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Research Article

Composite Algorithms for Minimization over the Solutions of Equilibrium Problems and Fixed Point Problems

Yonghong Yao[1] and Yeong-Cheng Liou[2]

[1] *Department of Mathematics, Tianjin Polytechnic University, Tianjin 300160, China*
[2] *Department of Information Management, Cheng Shiu University, Kaohsiung 833, Taiwan*

Correspondence should be addressed to Yonghong Yao, yaoyonghong@yahoo.cn

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The purpose of this paper is to solve the minimization problem of finding x^* such that $x^* = \arg\min_{x\in\Gamma}\|x\|^2$, where Γ stands for the intersection set of the solution set of the equilibrium problem and the fixed points set of a nonexpansive mapping. We first present two new composite algorithms (one implicit and one explicit). Further, we prove that the proposed composite algorithms converge strongly to x^*.

1. Introduction

In the present paper, our main purpose is to solve the minimization problem of finding x^* such that

$$x^* = \arg\min_{x\in\Gamma}\|x\|^2, \tag{1.1}$$

where Γ stands for the intersection set of the solution set of the equilibrium problem and the fixed points set of a nonexpansive mapping. This problem is motivated by the following least-squares solution to the constrained linear inverse problem:

$$\begin{aligned} Bx &= b, \\ x &\in C, \end{aligned} \tag{1.2}$$

where C is a nonempty closed convex subset of a real Hilbert space H, B is a bounded linear operator from H to another real Hilbert space H_1 and b is a given point in H_1. The least-squares solution to (1.2) is the least-norm minimizer of the minimization problem

$$\min_{x\in C}\|Bx-b\|^2. \tag{1.3}$$

Let S_b denote the solution set of (1.2) (or equivalently (1.3)). It is known that S_b is nonempty if and only if $P_{\overline{B(C)}}(b) \in B(C)$. In this case, S_b has a unique element with minimum norm (equivalently, (1.2) has a unique least-squares solution); that is, there exists a unique point $x^{\dagger} \in S_b$ satisfying

$$\left\|x^{\dagger}\right\| = \min\{\|x\| : x \in S_b\}. \tag{1.4}$$

The so-called C-constrained pseudoinverse of B is then defined as the operator $B_C^{\dagger}$ with domain and values given by

$$D\left(B_C^{\dagger}\right) = \left\{b \in H : P_{\overline{B(C)}}(b) \in B(C)\right\}, \quad B_C^{\dagger}(b) = x^{\dagger},\ b \in D\left(B_C^{\dagger}\right), \tag{1.5}$$

where $x^{\dagger} \in S_b$ is the unique solution to (1.4).

Note that the optimality condition for the minimization (1.3) is the variational inequality (VI)

$$\hat{x} \in C, \quad \langle B^*(B\hat{x}-b), x-\hat{x}\rangle \geq 0, \quad x \in C, \tag{1.6}$$

where B^* is the adjoint of B.

If $b \in D(B_C^{\dagger})$, then (1.3) is consistent and its solution set S_b coincides with the solution set of VI (1.6). On the other hand, VI (1.6) can be rewritten as

$$\hat{x} \in C, \quad \langle (\hat{x}-\lambda B^*(B\hat{x}-b)) - \hat{x}, x-\hat{x}\rangle \leq 0, \quad x \in C, \tag{1.7}$$

where $\lambda > 0$ is any positive scalar. In the terminology of projections, (1.7) is equivalent to the fixed point equation

$$\hat{x} = P_C(\hat{x}-\lambda B^*(B\hat{x}-b)). \tag{1.8}$$

It is not hard to find that for $0 < \lambda < 2/\|B\|^2$, the mapping $x \mapsto P_C(x-\lambda B^*(Bx-b))$ is nonexpansive. Therefore, finding the least-squares solution of the constrained linear inverse problem is equivalent to finding the minimum-norm fixed point of the nonexpansive mapping $x \mapsto P_C(x-\lambda B^*(Bx-b))$.

Based on the above facts, it is an interesting topic of finding the minimum norm fixed point of the nonexpansive mappings. In this paper, we will consider a general problem. We will focus on to solve the minimization problem (1.1). At this point, we first recall some definitions on the fixed point problem and the equilibrium problem as follows.

Let C be a nonempty closed convex subset of a real Hilbert space H. Recall that a mapping $A : C \to H$ is called α-inverse strongly monotone if there exists a positive real number α such that $\langle Ax - Ay, x - y\rangle \geq \alpha\|Ax - Ay\|^2$, for all $x, y \in C$. It is clear that any α-inverse strongly monotone mapping is monotone and $1/\alpha$-Lipschitz continuous. Let $f : C \to H$ be a ρ-contraction; that is, there exists a constant $\rho \in [0, 1)$ such that $\|f(x)-f(y)\| \leq \rho\|x-y\|$ for all $x, y \in C$. A mapping $S : C \to C$ is said to be nonexpansive if $\|Sx - Sy\| \leq \|x - y\|$, for all $x, y \in C$. Denote the set of fixed points of S by $\mathrm{Fix}(S)$.

Let $A : C \to H$ be a nonlinear mapping and $F : C \times C \to R$ be a bifunction. The equilibrium problem is to find $z \in C$ such that

$$F(z, y) + \langle Az, y - z\rangle \geq 0, \quad \forall y \in C. \tag{1.9}$$

The solution set of (1.9) is denoted by EP. If $A = 0$, then (1.9) reduces to the following equilibrium problem of finding $z \in C$ such that

$$F(z, y) \geq 0, \quad \forall y \in C. \tag{1.10}$$

If $F = 0$, then (1.9) reduces to the variational inequality problem of finding $z \in C$ such that

$$\langle Az, y - z\rangle \geq 0, \quad \forall y \in C. \tag{1.11}$$

We note that the problem (1.9) is very general in the sense that it includes, as special cases, optimization problems, variational inequalities, minimax problems, Nash equilibrium problem in noncooperative games and others, see, for example, [1–4].

We next briefly review some historic approaches which relate to the fixed point problems and the equilibrium problems.

In 2005, Combettes and Hirstoaga [5] introduced an iterative algorithm of finding the best approximation to the initial data and proved a strong convergence theorem. In 2007, by using the viscosity approximation method, S. Takahashi and W. Takahashi [6] introduced another iterative scheme for finding a common element of the set of solutions of the equilibrium problem and the set of fixed point points of a nonexpansive mapping. Subsequently, algorithms constructed for solving the equilibrium problems and fixed point problems have further developed by some authors. In particular, Ceng and Yao [7] introduced an iterative scheme for finding a common element of the set of solutions of the mixed equilibrium problem (1.9) and the set of common fixed points of finitely many nonexpansive mappings. Maingé and Moudafi [8] introduced an iterative algorithm for equilibrium problems and fixed point problems. Yao et al. [9] considered an iterative scheme for finding a common element of the set of solutions of the equilibrium problem and the set of common fixed points of an infinite nonexpansive mappings. Noor et al. [10] introduced an iterative method for solving fixed point problems and variational inequality problems. Their results extend and improve many results in the literature. Some works related to the equilibrium problem, fixed point problems, and the variational inequality problem in[1–45] and the references therein.

However, we note that all constructed algorithms in [2, 4, 6–10, 14, 15, 21, 23–40] do not work to find the minimum-norm solution of the corresponding fixed point problems and the equilibrium problems. It is our main purpose in this paper that we devote to construct some algorithms for finding the minimum-norm solution of the fixed point problems and the equilibrium problems. We first suggest two new composite algorithms (one implicit and one explicit) for solving the above minimization problem. Further, we prove that the proposed composite algorithms converge strongly to the minimum norm element x^*.

2. Preliminaries

Let C be a nonempty closed convex subset of a real Hilbert space H. Throughout this paper, we assume that a bifunction $F : C \times C \to R$ satisfies the following conditions:

(H1) $F(x, x) = 0$, for all $x \in C$;

(H2) F is monotone, that is, $F(x, y) + F(y, x) \leq 0$ for all $x, y \in C$;

(H3) for each $x, y, z \in C$, $\lim_{t\downarrow 0} F(tz + (1 - t)x, y) \leq F(x, y)$;

(H4) for each $x \in C$, $y \mapsto F(x, y)$ is convex and lower semicontinuous.

The metric (or nearest point) projection from H onto C is the mapping $P_C : H \to C$ which assigns to each point $x \in C$ the unique point $P_C x \in C$ satisfying the property

$$\|x - P_C x\| = \inf_{y\in C} \|x - y\| =: d(x, C). \tag{2.1}$$

It is well known that P_C is a nonexpansive mapping and satisfies

$$\langle x - y, P_C x - P_C y\rangle \geq \|P_C x - P_C y\|^2, \quad \forall x, y \in H. \tag{2.2}$$

We need the following lemmas for proving our main results.

Lemma 2.1 (see [5]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $F : C \times C \to R$ be a bifunction which satisfies conditions (H1)–(H4). Let $r > 0$ and $x \in C$. Then, there exists $z \in C$ such that*

$$F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C. \tag{2.3}$$

Further, if $T_r(x) = \{z \in C : F(z, y) + (1/r)\langle y - z, z - x\rangle \geq 0,\ \forall y \in C\}$, then the following hold:

(i) *T_r is single-valued and T_r is firmly nonexpansive, that is, for any $x, y \in H$, $\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle$;*

(ii) *EP is closed and convex and $EP = \mathrm{Fix}(T_r)$.*

Lemma 2.2 (see [17]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let the mapping $A : C \to H$ be α-inverse strongly monotone and $r > 0$ be a constant. Then, one has*

$$\|(I - rA)x - (I - rA)y\|^2 \leq \|x - y\|^2 + r(r - 2\alpha)\|Ax - Ay\|^2, \quad \forall x, y \in C. \tag{2.4}$$

In particular, if $0 \leq r \leq 2\alpha$, then $I - rA$ is nonexpansive.

Lemma 2.3 (see [28]). *Let C be a closed convex subset of a real Hilbert space H and let $S : C \to C$ be a nonexpansive mapping. Then, the mapping $I - S$ is demiclosed, that is, if $\{x_n\}$ is a sequence in C such that $x_n \to x^*$ weakly and $(I - S)x_n \to y$ strongly, then $(I - S)x^* = y$.*

Lemma 2.4 (see [22]). *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \le (1 - \gamma_n)a_n + \delta_n\gamma_n, \tag{2.5}$$

where $\{\gamma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that

(1) $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(2) $\limsup_{n\to\infty}\delta_n \le 0$ *or* $\sum_{n=1}^{\infty} |\delta_n\gamma_n| < \infty$.

Then $\lim_{n\to\infty}a_n = 0$.

3. Main Results

In this section we will introduce two algorithms for finding the minimum norm element x^* of $\Gamma := \mathrm{EP} \cap \mathrm{Fix}(S)$. Namely, we want to find the unique point x^* which solves the following minimization problem:

$$x^* = \arg \min_{x\in\Gamma}\|x\|^2. \tag{3.1}$$

Let $S : C \to C$ be a nonexpansive mapping and $A : C \to H$ be an α-inverse strongly monotone mapping. Let $F : C \times C \to R$ be a bifunction which satisfies conditions (H1)–(H4). Let r and μ be two constants such that $r \in (0, 2\alpha)$ and $\mu \in (0,1)$. In order to find a solution of the minimization problem (3.1), we construct the following implicit algorithm

$$x_t = \mu P_C[(1 - t)Sx_t] + (1 - \mu)T_r(x_t - rAx_t), \quad \forall t \in (0,1), \tag{3.2}$$

where T_r is defined as Lemma 2.1. We will show that the net $\{x_t\}$ defined by (3.2) converges to a solution of the minimization problem (3.1). As matter of fact, in this paper, we will study the following general algorithm.

Let $f : C \to H$ be a ρ-contraction. For each $t \in (0,1)$, we consider the following mapping W_t given by

$$W_tx = \mu P_C\left[tf(x) + (1 - t)Sx_t\right] + (1 - \mu)T_r(I - rA)x_t, \quad \forall x \in C. \tag{3.3}$$

Since the mappings S, P_C, T_r and $I - rA$ are nonexpansive, then we can check easily that $\|W_tx - W_ty\| \le [1 - (1 - \rho)\mu t]\|x - y\|$ which implies that W_t is a contraction. Using the Banach contraction principle, there exists a unique fixed point x_t of W_t in C, that is,

$$x_t = \mu P_C\left[tf(x_t) + (1 - t)Sx_t\right] + (1 - \mu)T_r(I - rA)x_t, \quad t \in (0,1). \tag{3.4}$$

In this point, we would like to point out that algorithm (3.4) includes algorithm (3.2) as a special case due to the contraction f is a possible nonself-mapping.

In the sequel, we assume

(1) C is a nonempty closed convex subset of a real Hilbert space H;

(2) $S : C \to C$ is a nonexpansive mapping, $A : C \to H$ is an α-inverse strongly monotone mapping and $f : C \to H$ is a ρ-contraction;

(3) $F : C \times C \to R$ is a bifunction which satisfies conditions (H1)–(H4);

(4) $\Gamma \neq \emptyset$.

In order to prove our first main result, we need the following lemmas.

Lemma 3.1. *The net $\{x_t\}$ generated by the implicit method* (3.4) *is bounded.*

Proof. Set $u_t = T_r(x_t - rAx_t)$ and $y_t = tf(x_t) + (1-t)Sx_t$ for all $t \in (0,1)$. Take $z \in \Gamma$. It is clear that $Sz = z = T_r(z - rAz)$. Since T_r is nonexpansive and A is α-inverse strongly monotone, we have from Lemma 2.2 that

$$\begin{aligned} \|u_t - z\|^2 &\leq \|x_t - rAx_t - (z - rAz)\|^2 \\ &\leq \|x_t - z\|^2 + r(r-2\alpha)\|Ax_t - Az\|^2 \\ &\leq \|x_t - z\|^2. \end{aligned} \tag{3.5}$$

So, we have that

$$\|u_t - z\| \leq \|x_t - z\|. \tag{3.6}$$

It follows from (3.4) that

$$\begin{aligned} \|x_t - z\| &= \left\|\mu P_C[y_t] + (1-\mu)u_t - z\right\| \\ &\leq \mu\left\|P_C[y_t] - z\right\| + (1-\mu)\|u_t - z\| \\ &\leq \mu\left(t\left\|f(x_t) - z\right\| + (1-t)\|Sx_t - z\|\right) + (1-\mu)\|x_t - z\| \\ &\leq \mu(t\rho + 1 - t)\|x_t - z\| + \mu t\left\|f(z) - z\right\| + (1-\mu)\|x_t - z\| \\ &= \left[1 - (1-\rho)\mu t\right]\|x_t - z\| + t\mu\left\|f(z) - z\right\|, \end{aligned} \tag{3.7}$$

that is,

$$\|x_t - z\| \leq \frac{\left\|f(z) - z\right\|}{1-\rho}. \tag{3.8}$$

So, $\{x_t\}$ is bounded. Hence $\{u_t\}$ and $\{f(x_t)\}$ are also bounded. This completes the proof. □

According to Lemma 3.1, we can choose some appropriate constant $M > 0$ such that M satisfies the following request.

Lemma 3.2. *The net* $\{x_t\}$ *generated by the implicit method* (3.4) *is relatively norm compact as* $t \to 0$.

Proof. From (3.4) and (3.5), we have

$$\begin{aligned}
\|x_t - z\|^2 &= \left\|\mu(P_C[y_t] - z) + (1-\mu)(u_t - z)\right\|^2 \\
&\le \mu\left\|P_C[y_t] - z\right\|^2 + (1-\mu)\|u_t - z\|^2 \\
&\le \mu\left\|t(f(x_t) - z) + (1-t)(Sx_t - z)\right\|^2 + (1-\mu)\|u_t - z\|^2 \\
&\le \mu\left[t\left\|f(x_t) - z\right\|^2 + (1-t)\|Sx_t - z\|^2\right] + (1-\mu)\|u_t - z\|^2 \\
&\le \mu\left[tM + (1-t)\|x_t - z\|^2\right] + (1-\mu)\|u_t - z\|^2.
\end{aligned} \tag{3.9}$$

It follows that

$$\begin{aligned}
\|x_t - z\|^2 &\le \frac{1-\mu}{1-\mu+\mu t}\|u_t - z\|^2 + \frac{\mu t M}{1-\mu+\mu t} \\
&\le \|u_t - z\|^2 + tM \\
&\le \|x_t - z\|^2 + r(r-2\alpha)\|Ax_t - Az\|^2 + tM,
\end{aligned} \tag{3.10}$$

that is,

$$r(2\alpha - r)\|Ax_t - Az\|^2 \le tM \longrightarrow 0. \tag{3.11}$$

Since $r(2\alpha - r) > 0$, we derive

$$\lim_{t\to 0}\|Ax_t - Az\| = 0. \tag{3.12}$$

From Lemmas 2.1 and 2.2, we obtain

$$\begin{aligned}
\|u_t - z\|^2 &= \|T_r(x_t - rAx_t) - T_r(z - rAz)\|^2 \\
&\le \langle (x_t - rAx_t) - (z - rAz), u_t - z\rangle \\
&= \frac{1}{2}\left(\|(x_t - rAx_t) - (z - rAz)\|^2 + \|u_t - z\|^2 - \|(x_t - z) - r(Ax_t - Az) - (u_t - z)\|^2\right) \\
&\le \frac{1}{2}\left(\|x_t - z\|^2 + \|u_t - z\|^2 - \|(x_t - u_t) - r(Ax_t - Az)\|^2\right) \\
&= \frac{1}{2}\left(\|x_t - z\|^2 + \|u_t - z\|^2 - \|x_t - u_t\|^2 + 2r\langle x_t - u_t, Ax_t - Az\rangle - r^2\|Ax_t - Az\|^2\right),
\end{aligned} \tag{3.13}$$

which implies that

$$\begin{aligned}\|u_t - z\|^2 &\le \|x_t - z\|^2 - \|x_t - u_t\|^2 + 2r\langle x_t - u_t, Ax_t - Az\rangle - r^2\|Ax_t - Az\|^2 \\ &\le \|x_t - z\|^2 - \|x_t - u_t\|^2 + 2r\|x_t - u_t\|\|Ax_t - Az\| \\ &\le \|x_t - z\|^2 - \|x_t - u_t\|^2 + M\|Ax_t - Az\|.\end{aligned} \tag{3.14}$$

By (3.10), and (3.14), we have

$$\begin{aligned}\|x_t - z\|^2 &\le \|u_t - z\|^2 + tM \\ &\le \|x_t - z\|^2 - \|x_t - u_t\|^2 + (\|Ax_t - Az\| + t)M.\end{aligned} \tag{3.15}$$

It follows that

$$\|x_t - u_t\|^2 \le (\|Ax_t - Az\| + t)M. \tag{3.16}$$

This together with (3.12) imply that

$$\lim_{t\to 0}\|x_t - u_t\| = 0. \tag{3.17}$$

It follows that

$$\lim_{t\to 0}\left\|x_t - P_C[y_t]\right\| = \lim_{t\to 0}\frac{1-\mu}{\mu}\|x_t - u_t\| = 0. \tag{3.18}$$

Hence,

$$\begin{aligned}\|x_t - Sx_t\| &\le \left\|x_t - P_C[y_t]\right\| + \left\|P_C[y_t] - Sx_t\right\| \\ &\le \left\|x_t - P_C[y_t]\right\| + \left\|y_t - Sx_t\right\| \\ &\le \left\|x_t - P_C[y_t]\right\| + t\left\|f(x_t)\right\| \longrightarrow 0.\end{aligned} \tag{3.19}$$

Next, we show that $\{x_t\}$ is relatively norm compact as $t \to 0$. Let $\{t_n\} \subset (0,1)$ be a sequence such that $t_n \to 0$ as $n \to \infty$. Put $x_n := x_{t_n}$ and $u_n := u_{t_n}$. From (3.19), we get

$$\|x_n - Sx_n\| \longrightarrow 0. \tag{3.20}$$

By (3.4), we deduce

$$\begin{aligned}\|x_t - z\|^2 &= \left\|\mu(P_C[y_t] - z) + (1-\mu)(u_t - z)\right\|^2 \\ &\le \mu\left\|P_C[y_t] - z\right\|^2 + (1-\mu)\|u_t - z\|^2 \\ &\le \mu\left\|y_t - z\right\|^2 + (1-\mu)\|x_t - z\|^2,\end{aligned} \tag{3.21}$$

that is,

$$\begin{aligned}
\|x_t - z\|^2 &\le \left\|t\left(f(x_t) - z\right) + (1-t)(Sx_t - z)\right\|^2 \\
&= (1-t)^2\|Sx_t - z\|^2 + 2t(1-t)\langle f(x_t) - f(z), Sx_t - z\rangle \\
&\quad + 2t(1-t)\langle f(z) - z, Sx_t - z\rangle + t^2\left\|f(x_t) - z\right\|^2 \\
&\le (1-t)^2\|x_t - z\|^2 + 2t(1-t)\rho\|x_t - z\|^2 \\
&\quad + 2t(1-t)\langle f(z) - z, Sx_t - z\rangle + t^2\left\|f(x_t) - z\right\|^2 \\
&\le \left[1 - 2(1-\rho)t\right]\|x_t - z\|^2 + 2t\langle f(z) - z, Sx_t - z\rangle + t^2 M.
\end{aligned} \tag{3.22}$$

It follows that

$$\|x_t - z\|^2 \le \frac{1}{1-\rho}\langle z - f(z), z - Sx_t\rangle + \frac{t}{2(1-\rho)}M. \tag{3.23}$$

In particular,

$$\|x_n - z\|^2 \le \frac{1}{1-\rho}\langle z - f(z), z - Sx_n\rangle + \frac{t_n}{2(1-\rho)}M, \quad z \in \Gamma. \tag{3.24}$$

Since $\{x_n\}$ is bounded, without loss of generality, we may assume that $\{x_n\}$ converges weakly to a point $x^* \in C$. Also $Sx_n \to x^*$ weakly. Noticing (3.20) we can use Lemma 2.3 to get $x^* \in \mathrm{Fix}(S)$.

Now, we show $x^* \in \mathrm{EP}$. Since $u_n = T_r(x_n - rAx_n)$, for any $y \in C$, we have

$$F(u_n, y) + \frac{1}{r}\langle y - u_n, u_n - (x_n - rAx_n)\rangle \ge 0. \tag{3.25}$$

From the monotonicity of F, we have

$$\frac{1}{r}\langle y - u_n, u_n - (x_n - rAx_n)\rangle \ge F(y, u_n), \quad \forall y \in C. \tag{3.26}$$

Hence,

$$\left\langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r} + Ax_{n_i}\right\rangle \ge F(y, u_{n_i}), \quad \forall y \in C. \tag{3.27}$$

Put $z_t = ty + (1-t)x^*$ for all $t \in (0,1]$ and $y \in C$. Then, we have $z_t \in C$. So, from (3.27), we have

$$\begin{aligned}\langle z_t - u_{n_i}, Az_t\rangle &\geq \langle z_t - u_{n_i}, Az_t\rangle - \left\langle z_t - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r} + Ax_{n_i}\right\rangle + F(z_t, u_{n_i}) \\ &= \langle z_t - u_{n_i}, Az_t - Au_{n_i}\rangle + \langle z_t - u_{n_i}, Au_{n_i} - Ax_{n_i}\rangle \\ &\quad - \left\langle z_t - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r}\right\rangle + F(z_t, u_{n_i}).\end{aligned} \tag{3.28}$$

Note that $\|Au_{n_i} - Ax_{n_i}\| \leq (1/\alpha)\|u_{n_i} - x_{n_i}\| \to 0$. Further, from monotonicity of A, we have $\langle z_t - u_{n_i}, Az_t - Au_{n_i}\rangle \geq 0$. Letting $i \to \infty$ in (3.28), we have

$$\langle z_t - x^*, Az_t\rangle \geq F(z_t, x^*). \tag{3.29}$$

From (H1), (H4), and (3.29), we also have

$$\begin{aligned}0 = F(z_t, z_t) &\leq tF(z_t, y) + (1-t)F(z_t, x^*) \\ &\leq tF(z_t, y) + (1-t)\langle z_t - x^*, Az_t\rangle \\ &= tF(z_t, y) + (1-t)t\langle y - x^*, Az_t\rangle\end{aligned} \tag{3.30}$$

and hence

$$0 \leq F(z_t, y) + (1-t)\langle Az_t, y - x^*\rangle. \tag{3.31}$$

Letting $t \to 0$ in (3.31), we have, for each $y \in C$,

$$0 \leq F(x^*, y) + \langle y - x^*, Ax^*\rangle. \tag{3.32}$$

This implies that $x^* \in \mathrm{EP}$. Therefore, $x^* \in \Gamma$.

We substitute x^* for z in (3.24) to get

$$\|x_n - x^*\|^2 \leq \frac{1}{1-\rho}\langle x^* - f(x^*), x^* - Sx_n\rangle + \frac{t_n}{2(1-\rho)}M. \tag{3.33}$$

Hence, the weak convergence of $\{Sx_n\}$ to x^* implies that $x_n \to x^*$ strongly. This has proved the relative norm compactness of the net $\{x_t\}$ as $t \to 0$. This completes the proof. □

Now, we show our first main result.

Theorem 3.3. *The net $\{x_t\}$ generated by the implicit method* (3.4) *converges in norm, as $t \to 0$, to the unique solution x^* of the following variational inequality:*

$$x^* \in \Gamma, \quad \langle (I-f)x^*, x - x^*\rangle \geq 0, \quad x \in \Gamma. \tag{3.34}$$

In particular, if we take $f = 0$*, then the net* $\{x_t\}$ *defined by* (3.2) *converges in norm, as* $t \to 0$*, to a solution of the minimization problem* (3.1)*.*

Proof. Now we return to (3.24) in Lemma 3.2 and take the limit as $n \to \infty$ to get

$$\|x^* - z\|^2 \leq \frac{1}{1-\rho}\langle z - f(z), z - x^*\rangle, \quad z \in \Gamma. \tag{3.35}$$

In particular, x^* solves the following variational inequality

$$x^* \in \Gamma, \quad \langle (I - f)z, z - x^*\rangle \geq 0, \quad z \in \Gamma \tag{3.36}$$

or the equivalent dual variational inequality:

$$x^* \in \Gamma, \quad \langle (I - f)x^*, z - x^*\rangle \geq 0, \quad z \in \Gamma. \tag{3.37}$$

Therefore, $x^* = (P_\Gamma f)x^*$. That is, x^* is the unique fixed point in Γ of the contraction $P_\Gamma f$. Clearly this is sufficient to conclude that the entire net $\{x_t\}$ converges in norm to x^* as $t \to 0$.

Finally, if we take $f = 0$, then (3.35) is reduced to

$$\|x^* - z\|^2 \leq \langle z, z - x^*\rangle, \quad z \in \Gamma. \tag{3.38}$$

Equivalently,

$$\|x^*\|^2 \leq \langle x^*, z\rangle, \quad z \in \Gamma. \tag{3.39}$$

This clearly implies that

$$\|x^*\| \leq \|z\|, \quad z \in \Gamma. \tag{3.40}$$

Therefore, x^* is a solution of minimization problem (3.1). This completes the proof. □

Next we introduce an explicit algorithm for finding a solution of minimization problem (3.1). This scheme is obtained by discretizing the implicit scheme (3.4).

Algorithm 3.4. Given that $x_0 \in C$ arbitrarily, let the sequence $\{x_n\}$ be generated iteratively by

$$x_{n+1} = \mu_n P_C\left[\alpha_n f(x_n) + (1 - \alpha_n)Sx_n\right] + (1 - \mu_n)T_r(x_n - rAx_n), \quad n \geq 0, \tag{3.41}$$

where $\{\alpha_n\}$ and $\{\mu_n\}$ are two sequences in $[0, 1]$.

Next, we give several lemmas in order to prove our second main result.

Lemma 3.5. *The sequence* $\{x_n\}$ *generated by* (3.41) *is bounded.*

Proof. Pick $z \in \Gamma$. Let $u_n = T_r(x_n - rAx_n)$ and $y_n = tf(x_n) + (1-t)Sx_n$ for all $n \geq 0$. From (3.41), we get

$$\begin{aligned}
\|u_n - z\| &= \|T_r(x_n - rAx_n) - T_r(z - rAz)\| \\
&\leq \|x_n - z\|, \\
\|x_{n+1} - z\| &= \left\|\mu_n\left(P_C\left[y_n\right] - z\right) + (1-\mu_n)(u_n - z)\right\| \\
&\leq \mu_n\left\|P_C\left[y_n\right] - z\right\| + (1-\mu_n)\|u_n - z\| \\
&\leq \mu_n\left\|y_n - z\right\| + (1-\mu_n)\|x_n - z\| \\
&\leq \mu_n\alpha_n\left\|f(x_n) - z\right\| + \mu_n(1-\alpha_n)\|Sx_n - z\| + (1-\mu_n)\|x_n - z\| \\
&\leq \left[1 - (1-\rho)\mu_n\alpha_n\right]\|x_n - z\| + \mu_n\alpha_n\left\|f(z) - z\right\|.
\end{aligned} \tag{3.42}$$

By induction, we obtain, for all $n \geq 0$,

$$\|x_n - z\| \leq \max\left\{\|x_0 - z\|, \frac{\|f(z) - z\|}{1-\rho}\right\}. \tag{3.43}$$

Hence, $\{x_n\}$ is bounded. Consequently, we deduce that $\{u_n\}$, $\{f(x_n)\}$ and $\{Ax_n\}$ are all bounded. This completes the proof. □

Lemma 3.6. *Assume the sequences* $\{\alpha_n\}$ *and* $\{\mu_n\}$ *satisfy the following conditions:*

(i) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\lim_{n\to\infty}(\alpha_{n+1}/\alpha_n) = 1$;

(ii) $0 < \liminf_{n\to\infty}\mu_n \leq \limsup_{n\to\infty}\mu_n < 1$ *and* $\lim_{n\to\infty}((\mu_{n+1} - \mu_n)/\alpha_{n+1}) = 1$.

Then $\lim_{n\to\infty}\|x_n - Sx_n\| = 0$.

Proof. From (3.41), we have

$$\begin{aligned}
\|x_{n+2} - x_{n+1}\| &= \left\|\mu_{n+1}P_C\left[y_{n+1}\right] + (1-\mu_{n+1})u_{n+1} - \mu_n P_C\left[y_n\right] - (1-\mu_n)u_n\right\| \\
&= \left\|\mu_{n+1}\left(P_C\left[y_{n+1}\right] - P_C\left[y_n\right]\right) + (\mu_{n+1} - \mu_n)P_C\left[y_n\right]\right. \\
&\quad \left.+ (1-\mu_{n+1})(u_{n+1} - u_n) + (\mu_n - \mu_{n+1})u_n\right\| \\
&\leq \mu_{n+1}\left\|y_{n+1} - y_n\right\| + (1-\mu_{n+1})\|u_{n+1} - u_n\| \\
&\quad + \left|\mu_{n+1} - \mu_n\right|\left(\left\|P_C\left[y_n\right]\right\| + \|u_n\|\right).
\end{aligned} \tag{3.44}$$

Next we estimate $\|y_{n+1} - y_n\|$ and $\|u_{n+1} - u_n\|$. We have

$$\begin{aligned}\left\|y_{n+1} - y_n\right\| &= \left\|\alpha_{n+1} f(x_{n+1}) + (1-\alpha_{n+1}) S x_{n+1} - \alpha_n f(x_n) + (1-\alpha_n) S x_n\right\| \\ &= \left\|\alpha_{n+1}\left(f(x_{n+1}) - f(x_n)\right) + (\alpha_{n+1} - \alpha_n) f(x_n)\right. \\ &\quad \left. + (1-\alpha_{n+1})(S x_{n+1} - S x_n) + (\alpha_n - \alpha_{n+1}) S x_n\right\| \\ &\leq \left[1 - (1-\rho)\alpha_{n+1}\right]\|x_{n+1} - x_n\| + |\alpha_{n+1} - \alpha_n|\left(\left\|f(x_n)\right\| + \|S x_n\|\right), \\ \|u_{n+1} - u_n\| &= \|T_r(x_{n+1} - r A x_{n+1}) - T_r(x_n - r A x_n)\| \\ &\leq \|(x_{n+1} - r A x_{n+1}) - (x_n - r A x_n)\| \\ &\leq \|x_{n+1} - x_n\|.\end{aligned} \tag{3.45}$$

Then, we obtain

$$\begin{aligned}\|x_{n+2} - x_{n+1}\| \leq &\left[1 - (1-\rho)\alpha_{n+1}\mu_{n+1}\right]\|x_{n+1} - x_n\| \\ &+ \left(|\alpha_{n+1} - \alpha_n| + \left|\mu_{n+1} - \mu_n\right|\right) M,\end{aligned} \tag{3.46}$$

where $M > 0$ is a constant satisfying

$$\sup_n\left\{\left\|f(x_n)\right\| + \|S x_n\|, \left\|P_C[y_n]\right\| + \|u_n\|, \left\|f(x_n) - z\right\|^2, 2\|x_n - z\|, 2r\|x_n - u_n\|\right\} \leq M. \tag{3.47}$$

This together with (i), (ii) and Lemma 2.4 imply that

$$\lim_{n \to \infty}\|x_{n+2} - x_{n+1}\| = 0. \tag{3.48}$$

By the convexity of the norm $\|\cdot\|$, we have

$$\begin{aligned}\|x_{n+1} - z\|^2 &= \left\|\mu_n(P_C[y_n] - z) + (1-\mu_n)(u_n - z)\right\|^2 \\ &\leq \mu_n\left\|P_C[y_n] - z\right\|^2 + (1-\mu_n)\|u_n - z\|^2 \\ &\leq \mu_n\left\|\alpha_n(f(x_n) - z) + (1-\alpha_n)(S x_n - z)\right\|^2 + (1-\mu_n)\|u_n - z\|^2 \\ &\leq \mu_n\left[\alpha_n\left\|f(x_n) - z\right\|^2 + (1-\alpha_n)\|S x_n - z\|^2\right] + (1-\mu_n)\|u_n - z\|^2 \\ &\leq \mu_n\left[\alpha_n M + (1-\alpha_n)\|x_n - z\|^2\right] + (1-\mu_n)\|u_n - z\|^2.\end{aligned} \tag{3.49}$$

From Lemma 2.2, we get

$$\begin{aligned}\|u_n - z\|^2 &= \|T_r(x_n - rAx_n) - T_r(z - rAz)\|^2 \\ &\le \|(x_n - rAx_n) - (z - rAz)\|^2 \\ &\le \|x_n - z\|^2 + r(r - 2\alpha)\|Ax_n - Az\|^2.\end{aligned} \tag{3.50}$$

Substituting (3.50) into (3.49), we have

$$\|x_{n+1} - z\|^2 \le (1 - \mu_n\alpha_n)\|x_n - z\|^2 + r(r - 2\alpha)(1 - \mu_n)\|Ax_n - Az\|^2 + \mu_n\alpha_n M. \tag{3.51}$$

Therefore,

$$\begin{aligned}r(2\alpha - r)(1 - \mu_n)\|Ax_n - Az\|^2 &\le (1 - \mu_n\alpha_n)\|x_n - z\|^2 - \|x_{n+1} - z\|^2 + \mu_n\alpha_n M \\ &\le (\|x_n - z\| + \|x_{n+1} - z\|)\|x_n - x_{n+1}\| + \alpha_n M \\ &\le (\|x_n - x_{n+1}\| + \alpha_n)M.\end{aligned} \tag{3.52}$$

Since $\liminf_{n\to\infty}(1 - \mu_n)r(2\alpha - r) > 0$, $\|x_n - x_{n+1}\| \to 0$ and $\alpha_n \to 0$, we derive

$$\lim_{n\to\infty}\|Ax_n - Az\| = 0. \tag{3.53}$$

From Lemma 2.1 and (3.41), we obtain

$$\begin{aligned}\|u_n - z\|^2 &= \|T_r(x_n - rAx_n) - T_r(z - rAz)\|^2 \\ &\le \langle (x_n - rAx_n) - (z - rAz), u_n - z\rangle \\ &= \frac{1}{2}\Big(\|(x_n - rAx_n) - (z - rAz)\|^2 + \|u_n - z\|^2 \\ &\quad - \|(x_n - z) - r(Ax_n - Az) - (u_n - z)\|^2\Big) \\ &\le \frac{1}{2}\Big(\|x_n - z\|^2 + \|u_n - z\|^2 - \|(x_n - u_n) - r(Ax_n - Az)\|^2\Big) \\ &= \frac{1}{2}\Big(\|x_n - z\|^2 + \|u_n - z\|^2 - \|x_n - u_n\|^2 + 2r\langle x_n - u_n, Ax_n - Az\rangle \\ &\quad - r^2\|Ax_n - Az\|^2\Big).\end{aligned} \tag{3.54}$$

Thus, we deduce

$$\begin{aligned}\|u_n - z\|^2 &\le \|x_n - z\|^2 - \|x_n - u_n\|^2 + 2r\|x_n - u_n\|\|Ax_n - Az\| \\ &\le \|x_n - z\|^2 - \|x_n - u_n\|^2 + M\|Ax_n - Az\|.\end{aligned} \tag{3.55}$$

By (3.49) and (3.55), we have

$$\begin{aligned}\|x_{n+1}-z\|^2 &\le \mu_n\left[\alpha_n M+(1-\alpha_n)\|x_n-z\|^2\right]+(1-\mu_n)\|u_n-z\|^2\\&\le \mu_n\left[\alpha_n M+(1-\alpha_n)\|x_n-z\|^2\right]\\&\quad+(1-\mu_n)\left[\|x_n-z\|^2-\|x_n-u_n\|^2+M\|Ax_n-Az\|\right]\\&\le \|x_n-z\|^2-(1-\mu_n)\|x_n-u_n\|^2+(\|Ax_n-Az\|+\alpha_n)M.\end{aligned} \tag{3.56}$$

It follows that

$$(1-\mu_n)\|x_n-u_n\|^2 \le (\|x_{n+1}-x_n\|+\|Ax_n-Az\|+\alpha_n)M. \tag{3.57}$$

Since $\liminf_{n\to\infty}(1-\mu_n)>0$, $\alpha_n \to 0$, $\|x_{n+1}-x_n\| \to 0$ and $\|Ax_n-Az\| \to 0$, we derive that

$$\lim_{n\to\infty}\|x_n-u_n\|=0. \tag{3.58}$$

Note that $x_{n+1}-x_n=\mu_n(P_C[y_n]-x_n)+(1-\mu_n)(u_n-x_n)$. Hence,

$$\left\|P_C[y_n]-x_n\right\| \longrightarrow 0. \tag{3.59}$$

Therefore,

$$\begin{aligned}\|Sx_n-x_n\| &\le \left\|Sx_n-P_C[y_n]\right\|+\left\|P_C[y_n]-x_n\right\|\\&\le \left\|Sx_n-y_n\right\|+\left\|P_C[y_n]-x_n\right\|\\&\le \alpha_n\left\|f(x_n)-Sx_n\right\|+\left\|P_C[y_n]-x_n\right\| \longrightarrow 0.\end{aligned} \tag{3.60}$$

This completes the proof. □

Now, we show the strong convergence of the sequence $\{x_n\}$ generated by (3.41).

Theorem 3.7. *Assume the sequences $\{\alpha_n\}$ and $\{\mu_n\}$ satisfy the following conditions:*

(i) $\lim_{n\to\infty}\alpha_n=0$, $\sum_{n=0}^{\infty}\alpha_n=\infty$ *and* $\lim_{n\to\infty}(\alpha_{n+1}/\alpha_n)=1$;

(ii) $0<\liminf_{n\to\infty}\mu_n\le\limsup_{n\to\infty}\mu_n<1$ *and* $\lim_{n\to\infty}((\mu_{n+1}-\mu_n)/\alpha_{n+1})=1$.

Then the sequence $\{x_n\}$ generated by (3.41) *converges strongly to x^* which is the unique solution of variational inequality* (3.34). *In particular, if $f=0$, then the sequence $\{x_n\}$ generated by*

$$x_{n+1}=\mu_n P_C[(1-\alpha_n)Sx_n]+(1-\mu_n)T_r(x_n-rAx_n), \quad n\ge 0, \tag{3.61}$$

converges strongly to a solution of the minimization problem (3.1).

Proof. We first prove

$$\limsup_{n\to\infty}\langle x^* - f(x^*), x^* - Sx_n\rangle \le 0 \tag{3.62}$$

where $x^* = P_\Gamma f(x^*)$.

Indeed, we can choose a subsequence $\{Sx_{n_i}\}$ of $\{Sx_n\}$ such that

$$\limsup_{n\to\infty}\langle x^* - f(x^*), x^* - Sx_n\rangle = \lim_{i\to\infty}\langle x^* - f(x^*), x^* - Sx_{n_i}\rangle. \tag{3.63}$$

Without loss of generality, we may further assume that $Sx_{n_i} \to \tilde{x}$ weakly. By the same argument as that of Theorem 3.3, we can deduce that $\tilde{x} \in \Gamma$. Therefore,

$$\limsup_{n\to\infty}\langle x^* - f(x^*), x^* - Sx_n\rangle = \langle x^* - f(x^*), x^* - \tilde{x}\rangle \le 0. \tag{3.64}$$

From (3.41), we have

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \left\|\mu_n(P_C[y_n] - z) + (1-\mu_n)(u_n - z)\right\|^2 \\
&\le \mu_n\left\|P_C[y_n] - z\right\|^2 + (1-\mu_n)\|u_n - z\|^2 \\
&\le \mu_n\left\|y_n - x^*\right\|^2 + (1-\mu_n)\|x_n - x^*\|^2 \\
&= \mu_n\Big[(1-\alpha_n)^2\|Sx_n - x^*\|^2 + 2\alpha_n(1-\alpha_n)\langle f(x_n) - f(x^*), Sx_n - x^*\rangle \\
&\qquad + 2\alpha_n(1-\alpha_n)\langle f(x^*) - x^*, Sx_n - x^*\rangle + \alpha_n^2\left\|f(x_n) - x^*\right\|^2\Big] \\
&\quad + (1-\mu_n)\|x_n - x^*\|^2 \\
&\le \mu_n\Big[(1-\alpha_n)^2\|x_n - x^*\|^2 + 2\alpha_n(1-\alpha_n)\rho\|x_n - x^*\|^2 \\
&\qquad + 2\alpha_n(1-\alpha_n)\langle f(x^*) - x^*, Sx_n - x^*\rangle + \alpha_n^2 M\Big] + (1-\mu_n)\|x_n - x^*\|^2 \\
&\le [1 - 2(1-\rho)\mu_n\alpha_n]\|x_n - x^*\|^2 + 2\alpha_n\mu_n(1-\alpha_n)\langle f(x^*) - x^*, Sx_n - x^*\rangle + 2\alpha_n^2\mu_n M \\
&= (1-\gamma_n)\|x_n - x^*\|^2 + \delta_n\gamma_n,
\end{aligned} \tag{3.65}$$

where $\gamma_n = 2(1-\rho)\mu_n\alpha_n$ and $\delta_n = ((1-\alpha_n)/(1-\rho))\langle f(x)^* - x^*, Sx_n - x^*\rangle + \alpha_n M/(1-\rho)$. It is clear that $\sum_{n=0}^{\infty}\gamma_n = \infty$ and $\limsup_{n\to\infty}\delta \le 0$. Hence, all conditions of Lemma 2.4 are satisfied. Therefore, we immediately deduce that $x_n \to x^*$.

Finally, if we take $f = 0$, by the similar argument as that Theorem 3.3, we deduce immediately that x^* is a minimum norm element in Γ. This completes the proof. □

4. Conclusions

Iterative methods for finding the common element of the equilibrium problem and the fixed point problem have been extensively studied, see, for example, [2, 4, 6, 7, 9, 14, 15, 21, 23–28]. However, iterative methods for finding the minimum norm solution of the equilibrium problem and the fixed point problem are far less developed than those for only finding the common element of the equilibrium problem and the fixed point problem. In the present paper, we suggest two algorithm, one implicit algorithm (3.4) and one explicit algorithm (3.41). We prove the strong convergence of the algorithms (3.4) and (3.41) to the common element of the equilibrium problem and the fixed points set of a nonexpansive mapping. As special cases, we prove that algorithms (3.2) and (3.61) converges to x^* which solves the minimization problem (3.1). It should be pointed out that our algorithms and our main results are new even if we assume f is a self-mapping on C.

Since in many problems, it is needed to find a solution with minimum norm. Hence, it is a very interesting problem to construct some algorithms for finding the minimum norm solution of some practical problem. The reader can develop iterative algorithms for solving some minimization problems by using our methods and technique contained in the present paper.

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