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Research Article

A New Subclass of Salagean-Type Harmonic Univalent Functions

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We define and investigate a new subclass of Salagean-type harmonic univalent functions. We obtain coefficient conditions, extreme points, distortion bounds, convolution, and convex combination for the above subclass of harmonic functions.

1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$.

We denote the subclass of A consisting of analytic and univalent functions $f(z)$ in the unit disk $\mathbb{U}$ by S.

The following classes of functions and many others are well known and have been studied repeatedly by many authors, namely, Sălăgean [1], Abdul Halim [2], and Darus [3] to mention but a few.

(i) $S_0 = \{f(z) \in \mathcal{A} : \text{Re}\{f(z)/z\} > 0,\ z \in \mathbb{U}\}$.

(ii) $B(\alpha) = \{f(z) \in \mathcal{A} : \text{Re}\{f(z)/z\} > \alpha,\ 0 \leq \alpha < 1,\ z \in \mathbb{U}\}$.

(iii) $\delta(\alpha) = \{f(z) \in \mathcal{A} : \operatorname{Re}\{f'(z)\} > \alpha,\ 0 \le \alpha < 1,\ z \in \mathbb{U}\}$.

(iv) $B_n(\beta) = \{f(z) \in \mathcal{A} : \operatorname{Re}\{D^n f(z)^\beta / z^\beta\} > 0,\ z \in \mathrm{U},\ n \in \mathbb{N}_0 = \mathrm{N} \cup \{0\},\ \beta > 0\}$.

In 1994, Opoola defined the class $T_n^\beta(\alpha)$ to be a subclass of A consisting of analytic functions satisfying the condition

$$\operatorname{Re}\left\{\frac{D^n f(z)^\beta}{z^\beta}\right\} > \alpha, \quad z \in \mathbb{U},\ n \in \mathbb{N}_0,\ 0 \le \alpha < 1,\ \beta > 0, \tag{1.2}$$

where D^n is the Salagean differential operator defined as follows:

$$\begin{aligned} D^0 f(z) &= f(z), \\ D^1 f(z) &= Df(z) = zf'(z), \\ D^n f(z) &= D\left(D^{n-1} f(z)\right) = z\left(D^{n-1} f(z)\right). \end{aligned} \tag{1.3}$$

We note that $T_n^\beta(\alpha)$ is a generalization of the classes of functions $S_0, B(\alpha), \delta(\alpha)$, and $B_n(\beta)$.

Some properties of this class of functions were established by Opoola [4] namely,

(i) $T_n^\beta(\alpha)$ is a subclass of univalent functions;

(ii) $T_{n+1}^\beta(\alpha) \subset T_n^\beta(\alpha)$;

(iii) if $f(z) \in T_n^\beta(\alpha)$, then the integral operator

$$F_c(z)^\beta = \frac{\beta + c}{z^\beta} \int_0^c t^{\beta-1} f(z)^\beta dt \quad (c \ge 0) \tag{1.4}$$

is also in $T_n^\beta(\alpha)$.

Now, by Binomial expansion, we have

$$\begin{aligned} f(z)^\beta = z^\beta + \beta a_2 z^{\beta+1} &+ \left[\beta a_3 + \frac{\beta(\beta-1)}{2!}\ a_2^3\right] z^{\beta+2} \\ &+ \left[\beta a_4 + \frac{\beta(\beta-1)}{2!}\ 2a_2 a_3 + \frac{\beta(\beta-1)(\beta-2)}{3!}\ a_2^3\right] z^{\beta+3} + \cdots . \end{aligned} \tag{1.5}$$

Hence, we define

$$\begin{aligned} f(z)^\beta &= z^\beta + \sum_{k=2}^{\infty} \beta a_k z^{\beta+k-1}, \quad \beta > 0, \\ D^n f(z)^\beta &= z^\beta + \sum_{k=2}^{\infty} \beta k^n a_k z^{\beta+k-1}, \quad n \in \mathbb{N}_0. \end{aligned} \tag{1.6}$$

2. Preliminaries

A continuous function $f = u + iv$ is a complex-valued harmonic function in a domain $D \subset \mathbb{C}$ if both u and v are real harmonic in D. In any simply connected domain, we can write

$$f = h + \overline{g}, \tag{2.1}$$

where h and g are analytic in D. We call h the analytic part and g the coanalytic part of f. A necessary and sufficient condition for f to be locally univalent and sense-preserving in D is that $|h'| > |g'|$ in D.

Denote by $S_{\mathcal{H}}$ the class of functions f of the form (2.1) that are harmonic univalent and sense-preserving in the unit disk $\mathbb{U}$. The subclasses of harmonic univalent functions have been studied by some authors for different purposes and different properties (see examples [5–12]). In this work, we may express the analytic functions h and g as

$$h(z)^{\beta} = z^{\beta} + \sum_{k=2}^{\infty} \beta a_k z^{\beta+k-1}, \quad g(z)^{\beta} = \sum_{k=1}^{\infty} \beta b_k z^{\beta+k-1}, \quad |b_1| < 1. \tag{2.2}$$

Therefore,

$$f(z)^{\beta} = h(z)^{\beta} + \overline{g(z)^{\beta}}. \tag{2.3}$$

We define the modified Salagean operator of f as

$$D^n f(z)^{\beta} = D^n h(z)^{\beta} + (-1)^n \overline{D^n g(z)^{\beta}}, \tag{2.4}$$

where

$$D^n h(z)^{\beta} = z^{\beta} + \sum_{k=2}^{\infty} \beta k^n a_k z^{\beta+k-1}, \qquad D^n g(z)^{\beta} = \sum_{k=1}^{\infty} \beta k^n b_k z^{\beta+k-1}. \tag{2.5}$$

We let $\mathcal{H}(n, \beta, \alpha)$ be the family of harmonic functions of the form (2.3) such that

$$\operatorname{Re}\left\{ \frac{D^{n+1} f(z)^{\beta}}{D^n f(z)^{\beta}} \right\} > \alpha, \quad \beta \geq 1, \ 0 \leq \alpha < 1, \ n \in \mathbb{N}_0, \tag{2.6}$$

where $D^n f(z)^{\beta}$ is defined by (2.4).

It is clear that the class $\mathcal{H}(n, \beta, \alpha)$ includes a variety of well-known subclasses of $S_{\mathcal{H}}$. For example, $\mathcal{H}(0, 1, \alpha) \equiv S^*_{\mathcal{H}}(\alpha)$ is the class of sense-preserving, harmonic univalent functions f which are starlike of order α in $\mathbb{U}$, that is, $\partial/\partial\theta\{\arg(f(re^{i\theta}))\} > \alpha$, and $\mathcal{H}(1, 1, \alpha) \equiv \mathcal{H}\mathcal{K}(\alpha)$ is the class of sense-preserving, harmonic univalent functions f which are convex of order α in $\mathbb{U}$, that is $\partial/\partial\theta\{\arg(\partial/\partial\theta f(re^{i\theta}))\} > \alpha$. Note that the classes $S^*_{\mathcal{H}}(\alpha)$ and $\mathcal{H}\mathcal{K}(\alpha)$

were introduced and studied by Jahangiri [5]. Also note that the class $\mathcal{H}(n,1,\alpha) \equiv \mathcal{HK}(\alpha)$ is the class of Salagean-type harmonic univalent functions introduced by Jahangiri et al. [13].

We let the subclass $\overline{\mathcal{H}}(n,\beta,\alpha)$ consist of harmonic functions $f_n = h + \overline{g_n}$ in $\mathcal{H}(n,\beta,\alpha)$ so h and g are of the form

$$h^{\beta}(z) = z^{\beta} - \sum_{k=2}^{\infty} |a_k|\, z^{\beta+k-1}, \qquad g_n^{\beta}(z) = (-1)^n \sum_{k=1}^{\infty} |b_k|\, z^{\beta+k-1}. \tag{2.7}$$

In 1984, Clunie and Sheil-Small [14] investigated the class $S_{\mathcal{H}}$ as well as its geometric subclasses and obtained some coefficient bounds. Since then, there have been several related papers on $S_{\mathcal{H}}$ and its subclasses such that Silverman [15], Silverman and Silvia [16], and Jahangiri [5, 17] studied the harmonic univalent functions. Jahangiri [5] proved the following theorem.

Theorem 2.1. *Let $f = h + \overline{g}$ where $h = z + \sum_{k=2}^{\infty} a_k z^k$ and $g = \sum_{k=1}^{\infty} b_k z^k$. If*

$$\sum_{k=1}^{\infty} \frac{k-\alpha}{1-\alpha}|a_k| + \frac{k+\alpha}{1-\alpha}|b_k| \le 2, \quad (0 \le \alpha < 1), \tag{2.8}$$

then f is sense-preserving, harmonic, and univalent in $\mathbb{U}$ and $f \in S^*_{\mathcal{H}}(\alpha)$. The condition (2.8) is also necessary if $f \in \mathcal{T}H(\alpha) \equiv \overline{\mathcal{H}}(0,1,\alpha)$.

In this paper, we will give the sufficient condition for functions $f^{\beta} = h^{\beta} + \overline{g^{\beta}}$ where h^{β} and g^{β} are given by (2.2) to be in the class $\mathcal{H}(n,\beta,\alpha)$ and it is shown that these coefficient conditions are also necessary for functions in the class $\overline{\mathcal{H}}(n,\beta,\alpha)$. Also, we obtain distortion theorems and characterize the extreme points for functions in $\overline{\mathcal{H}}(n,\beta,\alpha)$. Convolution and convex combination are also obtained.

3. Main Results

In this section, we prove the main results.

3.1. Coefficient Estimates

Theorem 3.1. *Let $f^{\beta} = h^{\beta} + \overline{g^{\beta}}$, where h^{β} and g^{β} are given by (2.2). If*

$$\sum_{k=1}^{\infty} [(k-\alpha)|a_k| + (k+\alpha)|b_k|]\beta k^n \le (1+\beta)(1-\alpha), \tag{3.1}$$

where $a_1 = 1$, $n \in \mathbb{N}_0$, $\beta \ge 1$, and $0 \le \alpha < 1$, then f^{β} is sense-preserving, harmonic univalent in U, and $f \in \mathcal{H}(n,\beta,\alpha)$.

Proof. If $z_1^\beta \neq z_2^\beta$, then

$$\left|\frac{f(z_1)^\beta - f(z_2)^\beta}{h(z_1)^\beta - h(z_2)^\beta}\right| \geq 1 - \left|\frac{g(z_1)^\beta - g(z_2)^\beta}{h(z_1)^\beta - h(z_2)^\beta}\right| = 1 - \left|\frac{\sum_{k=1}^{\infty} \beta b_k \left(z_1^{k+\beta-1} - z_2^{k+\beta-1}\right)}{\left(z_1^\beta - z_2^\beta\right) + \sum_{k=2}^{\infty} \beta a_k \left(z_1^{k+\beta-1} - z_2^{k+\beta-1}\right)}\right|$$
$$> 1 - \frac{\sum_{k=1}^{\infty}(k+\beta-1)b_k}{1-\sum_{k=2}^{\infty}(k+\beta-1)a_k} \geq 1 - \frac{\sum_{k=1}^{\infty}(k+\alpha)\beta k^n/(1-\alpha)|b_k|}{1-\sum_{k=2}^{\infty}(k-\alpha)\beta k^n/(1-\alpha)|a_k|} \geq 0, \tag{3.2}$$

which proves univalence. Note that f is sense-preserving in $\mathbb{U}$. This is because

$$\left|h'(z)^\beta\right| \geq \beta\left(|z|^{\beta-1} - \sum_{k=2}^{\infty}(k+\beta-1)|a_k||z|^{k+\beta-2}\right) > \beta\left(1 - \sum_{k=2}^{\infty}\frac{(k-\alpha)\beta k^n}{1-\alpha}|a_k|\right)$$
$$\geq \beta\left(\sum_{k=1}^{\infty}\frac{(k+\alpha)\beta k^n}{1-\alpha}|b_k|\right) \geq \sum_{k=1}^{\infty}\beta(k+\beta-1)|b_k||z|^{k+\beta-2} \geq \left|g'(z)^\beta\right|. \tag{3.3}$$

By (2.6),

$$\operatorname{Re}\left\{\frac{D^{n+1}f(z)^\beta}{D^n f(z)^\beta}\right\} = \operatorname{Re}\left\{\frac{D^{n+1}h(z)^\beta + \overline{(-1)^{n+1}D^{n+1}g(z)^\beta}}{D^n h(z)^\beta + \overline{(-1)^n D^n g(z)^\beta}}\right\} > \alpha. \tag{3.4}$$

Using the fact that $\operatorname{Re}(w) > \alpha$ if and only if $|1-\alpha+w| \geq |1+\alpha-w|$, it suffices to show that

$$\left|1-\alpha+\frac{D^{n+1}f(z)^\beta}{D^n f(z)^\beta}\right| - \left|1+\alpha-\frac{D^{n+1}f(z)^\beta}{D^n f(z)^\beta}\right| \geq 0, \tag{3.5}$$

$$\left|D^{n+1}f(z)^\beta + (1-\alpha)D^n f(z)^\beta\right| - \left|D^{n+1}f(z)^\beta - (1+\alpha)D^n f(z)^\beta\right| \geq 0. \tag{3.6}$$

Substituting for $D^{n+1}f(z)^\beta$, $D^n f(z)^\beta$ in (3.6), we have

$$\left|D^{n+1}h(z)^\beta + \overline{(-1)^{n+1}D^{n+1}g(z)^\beta} + (1-\alpha)\left[D^n h(z)^\beta + \overline{(-1)^n D^n g(z)^\beta}\right]\right|$$
$$-\left|D^{n+1}h(z)^\beta + \overline{(-1)^{n+1}D^{n+1}g(z)^\beta} - (1+\alpha)\left[D^n h(z)^\beta + \overline{(-1)^n D^n g(z)^\beta}\right]\right|$$

$$
\begin{aligned}
&= \left| z^{\beta} + \sum_{k=2}^{\infty} \beta k^{n+1} a_k z^{\beta+k-1} + (-1)^{n+1} \sum_{k=1}^{\infty} \beta k^{n+1} \overline{b_k z^{\beta+k-1}} \right.\\
&\qquad \left. +(1-\alpha)\left[z^{\beta} + \sum_{k=2}^{\infty} \beta k^{n} a_k z^{\beta+k-1} + (-1)^{n} \sum_{k=1}^{\infty} \beta k^{n} \overline{b_k z^{\beta+k-1}} \right] \right| \\
&\quad - \left| z^{\beta} + \sum_{k=2}^{\infty} \beta k^{n+1} a_k z^{\beta+k-1} + (-1)^{n+1} \sum_{k=1}^{\infty} \beta k^{n+1} \overline{b_k z^{\beta+k-1}} \right.\\
&\qquad \left. -(1+\alpha)\left[z^{\beta} + \sum_{k=2}^{\infty} \beta k^{n} a_k z^{\beta+k-1} + (-1)^{n} \sum_{k=1}^{\infty} \beta k^{n} \overline{b_k z^{\beta+k-1}} \right] \right| \\
&= \left| (2-\alpha) z^{\beta} + \sum_{k=2}^{\infty} \beta (k+1-\alpha) k^{n} a_k z^{\beta+k-1} - (-1)^{n} \sum_{k=1}^{\infty} \beta (k-1+\alpha) k^{n} b_k z^{\beta+k-1} \right| \\
&\quad - \left| (-\alpha) z^{\beta} + \sum_{k=2}^{\infty} \beta (k-1-\alpha) k^{n} a_k z^{\beta+k-1} - (-1)^{n} \sum_{k=1}^{\infty} \beta (k+1+\alpha) k^{n} b_k z^{\beta+k-1} \right| \\
&\geq 2(1-\alpha)|z|^{\beta} - \sum_{k=2}^{\infty} 2\beta k^{n}(k-\alpha)|a_k| \left| z^{\beta+k-1} \right| - \sum_{k=1}^{\infty} 2\beta k^{n}(k-\alpha)|b_k| \left| z^{\beta+k-1} \right| \\
&= 2(1-\alpha)\left[1 - \sum_{k=2}^{\infty} \beta k^{n} \frac{(k-\alpha)}{1-\alpha} |a_k| - \sum_{k=1}^{\infty} \beta k^{n} \frac{(k+\alpha)}{1-\alpha} |b_k| \right].
\end{aligned}
\tag{3.7}
$$

This last expression is nonnegative by (3.1), and so the proof is complete. □

The harmonic function

$$
f(z)^{\beta} = z^{\beta} + \sum_{k=2}^{\infty} \beta \frac{1-\alpha}{(k-\alpha)\beta k^{n}} x_k z^{k+\beta-1} + \sum_{k=1}^{\infty} \beta \frac{1-\alpha}{(k+\alpha)\beta k^{n}} \overline{y_k z^{k+\beta-1}}, \tag{3.8}
$$

where $n \in \mathbb{N}_0$, $\beta \geq 1$, $0 \leq \alpha < 1$, and $\sum_{k=2}^{\infty} |x_k| + \sum_{k=1}^{\infty} |y_k| = 1$, shows that the coefficient bound given by (3.1) is sharp. The functions of the form (3.8) are in $\mathscr{H}(n, \beta, \alpha)$ because

$$
\sum_{k=1}^{\infty} \left[\frac{k-\alpha}{1-\alpha} |a_k| + \frac{k+\alpha}{1-\alpha} |b_k| \right] \beta k^{n} = \beta + \sum_{k=2}^{\infty} |x_k| + \sum_{k=1}^{\infty} |y_k| = \beta + 1. \tag{3.9}
$$

In the following theorem, it is shown that the condition (3.1) is also necessary for functions $f_n^{\beta} = h^{\beta} + \overline{g_n^{\beta}}$ where h^{β} and g_n^{β} are of the form (2.7).

Theorem 3.2. *Let* $f_n^\beta = h^\beta + \overline{g_n^\beta}$ *be given by* (2.7). *Then* $f_n^\beta \in \overline{\mathcal{H}}(n,\beta,\alpha)$, *if and only if*

$$\sum_{k=1}^{\infty}[(k-\alpha)|a_k| + (k+\alpha)|b_k|]\beta k^n \le (1+\beta)(1-\alpha), \tag{3.10}$$

where $a_1 = 1$, $n \in \mathbb{N}_0$, $\beta \ge 1$, *and* $0 \le \alpha < 1$.

Proof. Since $\overline{\mathcal{H}}(n,\beta,\alpha) \subset \mathcal{H}(n,\beta,\alpha)$, we only need to prove the "only if" part of the theorem. To this end, for functions f_n^β of the form (2.7), we notice that the condition (2.6) is equivalent to

$$\operatorname{Re}\left\{\frac{(1-\alpha)z^\beta - \sum_{k=2}^{\infty}(k-\alpha)\beta k^n a_k z^{k+\beta-1} - (-1)^{2n}\sum_{k=1}^{\infty}(k+\alpha)\beta k^n b_k \overline{z^{k+\beta-1}}}{z^\beta - \sum_{k=2}^{\infty}\beta k^n a_k z^{k+\beta-1} + (-1)^n \sum_{k=1}^{\infty}\beta k^n b_k \overline{z^{k+\beta-1}}}\right\} \ge 0. \tag{3.11}$$

The above required condition (3.11) must hold for all values of z in $\mathbb{U}$. Upon choosing the values of z on the positive real axis where $0 \le z = r < 1$, we must have

$$\frac{1-\alpha - \sum_{k=2}^{\infty}(k-\alpha)\beta k^n a_k r^{k-1} - \sum_{k=1}^{\infty}(k+\alpha)\beta k^n b_k r^{k-1}}{1 - \sum_{k=2}^{\infty}\beta k^n a_k r^{k-1} + \sum_{k=1}^{\infty}\beta k^n \overline{b_k} r^{k-1}} \ge 0. \tag{3.12}$$

If the condition (3.10) does not hold, then the numerator in (3.12) is negative for r sufficiently close to 1. Hence there exist $z_0 = r_0$ in $(0,1)$ for which the quotient in (3.12) is negative. This contradicts the required condition for $f_n^\beta \in \overline{\mathcal{H}}(n,\lambda,\alpha)$ and so the proof is complete. □

3.2. Distortion Bounds and Extreme Points

In this section, first we will obtain distortion bounds for functions in $\overline{\mathcal{H}}(n,\beta,\alpha)$.

Theorem 3.3. *Let* $f_n^\beta \in \overline{\mathcal{H}}(n,\beta,\alpha)$. *Then for* $|z| = r < 1$, *we have*

$$\begin{aligned}\left|f_n(z)^\beta\right| &\le (1+|b_1|)r^\beta + \frac{1}{\beta 2^n}\left(\frac{1-\alpha}{2-\alpha} - \frac{1+\alpha}{2-\alpha}\beta|b_1|\right)r^{\beta+1},\\ \left|f_n(z)^\beta\right| &\ge (1-|b_1|)r^\beta - \frac{1}{\beta 2^n}\left(\frac{1-\alpha}{2-\alpha} - \frac{1+\alpha}{2-\alpha}\beta|b_1|\right)r^{\beta+1}.\end{aligned} \tag{3.13}$$

Proof. We only prove the right-hand inequality. The proof for the left-hand inequality is similar and will be omitted. Let $f_n^{\beta} \in \overline{\mathcal{H}}(n,\beta,\alpha)$. Taking the absolute value of f_n^{β}, we obtain

$$
\begin{aligned}
\left|f_n(z)^{\beta}\right| &= \left|z^{\beta} + \sum_{k=2}^{\infty} a_k z^{k+\beta-1} + (-1)^n \sum_{k=1}^{\infty} b_k \overline{z^{k+\beta-1}}\right| \\
&\leq (1+|b_1|)r^{\beta} + \sum_{k=2}^{\infty}(|a_k| + |b_k|) r^{k+\beta-1} \\
&\leq (1+|b_1|)r^{\beta} + r^{\beta+1}\sum_{k=2}^{\infty}(|a_k| + |b_k|) \\
&\leq (1+|b_1|)r^{\beta} + \frac{1-\alpha}{(2-\alpha)\beta 2^n}\left(\sum_{k=2}^{\infty} \frac{(2-\alpha)\beta 2^n}{1-\alpha}|a_k| + \frac{(2-\alpha)\beta 2^n}{1-\alpha}|b_k|\right) r^{\beta+1} \\
&\leq (1+|b_1|)r^{\beta} + \frac{1-\alpha}{(2-\alpha)\beta 2^n}\left(\sum_{k=2}^{\infty} \frac{(k-\alpha)\beta k^n}{1-\alpha}|a_k| + \frac{(k+\alpha)\beta k^n}{1-\alpha}|b_k|\right) r^{\beta+1} \\
&\leq (1+|b_1|)r^{\beta} + \frac{1-\alpha}{(2-\alpha)\beta 2^n}\left(1 - \frac{1+\alpha}{1-\alpha}\beta|b_1|\right) r^{\beta+1},
\end{aligned}
\tag{3.14}
$$

for $|b_1| < 1$. This shows that the bounds given in Theorem 3.3 are sharp. □

The following covering result follows from the left-hand inequality in Theorem 3.3.

Corollary 3.4. *If function $f_n^{\beta} = h^{\beta} + \overline{g^{\beta}}$, where h^{β} and g^{β} are given by* (2.7)*, is in $\overline{\mathcal{H}}(n,\beta,\alpha)$, then*

$$
\left\{ w : |w| < \frac{\beta 2^{n+1} - 1 - \left(\beta 2^n - 1\right)\alpha}{\beta 2^n (2-\alpha)} - \frac{2^{n+1}+1}{2^n(2-\alpha)}|b_1| \right\} \subset f_n(\mathbb{U}). \tag{3.15}
$$

Next we determine the extreme points of closed convex hulls of $\overline{\mathcal{H}}(n,\beta,\alpha)$ denoted by clco $\overline{\mathcal{H}}(n,\beta,\alpha)$.

Theorem 3.5. *Let $f_n^{\beta} = h^{\beta} + \overline{g^{\beta}}$, where h^{β} and g^{β} are given by* (2.7)*. Then $f_n^{\beta} \in \overline{\mathcal{H}}(n,\beta,\alpha)$ if and only if*

$$
f_n(z)^{\beta} = \sum_{k=1}^{\infty}\left(X_k h_k(z) + Y_k g_{n_k}(z)\right), \tag{3.16}
$$

where $h_1(z)^{\beta} = z^{\beta}$, $h_k(z)^{\beta} = z^{\beta} - (1-\alpha)/((k-\alpha)k^n) z^{k+\beta-1}$ $(k = 2,3,\ldots)$, $g_{n_k}(z)^{\beta} = z^{\beta} + (-1)^n(1-\alpha)/((k+\alpha)k^n)\overline{z^{k+\beta-1}}$ $(k = 1,2,3,..)$, and $\sum_{k=1}^{\infty}(X_k + Y_k) = 1$, $X_k \geq 0$, $Y_k \geq 0$. In particular, the extreme points of $\overline{\mathcal{H}}(n,\beta,\alpha)$ are $\{h_k\}$ and $\{g_{n_k}\}$.

Proof. For functions $f_n^\beta = h^\beta + \overline{g^\beta}$, where h^β and g^β are given by (3.16), we have

$$
\begin{aligned}
f_n(z)^\beta &= \sum_{k=1}^{\infty}\left(X_k h_k(z) + Y_k g_{n_k}(z)\right) \\
&= \sum_{k=1}^{\infty}(X_k + Y_k)z^\beta - \sum_{k=2}^{\infty}\frac{1-\alpha}{(k-\alpha)k^n}X_k z^{k+\beta-1} + (-1)^n\sum_{k=1}^{\infty}\frac{1-\alpha}{(k+\alpha)k^n}Y_k\overline{z^{k+\beta-1}}.
\end{aligned}
\tag{3.17}
$$

Then

$$
\sum_{k=2}^{\infty}\frac{(k-\alpha)\beta k^n}{1-\alpha}|a_k| + \sum_{k=1}^{\infty}\frac{(k+\alpha)\beta k^n}{1-\alpha}|b_k| = \sum_{k=2}^{\infty}X_k + \sum_{k=1}^{\infty}Y_k = 1 - X_1 \le 1,
\tag{3.18}
$$

and so $f_n^\beta \in \operatorname{clco}\ \overline{\mathcal{H}}(n,\beta,\alpha)$.

Conversely, suppose that $f_n^\beta \in \operatorname{clco}\ \overline{\mathcal{H}}(n,\beta,\alpha)$. Setting

$$
\begin{aligned}
X_k &= \frac{(k-\alpha)\beta k^n}{1-\alpha}|a_k| \quad 0 \le X_k \le 1\ (k = 2,3,\ldots), \\
Y_k &= \frac{(k+\alpha)\beta k^n}{1-\alpha}|b_k| \quad 0 \le Y_k \le 1\ (k = 1,2,3,\ldots),
\end{aligned}
\tag{3.19}
$$

and $X_1 = 1 - \sum_{k=2}^{\infty}X_k - \sum_{k=1}^{\infty}Y_k$; therefore, f_n^β can be written as

$$
\begin{aligned}
f_n(z)^\beta &= z^\beta - \sum_{k=2}^{\infty}\beta|a_k|z^{k+\beta-1} + (-1)^n\sum_{k=1}^{\infty}\beta|b_k|\overline{z^{k+\beta-1}} \\
&= z^\beta - \sum_{k=2}^{\infty}\frac{(1-\alpha)X_k}{(k-\alpha)k^n}z^{k+\beta-1} + (-1)^n\sum_{k=1}^{\infty}\frac{(1-\alpha)Y_k}{(k+\alpha)k^n}\overline{z^{k+\beta-1}} \\
&= z^\beta + \sum_{k=2}^{\infty}\left(h_k(z)^\beta - z^\beta\right)X_k + \sum_{k=1}^{\infty}\left(g_{n_k}(z)^\beta - z^\beta\right)Y_k \\
&= \sum_{k=2}^{\infty}h_k(z)^\beta X_k + \sum_{k=1}^{\infty}g_{n_k}(z)^\beta Y_k + z^\beta\left(1 - \sum_{k=2}^{\infty}X_k - \sum_{k=1}^{\infty}Y_k\right) \\
&= \sum_{k=1}^{\infty}\left(h_k(z)^\beta X_k + g_{n_k}(z)^\beta Y_k\right), \text{ as required.}
\end{aligned}
\tag{3.20}
$$

□

3.3. Convolution and Convex Combination

In this section, we show that the class $\overline{\mathcal{H}}(n,\beta,\alpha)$ is invariant under convolution and convex combination of its member.

For harmonic functions $f_n(z)^\beta = z^\beta - \sum_{k=2}^{\infty}|a_k|z^{k+\beta-1} + (-1)^n\sum_{k=1}^{\infty}|b_k|\overline{z^{k+\beta-1}}$ and $F_n(z)^\beta = z^\beta - \sum_{k=2}^{\infty}|A_k|z^{k+\beta-1} + (-1)^n\sum_{k=1}^{\infty}|B_k|\overline{z^{k+\beta-1}}$.

The convolution of f_n^β and F_n^β is given by

$$\left(f_n^\beta * F_n^\beta\right)(z) = f_n(z)^\beta * F_n(z)^\beta = z^\beta - \sum_{k=2}^{\infty}|a_k||A_k|z^{k+\beta-1} + (-1)^n\sum_{k=1}^{\infty}|b_k||B_k|\overline{z^{k+\beta-1}}. \tag{3.21}$$

Theorem 3.6. *For $0 \le \lambda \le \alpha < 1$, let $f_n^\beta \in \overline{\mathcal{H}}(n,\beta,\alpha)$ and $F_n^\beta \in \overline{\mathcal{H}}(n,\beta,\beta)$. Then $f_n^\beta * F_n^\beta \in \overline{\mathcal{H}}(n,\beta,\alpha) \subset \overline{\mathcal{H}}(n,\beta,\lambda)$.*

Proof. Let the functions $f_n(z)^\beta = z^\beta - \sum_{k=2}^{\infty}|a_k|z^{k+\beta-1} + (-1)^n\sum_{k=1}^{\infty}|b_k|\overline{z^{k+\beta-1}}$ be in the class $\overline{\mathcal{H}}(n,\beta,\alpha)$ and let the functions $F_n(z)^\beta = z^\beta - \sum_{k=2}^{\infty}|A_k|z^{k+\beta-1} + (-1)^n\sum_{k=1}^{\infty}|B_k|\overline{z^{k+\beta-1}}$ be in the class $\overline{\mathcal{H}}(n,\beta,\lambda)$. Then the convolution $f_n^\beta * F_n^\beta$ is given by (3.21). We wish to show that the coefficients of $f_n^\beta * F_n^\beta$ satisfy the required condition given in Theorem 3.2. For $F_n^\beta \in \overline{\mathcal{H}}(n,\beta,\lambda)$, we note that $|A_k| \le 1$ and $|B_k| \le 1$. Now, for the convolution function $f_n^\beta * F_n^\beta$, we obtain

$$\begin{aligned}
&\sum_{k=2}^{\infty}\frac{(k-\beta)\beta k^n}{1-\beta}|a_k||A_k| + \sum_{k=1}^{\infty}\frac{(k+\beta)\beta k^n}{1-\beta}|b_k||B_k| \\
&\quad\le \sum_{k=2}^{\infty}\frac{(k-\beta)\beta k^n}{1-\beta}|a_k| + \sum_{k=1}^{\infty}\frac{(k+\beta)\beta k^n}{1-\beta}|b_k| \\
&\quad\le \sum_{k=2}^{\infty}\frac{(k-\alpha)\beta k^n}{1-\alpha}|a_k| + \sum_{k=1}^{\infty}\frac{(k+\alpha)\beta k^n}{1-\alpha}|b_k| \le 1,
\end{aligned} \tag{3.22}$$

since $0 \le \lambda \le \alpha < 1$ and $f_n^\beta \in \overline{\mathcal{H}}(n,\beta,\alpha)$. Therefore, $f_n^\beta * F_n^\beta \in \overline{\mathcal{H}}(n,\beta,\alpha) \subset \overline{\mathcal{H}}(n,\beta,\lambda)$. □

We now examine the convex combination of $\overline{\mathcal{H}}(n,\beta,\alpha)$.

Let the functions $f_{n_j}(z)^\beta$ be defined, for $j = 1,2,\ldots$, by

$$f_{n_j}(z)^\beta = z^\beta - \sum_{k=2}^{\infty}\left|a_{k,j}\right|z^{k+\beta-1} + (-1)^n\sum_{k=1}^{\infty}\left|b_{k,j}\right|\overline{z^{k+\beta-1}}. \tag{3.23}$$

Theorem 3.7. *Let the functions $f_{n_j}(z)^\beta$ defined by* (3.23) *be in the class $\overline{\mathcal{H}}(n,\beta,\alpha)$ for every $j = 1,2,\ldots,m$. Then the functions $t_j(z)^\beta$ defined by*

$$t_j(z)^\beta = \sum_{j=1}^{m}c_j f_{n_j}(z) \quad (0 \le c_j \le 1) \tag{3.24}$$

are also in the class $\overline{\mathcal{H}}(n,\beta,\alpha)$ where $\sum_{j=1}^{m}c_j = 1$.

Proof. According to the definition of t^{β}, we can write

$$t(z)^{\beta} = z^{\beta} - \sum_{k=2}^{\infty}\left(\sum_{j=1}^{m} c_j a_{k,j}\right) z^{k+\beta-1} + (-1)^n \sum_{k=1}^{\infty}\left(\sum_{j=1}^{m} c_j b_{n,j}\right)\overline{z^{k+\beta-1}}. \quad (3.25)$$

Further, since $f_{n_j}(z)^{\beta}$ are in $\overline{\mathcal{H}}(n,\beta,\alpha)$ for every $(j = 1, 2, \ldots)$, then by (3.1) we have

$$\begin{aligned}
&\sum_{k=1}^{\infty}\left\{\left[(k-\alpha)\left(\sum_{j=1}^{m} c_j |a_{k,j}|\right) + (k+\alpha)\left(\sum_{j=1}^{m} c_j |b_{k,j}|\right)\right]\beta k^n\right\} \\
&\quad = \sum_{j=1}^{m} c_j\left(\sum_{k=1}^{\infty}\left[(k-\alpha)|a_{n,j}| + (k+\alpha)|b_{n,j}|\right]\beta k^n\right) \\
&\quad \le \sum_{j=1}^{m} c_j 2(1-\alpha) \le 2(1-\alpha).
\end{aligned} \quad (3.26)$$

Hence the theorem follows. □

Corollary 3.8. *The class $\overline{\mathcal{H}}(n,\beta,\alpha)$ is close under convex linear combination.*

Proof. Let the functions $f_{n_j}(z)^{\beta}(j = 1, 2)$ defined by (3.23) be in the class $M_{\overline{\mathcal{H}}}(n,\lambda,\alpha)$. Then the function $\Psi(z)^{\beta}$ defined by

$$\Psi(z)^{\beta} = \mu f_{n_1}(z)^{\beta} + (1-\mu) f_{n_2}(z)^{\beta} \quad (0 \le \mu \le 1) \quad (3.27)$$

is in the class $M_{\overline{\mathcal{H}}}(n,\lambda,\alpha)$. Also, by taking $m = 2$, $t_1 = \mu$, and $t_2 = (1-\mu)$ in Theorem 3.7, we have the above corollary. □

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References

[1] G. Ş. Sălăgean, "Subclasses of univalent functions," in *Complex Analysis—Fifth Romanian-Finnish Seminar*, vol. 1013 of *Lecture Notes in Mathematics*, pp. 362–372, Springer, Berlin, Germany, 1983.

[2] S. Abdul Halim, "On a class of analytic functions involving the Salagean differential operator," *Tamkang Journal of Mathematics*, vol. 23, no. 1, pp. 51–58, 1992.

[3] M Darus, "Fekete-Szego functional for functions in $B_n(\beta)$," *Institute of Mathematics & Computer Sciences. Mathematics Series*, vol. 17, no. 2, pp. 129–135, 2004.

[4] T. O. Opoola, "On a new subclass of univalent functions," *Mathematica*, vol. 36(59), no. 2, pp. 195–200, 1994.

[5] J. M. Jahangiri, "Harmonic functions starlike in the unit disk," *Journal of Mathematical Analysis and Applications*, vol. 235, no. 2, pp. 470–477, 1999.

[6] K. Al Shaqsi and M. Darus, "On subclass of harmonic starlike functions with respect to k -symmetric points," *International Mathematical Forum*, vol. 2, no. 57, pp. 2799–2805, 2007.

[7] K. Al-Shaqsi and M. Darus, "On harmonic univalent functions with respect to k -symmetric points," *International Journal of Contemporary Mathematical Sciences*, vol. 3, no. 3, pp. 111–118, 2008.

[8] S. Yalçin, M. Öztürk, and M. Yamankaradeniz, "On the subclass of Salagean-type harmonic univalent functions," *Journal of Inequalities in Pure and Applied Mathematics*, vol. 8, no. 2, article 54, pp. 1–17, 2007.

[9] M. Darus and K. Al Shaqsi, "On harmonic univalent functions defined by a generalized Ruscheweyh derivatives operator," *Lobachevskii Journal of Mathematics*, vol. 22, pp. 19–26, 2006.

[10] K. Al Shaqsi and M. Darus, "A new class of multivalent harmonic functions," *General Mathematics*, vol. 14, no. 4, pp. 37–46, 2006.

[11] M. Darus and K. Al-Shaqsi, "On certain subclass of harmonic univalent functions," *Journal of Analysis and Applications*, vol. 6, no. 1, pp. 17–28, 2008.

[12] K. Al-Shaqsi and M. Darus, "On harmonic functions defined by derivative operator," *Journal of Inequalities and Applications*, vol. 2008, Article ID 263413, 10 pages, 2008.

[13] J. M. Jahangiri, G. Murugusundaramoorthy, and K. Vijaya, "Salagean-type harmonic univalent functions," *Southwest Journal of Pure and Applied Mathematics*, vol. 2, pp. 77–82, 2002.

[14] J. Clunie and T. Sheil-Small, "Harmonic univalent functions," *Annales Academiae Scientiarum Fennicae. Series A I. Mathematica*, vol. 9, pp. 3–25, 1984.

[15] H. Silverman, "Harmonic univalent functions with negative coefficients," *Journal of Mathematical Analysis and Applications*, vol. 51, no. 1, pp. 283–289, 1998.

[16] H. Silverman and E. M. Silvia, "Subclasses of harmonic univalent functions," *New Zealand Journal of Mathematics*, vol. 28, no. 2, pp. 275–284, 1999.

[17] Jay M. Jahangiri, "Coefficient bounds and univalence criteria for harmonic functions with negative coefficients," *Annales Universitatis Mariae Curie-Skłodowska. Sectio A*, vol. 52, no. 2, pp. 57–66, 1998.