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*Research Article*

# Technical Note on $(Q, r, L)$ Inventory Model with Defective Items

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Under a reasonable assumption, we derive an analytical approach that verifies uniqueness of the optimal solution for stochastic inventory models with defective items. Our approach implies a robust method to find the optimal solution.

## 1. Introduction

Wu and Ouyang [1] considered the imperfect production of the supplier and/or damage in delivery so that an arrival order lot may contain defective items. They extended the inventory model presented by Paknejad et al. [2] with constant lead time and fixed defective rate in an order lot to stochastic inventory models with crashable lead time and a random number of defective items. Wu and Ouyang [1] assumed that the purchasers inspect all the items they have ordered. Inspection is proposed to be nondestructive and error-free. All defective items are detected and will be returned to the vendor at the time of delivery of the next lot. The inventory model is continuously reviewed and an order of size, $Q$, is made whenever the inventory level falls to the reorder point $r$. In essence, Wu and Ouyang [1] applied the minimax approach for the stochastic inventory models with distribution-free demand to derive a mixture inventory model with back orders and lost sales in which an arrival order lot may contain defective items and the number of defective items is a random variable. The decision variables include the order quantity, the reorder point, and the lead time. Wu and Ouyang [1] developed two inventory models: for the first model the lead time demand follows a normal distribution, and for the second model, the distribution of the lead time demand is unknown except for the finite first and second moments. Because the information about the form of the probability distribution of lead time demand is often limited in practice, we consider in this paper only the second model of Wu and Ouyang [1] in which the minimax approach proposed by Moon and Gallego [3] is applied. Wu and Ouyang [1] claimed that they proposed an algorithm procedure to obtain the optimal ordering strategy. However, we have found and shown that the iterative method they employed in the attempt to minimize the average cost may not be able to locate or guarantee the optimal solution. For rectification and improvement, the aim of this study is to develop an analytical approach to find an upper bound and a lower bound for the order quantity. Our approach will ensure the uniqueness for the optimal solution. The same problem as Wu and Ouyang [1] is examined to demonstrate our proposed approach.

## 2. Review of Previous Results

To be compatible with the results of Wu and Ouyang [1], we use the same notation and assumptions as them. For the distribution-free model, we directly quote the objective function of Wu and Ouyang [1]

$$\begin{aligned}
\mathrm{EAC}^u(Q,k,L) = {} & \frac{AD}{Q(1-E(p))} + \frac{h}{2}\left\{Q(1-E(p)) + Q\frac{E(p^2)-E^2(p)}{1-E(p)} + \frac{E(p(1-p))}{1-E(p)}\right\} \\
& + h\left\{k\sigma\sqrt{L} + \frac{1-\beta}{2}\sigma\sqrt{L}\left(\sqrt{1+k^2}-k\right)\right\} \\
& + \frac{D(\pi+\pi_0(1-\beta))}{2Q(1-E(p))}\sigma\sqrt{L}\left(\sqrt{1+k^2}-k\right) + (Q-1)h'\frac{E(p(1-p))}{1-E(p)} \\
& + \frac{Dv}{1-E(p)} + \frac{D}{Q(1-E(p))}\left(c_i(L_{i-1}-L) + \sum_{j=1}^{i-1} c_j(b_j-a_j)\right),
\end{aligned} \tag{2.1}$$

for $L \in [L_i, L_{i-1}]$, where $\mathrm{EAC}^u(Q,k,L)$ is the least upper bound of $\mathrm{EAC}(Q,k,L)$. Wu and Ouyang derived that $\mathrm{EAC}^u(Q,k,L)$ is a concave function on $[L_i, L_{i-1}]$ so the minimum must occur at boundary point $L_i$ or $L_{i-1}$. To simplify the expression, we use $L$ instead of $L_i$ or $L_{i-1}$. For $\mathrm{EAC}^u(Q,k,L)$, they computed the first partial derivative with respect to $Q$ and $k$, from $(\partial/\partial Q)(\mathrm{EAC}^u(Q,k,L)) = 0$ and $(\partial/\partial k)(\mathrm{EAC}^u(Q,k,L)) = 0$ to imply that

$$Q = \left[\frac{2D}{h\delta}\left\{A + c_i(L_{i-1}-L) + \sum_{j=1}^{i-1} c_j(b_j-a_j) + \frac{\pi+\pi_0(1-\beta)}{2}\sigma\sqrt{L}\left(\sqrt{1+k^2}-k\right)\right\}\right]^{1/2}, \tag{2.2}$$

where $\delta = 1 - 2E(p) + E(p^2) + 2(h'/h)E(p(1-p))$, and

$$\frac{2\sqrt{1+k^2}}{\sqrt{1+k^2}-k} = 1-\beta + \frac{D(\pi+\pi_0(1-\beta))}{hQ(1-E(p))}. \tag{2.3}$$

Wu and Ouyang [1] stated that the optimal solution can be obtained by the iterative method. Nevertheless, we point out that two sequences, generated by Wu and Ouyang [1], are not guaranteed to converge. On the other hand, even when the sequence converges, the reason why the limit is the optimal solution is not discussed or verified. In Section 3, we will develop an approach to show that there is an optimal order quantity under a reasonable assumption, based on which the feasible domain with an upper bound and a lower bound can be derived.

## 3. Our Revision

We simplify the expression in (2.2) and (2.3) as follows:

$$Q^2 = \alpha_1 + \alpha_2\left(\sqrt{1+k^2} - k\right), \tag{3.1}$$

$$\frac{\sqrt{1+k^2} - k}{\sqrt{1+k^2}} = \frac{2\alpha_3 Q}{(1-\beta)\alpha_3 Q + \alpha_4}, \tag{3.2}$$

where $\alpha_1 = (2D/h\delta)(A + c_i(L_{i-1} - L) + \sum_{j=1}^{i-1} c_j(b_j - a_j))$, $\alpha_2 = (D/h\delta)(\pi + \pi_0(1-\beta))\sigma\sqrt{L}$, $\alpha_3 = h(1 - E(p))$, and $\alpha_4 = D(\pi + \pi_0(1-\beta))$. Owing to (3.2), we compute

$$1 - \frac{\sqrt{1+k^2} - k}{\sqrt{1+k^2}} = 1 - \frac{2\alpha_3 Q}{(1-\beta)\alpha_3 Q + \alpha_4}, \tag{3.3}$$

to imply that

$$\frac{k}{\sqrt{1+k^2}} = \frac{\alpha_4 - (1+\beta)\alpha_3 Q}{(1-\beta)\alpha_3 Q + \alpha_4}. \tag{3.4}$$

If we observe the left hand side of (3.4), then $k$ is a safe factor with $k \geq 0$. We try to locate interior solution for the first partial derivatives system, so we only consider $k > 0$. It turns out that an upper bound for the lot size, $Q$, is derived

$$\alpha_4 > (1+\beta)\alpha_3 Q. \tag{3.5}$$

If we flip over and take square and then minus one on both side of (3.4), it yields that

$$\frac{1}{k^2} = \frac{4\alpha_3 Q(\alpha_4 - \beta\alpha_3 Q)}{(\alpha_4 - (1+\beta)\alpha_3 Q)^2}. \tag{3.6}$$

According to (3.5), the relation, $\alpha_4 - \beta\alpha_3 Q > 0$, holds such that (3.6) is well defined and based on which we find a relation to express $k$ as a function in $Q$ so that

$$k = \frac{\alpha_4 - (1+\beta)\alpha_3 Q}{2\sqrt{\alpha_3 Q(\alpha_4 - \beta\alpha_3 Q)}}. \tag{3.7}$$

**Table 1:** The ratio of $\alpha_4/(1+\beta)\alpha_3\sqrt{\alpha_1+\alpha_2}$.

| | $L = L_0$ | $L = L_1$ | $L = L_2$ | $L = L_3$ |
|---|---|---|---|---|
| $\beta = 0$ | 18.855 | 20.090 | 21.856 | 22.962 |
| $\beta = 0.5$ | 9.675 | 10.264 | 11.073 | 11.513 |
| $\beta = 0.8$ | 6.211 | 6.552 | 6.994 | 7.180 |
| $\beta = 1$ | 4.174 | 4.369 | 4.600 | 4.648 |

Based on (3.7), we derive that

$$\sqrt{1+k^2} = \left(1 + \frac{\left(\alpha_4 - (1+\beta)\alpha_3 Q\right)^2}{4\alpha_3 Q(\alpha_4 - \beta\alpha_3 Q)}\right)^{1/2} = \frac{\alpha_4 + (1-\beta)\alpha_3 Q}{2\sqrt{\alpha_3 Q(\alpha_4 - \beta\alpha_3 Q)}}. \tag{3.8}$$

Using (3.7) and (3.8), equation (3.1) turns into

$$Q^2 = \alpha_1 + \alpha_2\sqrt{\frac{\alpha_3 Q}{\alpha_4 - \beta\alpha_3 Q}}. \tag{3.9}$$

From (3.9), we obtain the lower bound

$$Q > \sqrt{\alpha_1}. \tag{3.10}$$

Equation (3.5) yields $\sqrt{\alpha_3 Q/(\alpha_4 - \beta\alpha_3 Q)} < 1$, and therefore we have (3.11) owing to (3.9)

$$Q < \sqrt{\alpha_1 + \alpha_2}. \tag{3.11}$$

Next, we will compare the two upper bounds $\alpha_4/(1+\beta)\alpha_3$ from (3.5) and $\sqrt{\alpha_1+\alpha_2}$ from (3.11). We use the same data as that in Wu and Ouyang [1]. We compute the two upper bounds for different values of $\beta$ and $L_i$, and then list the ratio of $\alpha_4/(1+\beta)\alpha_3$ over $\sqrt{\alpha_1+\alpha_2}$ in Table 1.

From Table 1, it is clear that $\alpha_4/(1+\beta)\alpha_3 > \sqrt{\alpha_1+\alpha_2}$. Hence, by (3.5), (3.10) and (3.11), and our comparison in Table 1, we conclude that the feasible domain for the lot size, $Q$, is

$$\sqrt{\alpha_1} < Q < \sqrt{\alpha_1+\alpha_2}. \tag{3.12}$$

We now solve (3.9) under the restriction of (3.12) by assuming the auxiliary function,

$$f(Q) = \left(Q^2 - \alpha_1\right)\sqrt{\alpha_4 - \beta\alpha_3 Q} - \alpha_2\sqrt{\alpha_3 Q}. \tag{3.13}$$

It follows that

$$\begin{aligned} f''(Q) &= 2(\alpha_4 - \beta\alpha_3 Q)^{-0.5}(\alpha_4 - 2\beta\alpha_3 Q) \\ &\quad + \frac{1}{4}(\alpha_4 - \beta\alpha_3 Q)^{-1.5} Q^{-1.5}\left[\alpha_2\sqrt{\alpha_3}(\alpha_4 - \beta\alpha_3 Q)^{1.5} - Q^{1.5}(\beta\alpha_3)^2\left(Q^2 - \alpha_1\right)\right]. \end{aligned} \tag{3.14}$$

From (3.5), we see that

$$\begin{aligned} \alpha_4 &\geq (1+\beta)\alpha_3 Q \geq 2\beta\alpha_3 Q, \\ \sqrt{\alpha_3}(\alpha_4 - \beta\alpha_3 Q)^{1.5} &\geq \alpha_3^2 Q^{1.5} \geq Q^{1.5}(\beta\alpha_3)^2. \end{aligned} \tag{3.15}$$

Together with (3.12), it yields that

$$\alpha_2 > \left(Q^2 - \alpha_1\right). \tag{3.16}$$

By combining the results of (3.14) through (3.16), we obtain

$$f''(Q) > 0. \tag{3.17}$$

Therefore, $f(Q)$ is a convex function. The auxiliary function can be rewritten as

$$f(Q) = \sqrt{\alpha_4 - \beta\alpha_3 Q}\left[Q^2 - \alpha_1 - \alpha_2\sqrt{\frac{\alpha_3 Q}{\alpha_4 - \beta\alpha_3 Q}}\right]. \tag{3.18}$$

Next, we will show that $f(\sqrt{\alpha_1 + \alpha_2}) > 0$. If $Q = \sqrt{\alpha_1 + \alpha_2}$, based on Table 1, the following expression can be derived,

$$Q = \sqrt{\alpha_1 + \alpha_2} < \frac{\alpha_4}{(1+\beta)\alpha_3}. \tag{3.19}$$

Hence, if $Q = \sqrt{\alpha_1 + \alpha_2}$, then $\alpha_3 Q < \alpha_4 - \beta\alpha_3 Q$ and then

$$f\left(\sqrt{\alpha_1 + \alpha_2}\right) > \sqrt{\alpha_4 - \beta\alpha_3\sqrt{\alpha_1 + \alpha_2}}\left[\left(\sqrt{\alpha_1 + \alpha_2}\right)^2 - \alpha_1 - \alpha_2\right] = 0. \tag{3.20}$$

From $f(\sqrt{\alpha_1}) = -\alpha_2\sqrt{\sqrt{\alpha_1}\alpha_3} < 0$ to $f(\sqrt{\alpha_1 + \alpha_2}) > 0$, $f(Q)$ is a convex function. We may separate the graph of $f(Q)$ into two parts. On the left wing, $f(Q)$ decreases to its minimum. On the right wing, $f(Q)$ increases from its minimum. Accordingly, we divide the problem into two cases.

**Table 2:** Our minimum solution for $L_i$ with $i = 0, 1, 2, 3$.

| $\beta$ | $i$ | $L_i$ | $Q_i$ | $k_i$ | $\mathrm{EAC}^u(Q_i, k_i, L_i)$ |
|---|---|---|---|---|---|
| 0 | 0 | 8 | 193.975460 | 3.028636 | 6439.612786 |
| | 1 | 6 | 185.764855 | 3.098322 | 6128.294847 |
| | 2 | 4 | 178.358645 | 3.165197 | 5793.589860 |
| | 3 | 3 | 179.736103 | 3.152452 | 5697.410674 |
| 0.5 | 0 | 8 | 179.636047 | 2.430715 | 5894.226414 |
| | 1 | 6 | 173.251839 | 2.479599 | 5647.166278 |
| | 2 | 4 | 168.167444 | 2.520443 | 5394.230294 |
| | 3 | 3 | 171.139009 | 2.496358 | 5353.564526 |
| 0.8 | 0 | 8 | 168.715543 | 1.939091 | 5463.904571 |
| | 1 | 6 | 163.754138 | 1.973400 | 5268.847034 |
| | 2 | 4 | 160.468314 | 1.996945 | 5081.775547 |
| | 3 | 3 | 164.670385 | 1.966953 | 5085.625751 |
| 1.0 | 0 | 8 | 159.607557 | 1.486658 | 5086.909364 |
| | 1 | 6 | 155.853649 | 1.510064 | 4938.486078 |
| | 2 | 4 | 154.088812 | 1.521335 | 4810.142818 |
| | 3 | 3 | 159.331609 | 1.488353 | 4853.426462 |

For the first case, $\sqrt{\alpha_1}$ is on the left wing, so that from $f(\sqrt{\alpha_1}) < 0$, $f(Q)$ continuously decreases to its minimum, say $f(Q_{\min})$, with $f(Q_{\min}) < 0$ and then $f(Q)$ changes, to increase. Therefore, there is a unique point, say $Q^*$, that satisfies

$$f(Q^*) = 0 \tag{3.21}$$

with $\sqrt{\alpha_1} \le Q_{\min} < Q^* < \sqrt{\alpha_1 + \alpha_2}$.

For the second case, $\sqrt{\alpha_1}$ is on the right wing, with $Q_{\min} \le \sqrt{\alpha_1}$ so that $f(Q)$ increases from $f(\sqrt{\alpha_1}) < 0$ to $f(\sqrt{\alpha_1 + \alpha_2}) > 0$ such that there is a unique point, (we still) say $Q^*$, that satisfies

$$f(Q^*) = 0, \tag{3.22}$$

with $Q_{\min} \le \sqrt{\alpha_1} < Q^* < \sqrt{\alpha_1 + \alpha_2}$.

We are handling a minimum problem that is bounded below by zero so that the existence of the optimal solution is trivial.

Under our assumption, $\alpha_4 > \sqrt{\alpha_1 + \alpha_2}(1 + \beta)\alpha_3$, we have shown that there is a unique solution, $Q^*$ from (3.21) or (3.22), respectively, for the first derivative system such that $Q^*$ and $k(Q^*)$, derived by (3.7), are the optimal quantity and the minimum solution for the stochastic inventory model, respectively.

## 4. Numerical Examples

In order to demonstrate that our approach can derive the minimum solution, we refer to the previous example and list our computed results in Table 2. $Q_i$ is the unique solution for $f(Q) = 0$ of (3.13). $k_i$ and $\mathrm{EAC}^u(Q_i, k_i, L_i)$ are derived by (3.7) and (2.1), respectively.

**Table 3:** The comparison between Wu and Ouyang [1] and ours.

| | Wu and Ouyang ([1]) | | Our results | |
|---|---|---|---|---|
| $\beta$ | $Q$ | $\mathrm{EAC}^u(Q,k,L)$ | $Q$ | $\mathrm{EAC}^u(Q,k,L)$ |
| 0 | 183 | 5697.95 | 179.74 | 5697.41 |
| 0.5 | 174 | 5354.01 | 171.14 | 5353.56 |
| 0.8 | 163 | 5082.14 | 160.47 | 5081.78 |
| 1.0 | 157 | 4810.65 | 154.09 | 4810.14 |

**Table 4:** The iterative method of Wu and Ouyang [1].

| | $n=1$ | $n=2$ | $n=3$ | $n=4$ | $n=5$ | $n=6$ |
|---|---|---|---|---|---|---|
| $Q_n$ | 315.62 | 194.45 | 181.25 | 179.90 | 179.75 | 179.74 |
| $k_n$ | 2.2916 | 3.0248 | 3.1386 | 3.1510 | 3.1523 | 3.1524 |

From Table 2, we know the minimum for each backordered rate, and then we list the comparison between our findings and that of Wu and Ouyang [1] in Table 3.

By comparing the third and the fifth column of Table 3, it shows that our findings are slightly better than those of Wu and Ouyang [1]. Moreover, we try to consider the iterative method, which is the solution algorithm of Wu and Ouyang [1]. In (2.2), $Q$ is an explicit function of $k$. By plugging, into a value of $k$, we derive the corresponding value of $Q$.

However, in (2.3), $k$ is expressed as an implicit function of $Q$. When we plug into a value of $Q$, then (2.3) only derives the value of $\sqrt{1+k^2}/(\sqrt{1+k^2}-k)$. Hence, we will not use (2.3). Instead, we apply the equivalent relation in (3.7). Consequently, we consider (2.2) and (3.7) in studying the iterative algorithm of Wu and Ouyang [1].

With $k_0 = 0$ (proposed by them), and for example, $\beta = 0$ and $i = 3$, it yields that

$$Q_{n+1} = \left[\theta_1 + \theta_2\left(\sqrt{1+k_n^2} - k_n\right)\right]^{1/2}, \qquad k_{n+1} = \frac{1-\theta_3 Q_n}{2\sqrt{\theta_3 Q_n}}, \tag{4.1}$$

with $\theta_1 = (2D/h\delta)(A+\sum_{j=1}^{3} c_j(b_j-a_j))$, $\theta_2 = (D\sigma/h\delta)(\pi+\pi_0)\sqrt{L_3}$, and $\theta_3 = h(1-E(p))/D(\pi+\pi_0)$. The computation results for two sequences $(Q_n)$ and $(k_n)$ are listed in Table 4.

From Table 4, we see that by iterative algorithm proposed by Wu and Ouyang [1], for $\beta = 0$ and lead time, $L_3$, Wu and Ouyang [1] should derive that $k^* = 3.15$ and $Q^* = 179.74$. However, they derived that $Q^* = 183$. It may indicate that the iterative method with two generated sequences may be too complex to execute, such that our approach provides an improvement to locate the optimal solution.


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