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Research Article

Fixed Points and the Stability of an AQCQ-Functional Equation in Non-Archimedean Normed Spaces

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Using fixed point method, we prove the generalized Hyers-Ulam stability of the following additive-quadratic-cubic-quartic functional equation $f(x+2y)+f(x-2y)=4f(x+y)+4f(x-y)-6f(x)+f(2y)+f(-2y)-4f(y)-4f(-y)$ in non-Archimedean Banach spaces.

1. Introduction and Preliminaries

A *valuation* is a function $|\cdot|$ from a field K into $[0,\infty)$ such that 0 is the unique element having the 0 valuation, $|rs|=|r|\cdot|s|$, and the triangle inequality holds, that is,

$$|r+s| \leq |r|+|s|, \quad \forall r,s \in K. \tag{1.1}$$

A field K is called a *valued field* if K carries a valuation. The usual absolute values of $\mathbb{R}$ and $\mathbb{C}$ are examples of valuations.

Let us consider a valuation which satisfies a stronger condition than the triangle inequality. If the triangle inequality is replaced by

$$|r+s| \leq \max\{|r|,|s|\}, \quad \forall r,s \in K, \tag{1.2}$$

then the function $|\cdot|$ is called a *non-Archimedean valuation*, and the field is called a *non-Archimedean field*. Clearly $|1|=|-1|=1$ and $|n|\leq 1$ for all $n\in\mathbb{N}$. A trivial example of a non-Archimedean valuation is the function $|\cdot|$ taking everything except for 0 into 1 and $|0| = 0$.

Throughout this paper, we assume that the base field is a non-Archimedean field, hence call it simply a field.

Definition 1.1 (see [1]). Let X be a vector space over a field K with a non-Archimedean valuation $|\cdot|$. A function $\|\cdot\| : X \to [0,\infty)$ is said to be a *non-Archimedean norm* if it satisfies the following conditions:

(i) $\|x\| = 0$ if and only if $x = 0$;

(ii) $\|rx\| = |r|\|x\|$ $(r \in K, x \in X)$;

(iii) the strong triangle inequality

$$\|x + y\| \le \max\{\|x\|, \|y\|\}, \quad \forall x, y \in X \tag{1.3}$$

holds. Then $(X, \|\cdot\|)$ is called a *non-Archimedean normed space*.

Definition 1.2. (i) Let $\{x_n\}$ be a sequence in a non-Archimedean normed space X. Then the sequence $\{x_n\}$ is called *Cauchy* if for a given $\varepsilon > 0$, there is a positive integer N such that

$$\|x_n - x_m\| \le \varepsilon \tag{1.4}$$

for all $n, m \ge N$.

(ii) Let $\{x_n\}$ be a sequence in a non-Archimedean normed space X. Then the sequence $\{x_n\}$ is called *convergent* if for a given $\varepsilon > 0$, there are a positive integer N and an $x \in X$ such that

$$\|x_n - x\| \le \varepsilon \tag{1.5}$$

for all $n \ge N$. Then we call $x \in X$ a limit of the sequence $\{x_n\}$, and denote it by $\lim_{n\to\infty} x_n = x$.

(iii) If every Cauchy sequence in X converges, then the non-Archimedean normed space X is called a *non-Archimedean Banach space*.

The stability problem of functional equations originated from a question of Ulam [2] concerning the stability of group homomorphisms. Hyers [3] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [4] for additive mappings and by Rassias [5] for linear mappings by considering an unbounded Cauchy difference. The paper of Rassias [5] has provided a lot of influence in the development of what we call *generalized Hyers-Ulam stability* or as *Hyers-Ulam-Rassias stability* of functional equations. A generalization of the Rassias theorem was obtained by Găvruţa [6] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach.

The functional equation

$$f(x + y) + f(x - y) = 2f(x) + 2f(y) \tag{1.6}$$

is called a *quadratic functional equation*. In particular, every solution of the quadratic functional equation is said to be a *quadratic mapping*. A generalized Hyers-Ulam stability problem for the quadratic functional equation was proved by Skof [7] for mappings $f : X \to Y$, where X is a normed space and Y is a Banach space. Cholewa [8] noticed that the theorem of Skof is still true if the relevant domain X is replaced by an Abelian group. Czerwik [9] proved the generalized Hyers-Ulam stability of the quadratic functional equation.

In [10], Jun and Kim considered the following cubic functional equation:

$$f(2x+y) + f(2x-y) = 2f(x+y) + 2f(x-y) + 12f(x), \tag{1.7}$$

which is called a *cubic functional equation* and every solution of the cubic functional equation is said to be a *cubic mapping*. In [11], Lee et al. considered the following quartic functional equation:

$$f(2x+y) + f(2x-y) = 4f(x+y) + 4f(x-y) + 24f(x) - 6f(y), \tag{1.8}$$

which is called a *quartic functional equation* and every solution of the quartic functional equation is said to be a *quartic mapping*.

The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [12–27]).

Let X be a set. A function $d : X \times X \to [0, \infty]$ is called a *generalized metric* on X if d satisfies

(1) $d(x, y) = 0$ if and only if $x = y$;

(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(3) $d(x, z) \le d(x, y) + d(y, z)$ for all $x, y, z \in X$.

We recall a fundamental result in fixed point theory.

Theorem 1.3 (see [28, 29]). *Let (X, d) be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then for each given element $x \in X$, either*

$$d\left(J^n x, J^{n+1} x\right) = \infty \tag{1.9}$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(1) $d(J^n x, J^{n+1} x) < \infty$, *for all* $n \ge n_0$;

(2) *the sequence* $\{J^n x\}$ *converges to a fixed point* y^* *of* J;

(3) y^* *is the unique fixed point of* J *in the set* $Y = \{y \in X \mid d(J^{n_0} x, y) < \infty\}$;

(4) $d(y, y^*) \le (1/(1-L)) d(y, Jy)$ *for all* $y \in Y$.

In 1996, Isac and Rassias [30] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see [31–36]).

This paper is organized as follows: in Section 2, using the fixed point method, we prove the generalized Hyers-Ulam stability of the additive-quadratic-cubic-quartic functional equation

$$f(x+2y)+f(x-2y)=4f(x+y)+4f(x-y)-6f(x)+f(2y)+f(-2y)-4f(y)-4f(-y) \tag{1.10}$$

in non-Archimedean Banach spaces for an odd case. In Section 3, using the fixed point method, we prove the generalized Hyers-Ulam stability of the additive-quadratic-cubic-quartic functional equation (1.10) in non-Archimedean Banach spaces for an even case.

Throughout this paper, assume that X is a non-Archimedean normed vector space and that Y is a non-Archimedean Banach space.

2. Generalized Hyers-Ulam Stability of the Functional Equation (1.10): An Odd Case

One can easily show that an odd mapping $f : X \to Y$ satisfies (1.10) if and only if the odd mapping $f : X \to Y$ is an additive-cubic mapping, that is,

$$f(x+2y)+f(x-2y)=4f(x+y)+4f(x-y)-6f(x). \tag{2.1}$$

It was shown in Lemma 2.2 of [37] that $g(x) := f(2x)-2f(x)$ and $h(x) := f(2x)-8f(x)$ are cubic and additive, respectively, and that $f(x) = (1/6)g(x)-(1/6)h(x)$.

One can easily show that an even mapping $f : X \to Y$ satisfies (1.10) if and only if the even mapping $f : X \to Y$ is a quadratic-quartic mapping, that is,

$$f(x+2y)+f(x-2y)=4f(x+y)+4f(x-y)-6f(x)+2f(2y)-8f(y). \tag{2.2}$$

It was shown in Lemma 2.1 of [38] that $g(x) := f(2x)-4f(x)$ and $h(x) := f(2x)-16f(x)$ are quartic and quadratic, respectively, and that $f(x) = (1/12)g(x)-(1/12)h(x)$.

For a given mapping $f : X \to Y$, we define

$$\begin{aligned} Df(x,y) := {} & f(x+2y)+f(x-2y)-4f(x+y)-4f(x-y)+6f(x) \\ & -f(2y)-f(-2y)+4f(y)+4f(-y) \end{aligned} \tag{2.3}$$

for all $x, y \in X$.

We prove the generalized Hyers-Ulam stability of the functional equation $Df(x,y) = 0$ in non-Archimedean Banach spaces: an odd case.

Theorem 2.1. *Let $\varphi : X^2 \to [0,\infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x,y) \le \frac{L}{|8|}\varphi(2x,2y) \tag{2.4}$$

for all $x, y \in X$. Let $f : X \to Y$ be an odd mapping satisfying

$$\|Df(x,y)\| \leq \varphi(x,y) \tag{2.5}$$

for all $x, y \in X$. Then there is a unique cubic mapping $C : X \to Y$ such that

$$\|f(2x) - 2f(x) - C(x)\| \leq \frac{L}{|8| - |8|L} \max\{|4|\varphi(x,x), \varphi(2x,x)\} \tag{2.6}$$

for all $x \in X$.

Proof. Letting $x = y$ in (2.5), we get

$$\|f(3y) - 4f(2y) + 5f(y)\| \leq \varphi(y,y) \tag{2.7}$$

for all $y \in X$.

Replacing x by $2y$ in (2.5), we get

$$\|f(4y) - 4f(3y) + 6f(2y) - 4f(y)\| \leq \varphi(2y,y) \tag{2.8}$$

for all $y \in X$.

By (2.7) and (2.8),

$$\begin{aligned} &\|f(4y) - 10f(2y) + 16f(y)\| \\ &\quad \leq \max\{\|4(f(3y) - 4f(2y) + 5f(y))\|, \|f(4y) - 4f(3y) + 6f(2y) - 4f(y)\|\} \\ &\quad \leq \max\{|4| \cdot \|f(3y) - 4f(2y) + 5f(y)\|, \|f(4y) - 4f(3y) + 6f(2y) - 4f(y)\|\} \\ &\quad \leq \max\{|4|\varphi(y,y), \varphi(2y,y)\} \end{aligned} \tag{2.9}$$

for all $y \in X$.

Letting $y := x/2$ and $g(x) := f(2x) - 2f(x)$ for all $x \in X$, we get

$$\left\|g(x) - 8g\left(\frac{x}{2}\right)\right\| \leq \max\left\{|4|\varphi\left(\frac{x}{2}, \frac{x}{2}\right), \varphi\left(x, \frac{x}{2}\right)\right\} \tag{2.10}$$

for all $x \in X$.

Consider the set

$$S := \{g : X \longrightarrow Y\}, \tag{2.11}$$

and introduce the generalized metric on S

$$d(g,h) = \inf\{\mu \in \mathbb{R}_+ : \|g(x) - h(x)\| \leq \mu(\max\{|4|\varphi(x,x), \varphi(2x,x), \forall x \in X\})\}, \tag{2.12}$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that (S, d) is complete. (See the proof of Lemma 2.1 of [39].)

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 8g\left(\frac{x}{2}\right) \tag{2.13}$$

for all $x \in X$.

Let $g, h \in S$ be given such that $d(g, h) = \varepsilon$. Then

$$\|g(x) - h(x)\| \le \varepsilon \cdot \max\{|4|\varphi(x, x), \varphi(2x, x)\} \tag{2.14}$$

for all $x \in X$. Hence

$$\|Jg(x) - Jh(x)\| = \left\|8g\left(\frac{x}{2}\right) - 8h\left(\frac{x}{2}\right)\right\| \le |8|\varepsilon\frac{L}{|8|}\max\{|4|\varphi(x, x), \varphi(2x, x)\} \tag{2.15}$$

for all $x \in X$. So $d(g, h) = \varepsilon$ implies that $d(Jg, Jh) \le L\varepsilon$. This means that

$$d(Jg, Jh) \le Ld(g, h) \tag{2.16}$$

for all $g, h \in S$.

It follows from (2.10) that

$$\left\|g(x) - 8g\left(\frac{x}{2}\right)\right\| \le \frac{L}{|8|}(\max\{|4|\varphi(x, x), \varphi(2x, x)\}) \tag{2.17}$$

for all $x \in X$. So $d(g, Jg) \le L/|8|$.

By Theorem 1.3, there exists a mapping $C : X \to Y$ satisfying the following.

(1) C is a fixed point of J, that is,

$$C\left(\frac{x}{2}\right) = \frac{1}{8}C(x) \tag{2.18}$$

for all $x \in X$. The mapping C is a unique fixed point of J in the set

$$M = \{h \in S : d(g, h) < \infty\}. \tag{2.19}$$

This implies that C is a unique mapping satisfying (2.18) such that there exists a $\mu \in (0, \infty)$ satisfying

$$\|g(x) - C(x)\| \le \mu \cdot \max\{|4|\varphi(x, x), \varphi(2x, x)\} \tag{2.20}$$

for all $x \in X$; since $g : X \to Y$ is odd, $C : X \to Y$ is an odd mapping.

(2) $d(J^n g, C) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} 8^n g\left(\frac{x}{2^n}\right) = C(x) \tag{2.21}$$

for all $x \in X$.

(3) $d(g, C) \leq (1/(1-L))d(g, Jg)$, which implies the inequality

$$d(g, C) \leq \frac{L}{|8| - |8|L}. \tag{2.22}$$

This implies that the inequality (2.6) holds.

By (2.5),

$$\left\|8^n Dg\left(\frac{x}{2^n}, \frac{y}{2^n}\right)\right\| \leq |8|^n \max\left\{\varphi\left(\frac{2x}{2^n}, \frac{2y}{2^n}\right), |2|\varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right)\right\} \tag{2.23}$$

for all $x, y \in X$ and all $n \in \mathbb{N}$. So

$$\left\|8^n Dg\left(\frac{x}{2^n}, \frac{y}{2^n}\right)\right\| \leq |8|^n \frac{L^n}{|8|^n} \max\{\varphi(2x, 2y), |2|\varphi(x, y)\} \tag{2.24}$$

for all $x, y \in X$ and all $n \in \mathbb{N}$. So

$$\|DC(x, y)\| = 0 \tag{2.25}$$

for all $x, y \in X$. Thus the mapping $C : X \to Y$ is cubic, as desired. □

Corollary 2.2. *Let θ and p be positive real numbers with $p < 3$. Let $f : X \to Y$ be an odd mapping satisfying*

$$\|Df(x, y)\| \leq \theta(\|x\|^p + \|y\|^p) \tag{2.26}$$

for all $x, y \in X$. Then there exists a unique cubic mapping $C : X \to Y$ such that

$$\|f(2x) - 2f(x) - C(x)\| \leq \max\{2 \cdot |4|, |2|^p + 1\} \frac{\theta}{|2|^p - |8|} \|x\|^p \tag{2.27}$$

for all $x \in X$.

Proof. The proof follows from Theorem 2.1 by taking

$$\varphi(x, y) := \theta(\|x\|^p + \|y\|^p) \tag{2.28}$$

for all $x, y \in X$. Then we can choose $L = |8|/|2|^p$ and we get the desired result. □

Theorem 2.3. *Let $\varphi : X^2 \to [0,\infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x,y) \le |8|L\varphi\left(\frac{x}{2},\frac{y}{2}\right) \tag{2.29}$$

for all $x, y \in X$. Let $f : X \to Y$ be an odd mapping satisfying (2.5)*. Then there is a unique cubic mapping $C : X \to Y$ such that*

$$\|f(2x) - 2f(x) - C(x)\| \le \frac{1}{|8| - |8|L} \max\{|4|\varphi(x,x), \varphi(2x,x)\} \tag{2.30}$$

for all $x \in X$.

Proof. It follows from (2.10) that

$$\left\|g(x) - \frac{1}{8}g(2x)\right\| \le \frac{1}{|8|} \max\{|4|\varphi(x,x), \varphi(2x,x)\} \tag{2.31}$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 2.1. □

Theorem 2.4. *Let $\varphi : X^2 \to [0,\infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x,y) \le \frac{L}{|2|}\varphi(2x,2y) \tag{2.32}$$

for all $x, y \in X$. Let $f : X \to Y$ be an odd mapping satisfying (2.5)*. Then there is a unique additive mapping $A : X \to Y$ such that*

$$\|f(2x) - 8f(x) - A(x)\| \le \frac{L}{|2| - |2|L} \max\{|4|\varphi(x,x), \varphi(2x,x)\} \tag{2.33}$$

for all $x \in X$.

Proof. Letting $y := x/2$ and $g(x) := f(2x) - 8f(x)$ for all $x \in X$ in (2.9), we get

$$\left\|g(x) - 2g\left(\frac{x}{2}\right)\right\| \le \max\left\{|4|\varphi\left(\frac{x}{2},\frac{x}{2}\right), \varphi\left(x,\frac{x}{2}\right)\right\} \tag{2.34}$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 2.1. □

Corollary 2.5. *Let θ and p be positive real numbers with $p < 1$. Let $f : X \to Y$ be an odd mapping satisfying* (2.26). *Then there exists a unique additive mapping $C : X \to Y$ such that*

$$\left\| f(2x) - 8f(x) - A(x) \right\| \leq \max\{2 \cdot |4|, |2|^p + 1\} \frac{\theta}{|2|^p - |2|} \|x\|^p \tag{2.35}$$

for all $x \in X$.

Proof. The proof follows from Theorem 2.4 by taking

$$\varphi(x, y) := \theta\left(\|x\|^p + \|y\|^p\right) \tag{2.36}$$

for all $x, y \in X$. Then we can choose $L = |2|/|2|^p$ and we get the desired result. □

Theorem 2.6. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x, y) \leq |2| L \varphi\left(\frac{x}{2}, \frac{y}{2}\right) \tag{2.37}$$

for all $x, y \in X$. Let $f : X \to Y$ be an odd mapping satisfying (2.5). *Then there is a unique additive mapping $A : X \to Y$ such that*

$$\left\| f(2x) - 8f(x) - A(x) \right\| \leq \frac{1}{|2| - |2|L} \max\{|4|\varphi(x, x), \varphi(2x, x)\} \tag{2.38}$$

for all $x \in X$.

Proof. It follows from (2.34) that

$$\left\| g(x) - \frac{1}{2} g(2x) \right\| \leq \frac{1}{|2|} \max\{|4|\varphi(x, x), \varphi(2x, x)\} \tag{2.39}$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 2.1. □

3. Generalized Hyers-Ulam Stability of the Functional Equation (1.10): An Even Case

Now we prove the generalized Hyers-Ulam stability of the functional equation $Df(x, y) = 0$ in non-Archimedean Banach spaces: an even case.

Theorem 3.1. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x, y) \leq \frac{L}{|16|} \varphi(2x, 2y) \tag{3.1}$$

for all $x, y \in X$. Let $f : X \to Y$ be an even mapping satisfying (2.5) *and $f(0) = 0$. Then there is a unique quartic mapping $Q : X \to Y$ such that*

$$\|f(2x) - 4f(x) - Q(x)\| \le \frac{L}{|16| - |16|L} \max\{|4|\varphi(x,x), \varphi(2x,x)\} \tag{3.2}$$

for all $x \in X$.

Proof. Letting $x = y$ in (2.5), we get

$$\|f(3y) - 6f(2y) + 15f(y)\| \le \varphi(y,y) \tag{3.3}$$

for all $y \in X$.

Replacing x by $2y$ in (2.5), we get

$$\|f(4y) - 4f(3y) + 4f(2y) + 4f(y)\| \le \varphi(2y,y) \tag{3.4}$$

for all $y \in X$.

By (3.3) and (3.4),

$$\begin{aligned}
&\|f(4y) - 20f(2y) + 64f(y)\| \\
&\quad \le \max\{\|4(f(3y) - 6f(2y) + 15f(y))\|, \|f(4y) - 4f(3y) + 4f(2y) + 4f(y)\|\} \\
&\quad \le \max\{|4| \cdot \|f(3y) - 6f(2y) + 15f(y)\|, \|f(4y) - 4f(3y) + 4f(2y) + 4f(y)\|\} \\
&\quad \le \max\{|4|\varphi(y,y), \varphi(2y,y)\}
\end{aligned} \tag{3.5}$$

for all $y \in X$.

Letting $y := x/2$ and $g(x) := f(2x) - 4f(x)$ for all $x \in X$, we get

$$\left\|g(x) - 16g\left(\frac{x}{2}\right)\right\| \le \max\left\{|4|\varphi\left(\frac{x}{2}, \frac{x}{2}\right), \varphi\left(x, \frac{x}{2}\right)\right\} \tag{3.6}$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 2.1. □

Corollary 3.2. *Let θ and p be positive real numbers with $p < 4$. Let $f : X \to Y$ be an even mapping satisfying* (2.26) *and $f(0) = 0$. Then there exists a unique quartic mapping $Q : X \to Y$ such that*

$$\|f(2x) - 4f(x) - Q(x)\| \le \max\{2 \cdot |4|, |2|^p + 1\} \frac{\theta}{|2|^p - |16|} \|x\|^p \tag{3.7}$$

for all $x \in X$.

Proof. The proof follows from Theorem 3.1 by taking

$$\varphi(x, y) := \theta(\|x\|^p + \|y\|^p) \tag{3.8}$$

for all $x, y \in X$. Then we can choose $L = |16|/|2|^p$ and we get the desired result. □

Theorem 3.3. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x, y) \leq |16| L \varphi\left(\frac{x}{2}, \frac{y}{2}\right) \tag{3.9}$$

for all $x, y \in X$. Let $f : X \to Y$ be an even mapping satisfying (2.5) *and $f(0) = 0$. Then there is a unique quartic mapping $Q : X \to Y$ such that*

$$\|f(2x) - 4f(x) - Q(x)\| \leq \frac{1}{|16| - |16|L} \max\{|4|\varphi(x, x), \varphi(2x, x)\} \tag{3.10}$$

for all $x \in X$.

Proof. It follows from (3.6) that

$$\left\| g(x) - \frac{1}{16} g(2x) \right\| \leq \frac{1}{|16|} \max\{|4|\varphi(x, x), \varphi(2x, x)\} \tag{3.11}$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 2.1. □

Theorem 3.4. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x, y) \leq \frac{L}{|4|} \varphi(2x, 2y) \tag{3.12}$$

for all $x, y \in X$. Let $f : X \to Y$ be an even mapping satisfying (2.5) *and $f(0) = 0$. Then there is a unique quadratic mapping $T : X \to Y$ such that*

$$\|f(2x) - 16f(x) - T(x)\| \leq \frac{L}{|4| - |4|L} \max\{|4|\varphi(x, x), \varphi(2x, x)\} \tag{3.13}$$

for all $x \in X$.

Proof. Letting $y := x/2$ and $g(x) := f(2x) - 16f(x)$ for all $x \in X$ in (3.5), we get

$$\left\| g(x) - 4g\left(\frac{x}{2}\right) \right\| \leq \max\left\{ |4|\varphi\left(\frac{x}{2}, \frac{x}{2}\right), \varphi\left(x, \frac{x}{2}\right) \right\} \tag{3.14}$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 2.1. □

Corollary 3.5. *Let θ and p be positive real numbers with $p < 2$. Let $f : X \to Y$ be an even mapping satisfying* (2.26) *and $f(0) = 0$. Then there exists a unique quadratic mapping $T : X \to Y$ such that*

$$\|f(2x) - 16f(x) - T(x)\| \le \max\{2 \cdot |4|, |2|^p + 1\} \frac{\theta}{|2|^p - |4|} \|x\|^p \tag{3.15}$$

for all $x \in X$.

Proof. The proof follows from Theorem 3.4 by taking

$$\varphi(x, y) := \theta(\|x\|^p + \|y\|^p) \tag{3.16}$$

for all $x, y \in X$. Then we can choose $L = |4|/|2|^p$ and we get the desired result. □

Theorem 3.6. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x, y) \le |4| L \varphi\left(\frac{x}{2}, \frac{y}{2}\right) \tag{3.17}$$

for all $x, y \in X$. Let $f : X \to Y$ be an even mapping satisfying (2.5) *and $f(0) = 0$. Then there is a unique quadratic mapping $T : X \to Y$ such that*

$$\|f(2x) - 16f(x) - T(x)\| \le \frac{1}{|4| - |4|L} \max\{|4|\varphi(x, x), \varphi(2x, x)\} \tag{3.18}$$

for all $x \in X$.

Proof. It follows from (3.14) that

$$\left\|g(x) - \frac{1}{4} g(2x)\right\| \le \frac{1}{|4|} \max\{|4|\varphi(x, x), \varphi(2x, x)\} \tag{3.19}$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 2.1. □

For a given f, let $f_o(x) := ((f(x) - f(-x))/2)$ and $f_e(x) := ((f(x) + f(-x))/2)$. Then f_o is odd and f_e is even. Let $g_o(x) := f_o(2x) - 2f_o(x)$ and $h_o(x) := f_o(2x) - 8f_o(x)$. Then $f_o(x) = (1/6)g_o(x) - (1/6)h_o(x)$. Let $g_e(x) := f_e(2x) - 4f_e(x)$ and $h_e(x) := f_e(2x) - 16f_e(x)$. Then $f_e(x) = (1/12)g_e(x) - (1/12)h_e(x)$. Thus

$$f(x) = \frac{1}{6} g_o(x) - \frac{1}{6} h_o(x) + \frac{1}{12} g_e(x) - \frac{1}{12} h_e(x). \tag{3.20}$$

Theorem 3.7. *Let $\varphi : X^2 \to [0,\infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x,y) \leq \frac{L}{|2|}\varphi(2x,2y) \tag{3.21}$$

for all $x, y \in X$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (2.5). Then there exist an additive mapping $A : X \to Y$, a quadratic mapping $T : X \to Y$, a cubic mapping $C : X \to Y$, and a quartic mapping $Q : X \to Y$ such that

$$\begin{aligned} &\left\| f(x) - \frac{1}{6}A(x) - \frac{1}{12}T(x) - \frac{1}{6}C(x) - \frac{1}{12}Q(x) \right\| \\ &\quad \leq \max\left\{ \frac{L}{|6|\cdot|2|(1-L)}, \frac{L}{|12|\cdot|4|(1-L)}, \frac{L}{|6|\cdot|8|(1-L)}, \frac{L}{|12|\cdot|16|(1-L)} \right\} \\ &\qquad \cdot \frac{1}{|2|}\max\{|4|\varphi(x,x), \varphi(2x,x), |4|\varphi(-x,-x), \varphi(-2x,-x)\} \\ &\quad \leq \frac{L}{|12|\cdot|16|\cdot|2|(1-L)} \cdot \max\{|4|\varphi(x,x), \varphi(2x,x), |4|\varphi(-x,-x), \varphi(-2x,-x)\} \end{aligned} \tag{3.22}$$

for all $x \in X$.

Corollary 3.8. *Let θ and p be positive real numbers with $p < 1$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (2.5). Then there exist an additive mapping $A : X \to Y$, a quadratic mapping $T : X \to Y$, a cubic mapping $C : X \to Y$ and a quartic mapping $Q : X \to Y$, such that*

$$\left\| f(x) - \frac{1}{6}A(x) - \frac{1}{12}T(x) - \frac{1}{6}C(x) - \frac{1}{12}Q(x) \right\| \leq \max\{2\cdot|4|, |2|^p + 1\} \cdot \frac{\theta}{|12|(|2|^p - |2|)}\|x\|^p \tag{3.23}$$

for all $x \in X$.

Proof. The proof follows from Theorem 3.7 by taking

$$\varphi(x,y) := \theta(\|x\|^p + \|y\|^p) \tag{3.24}$$

for all $x, y \in X$. Then we can choose $L = |2|/|2|^p$ and we get the desired result. □

Theorem 3.9. *Let $\varphi : X^2 \to [0,\infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x,y) \leq |16|L\varphi\left(\frac{x}{2}, \frac{y}{2}\right) \tag{3.25}$$

for all $x, y \in X$. *Let* $f : X \to Y$ *be a mapping satisfying* $f(0) = 0$ *and* (2.5). *Then there exist an additive mapping* $A : X \to Y$, *a quadratic mapping* $T : X \to Y$, *a cubic mapping* $C : X \to Y$ *and a quartic mapping* $Q : X \to Y$ *such that*

$$\begin{aligned}
&\left\| f(x) - \frac{1}{6}A(x) - \frac{1}{12}T(x) - \frac{1}{6}C(x) - \frac{1}{12}Q(x) \right\| \\
&\quad \le \max\left\{ \frac{1}{|6|\cdot|2|(1-L)}, \frac{1}{|12|\cdot|4|(1-L)}, \frac{1}{|6|\cdot|8|(1-L)}, \frac{1}{|12|\cdot|16|(1-L)} \right\} \\
&\qquad \cdot \frac{1}{|2|}\max\{|4|\varphi(x,x), \varphi(2x,x), |4|\varphi(-x,-x), \varphi(-2x,-x)\} \\
&\quad \le \frac{1}{|12|\cdot|16|\cdot|2|(1-L)} \cdot \max\{|4|\varphi(x,x), \varphi(2x,x), |4|\varphi(-x,-x), \varphi(-2x,-x)\}
\end{aligned} \tag{3.26}$$

for all $x \in X$.

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