad (5.16)$$

That implies

$$\lim_{t \to \infty} \big\|y(t, \tau_0, y_0) - x(t - \eta, t_0, x_0)\big\| = 0 \quad (5.17)$$

since $a \in \mathcal{K}$. Hence, $y(t, \tau_0, y_0)$ the solution of fractional differential equation (2.18) is asymptotically stable with respect to $\tilde{x}(t) = x(t - \eta, t_0, x_0)$, where $x(t, t_0, x_0)$ is any solution of the system (2.16). Since V is decrescent and $b \in \mathcal{K}$ by (i) we have

$$V(t, y - \tilde{x}) \leq b\big(\big\|(y - \tilde{x})\big\|\big) \quad \text{for } (t, \tilde{x}), (t, y) \in \mathbb{R}_+ \times \mathbb{R}^n \quad (5.18)$$

and the choice of $\delta = \delta(\epsilon)$ is independent of τ_0.

Thus, uniform stability and uniform asymptotic stability of the fractional comparison system of (5.12) with Caputo's derivative imply the corresponding uniform stability and uniform asymptotic stability of the fractional solution of (2.18) in Caputo's sense with respect to $\tilde{x}(t) = x(t - \eta, t_0, x_0)$, where $x(t, t_0, x_0)$ is any solution of the fractional system of (2.16) in Caputo's sense. Hence, $x(t, t_0, x_0)$ is any solution of the fractional system of (2.16) in Caputo's sense that is uniformly stable and uniformly asymptotically stable. □

Acknowledgments

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